Version: May 31, 2013

Lecture 1

1. Introduction

- Me: Thomas Kragh, office: Å14261, thomas.kragh@math.uu.se.
- Course web page: `www.uu.se/~tkragh/1MA259-2013/Kurs.html`
- Differential topology: Introduction to smooth manifolds (locally looks [smoothly] like $\mathbb{R}^n$).
- Examples: physics phase-space, solutions to $f = 0$ of smooth maps $f\colon \mathbb{R}^n \to \mathbb{R}^m$ (under certain conditions).
- Plan:
  - These notes.
  - Milnors book [3] (these notes will continue as supplements).

Some notation:

- $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$
- $D^n_r = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$
- $\mathring{A}$ denotes the interior of A.
- So $\mathring{D}^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$
- $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\} = \partial D^n$
- $U \mathring{\subset} X$ means U is an open subset of X.

2. Manifolds

Definition 2.1. An n-dimensional topological manifold M is a topological space such that

M1: The space M is Hausdorff.
M2: The space M is second countable.
M3: For any point $x \in M$ there is a neighborhood $V \ni x$ such that V is homeomorphic to $U \mathring{\subset} \mathbb{R}^n$.

Examples:

- $\mathbb{R}^n$. Indeed, M1 follows because it is a metric space. M2 because the set of all open balls with rational center and rational radius is a countable basis.
- $U \mathring{\subset} \mathbb{R}^n$. Indeed, properties M1 and M2 are inherited by sub-spaces.
- S^n. Again M1 and M2 follows because it is a sub-space in $\mathbb{R}^{n+1}$. To see M3 we may use that the function

$$\varphi\colon \mathring{D}^n \to S^n$$

given by

$$\varphi(x_1, \dots, x_n) = \left(x_1, \dots, x_n, \sqrt{1 - (x_1^2 + x_2^2 + \cdots + x_n^2)}\right)$$

defines a homeomorphism from the open unit disc in $\mathbb{R}^n$ onto the open subset (upper hemisphere) $S^n \cap \{x_{n+1} > 0\}$. Indeed, the inverse is

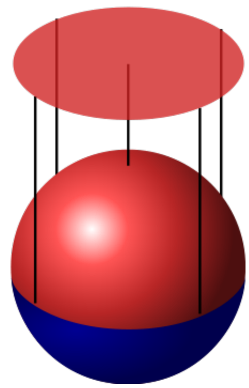


FIGURE 1. Map from disc to upper hemisphere (source:Wikipedia)

given by projection $\psi(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n)$ and is continuous. So all points in this set has a neighborhood as in M3. We may similarly prove M3 for the set $S^n \cap \{x_{n+1} < 0\}$ by replacing φ with

$$\varphi(x_1, \ldots, x_n) = (x_1, \ldots, x_n, -\sqrt{1 - (x_1^2 + x_2^2 + \cdots + x_n^2)})$$

So we are left with proving M3 for all points where $x_{n+1} = 0$. However, for any $x \in S^n$ some x_i is non-zero and using same trick with x_i instead of x_{n+1} proves that we have a neighborhood homeomorphic to $\mathring{D}^n$.

Such maps φ (and their inverses ψ) are called coordinate charts, and what we just did is called covering S^n with charts. In fact we covered S^n with $2(n+1)$ charts.

Exercise 2.2. The Cartesian product $M_1 \times M_2$ of two manifolds M_1 and M_2 is a manifold.

Example 2.3. The following is a space that satisfies M2 and M3 but not M1. Define:

$$X = \mathbb{R} \times \{0, 1\}/\sim$$

Here

$$(x, t) \sim (y, s) \qquad \text{if} \qquad \begin{cases} (x, t) = (y, s) & \text{(to ensure reflexivity) or} \\ x = y \neq 0 \end{cases}$$

Exercise: This satisfies M2 and M3, but not M1.

Convention: From now on everything will be Hausdorff unless otherwise stated, and thus compact will mean compact Hausdorff.

What about M2? Why is this included in the definition. This is to make manifolds "small". This idea is explained in the following lemma and example, but other interpretations/consequences of this "smallness" will be seen later.

Lemma 2.4. Any n-dimensional manifold M has a countable dense subset.

Proof. Take a countable base $(U_i)_{i\in\mathbb{N}}$ for the topology of M pick a point in each, say $x_i \in U_i$. These points form a dense subset. Indeed, any open set contains a U_i in its interior - so it contains a point x_i. So the largest open set not containing any x_i's is the empty set - hence the closure of $\cup_i\{x_i\}$ is all of M. □

Example 2.5. A space that satisfies M1 and M3 but not M2:

$$X = \bigsqcup_{r\in\mathbb{R}} \mathbb{R}^n \qquad (\text{uncountably many disjoint copies of } \mathbb{R}^n)\,.$$

Note in contrast that the space

$$M = \bigsqcup_{i\in\mathbb{N}} \mathbb{R}^n \qquad (\text{countably many disjoint copies of } \mathbb{R}^n)\,.$$

is a manifold, and the uncountability of $\mathbb{R}$ is the point in this example.

Lemma 2.6. Every n-dimensional topological manifold M can be written as a countable union

$$M = \bigcup_{k=0}^{\infty} K_k$$

of increasing compact sub-spaces $K_0 \subset K_1 \subset \cdots \subset K_k \subset \cdots \subset M$ such that $K_k \subset \mathring{K}_{k+1}$.

Proof. This is true for any $U \overset{\circ}{\subset} \mathbb{R}^n$. Indeed, by triangle inequality the distance function $d_A(x) = \inf_{y\in A}\|x-y\|$ to a set $A \subset \mathbb{R}^n$ is continuous (in x). So we may define

$$K_k(U) = D^n_k \cap \{x \in \mathbb{R}^n \mid d_{U^c}(x) \geq 1/k\}$$

which is closed and bounded in $\mathbb{R}^n$ and hence compact. Here $U^c = \mathbb{R}^n - U$ is the complement. Since U is open we see that

$$U = \bigcup_{k\in\mathbb{N}} K_k(U).$$

Indeed, since U is open any point $x \in U$ has a positive distance to U^c (and it has some norm $\|x\|$). Also we see that $K_k(U) \subset \mathring{K}_{k+1}(U)$.

Now cover M by charts. Using Lemma 2.4 we can refine this cover to a countable cover (take a single chart for each x_i containing it). This gives a countable set of maps:

$$\psi_i \colon U_i \to M$$

for each $i \in \mathbb{N}$. All of these are homeomorphisms onto their open images - so they are open maps. I.e. ψ_i sends open subsets of U_i to open subsets of M. Now define

$$K_k = \bigcup_{i=1}^{k} \psi_i(K_k(U_i)).$$

These are compact because it is a finite union of compact sets $\psi_i(K_k(U_i))$.

They satisfy the additional property because the set

$$V_{k+1} = \bigcup_{i=1}^{k} \psi_i(\mathring{K}_{k+1}(U_i))$$

is open and satisfy $K_k \subset V_{k+1} \subset K_{k+1}$. □

3. Smoothness

Definition 3.1. A map $f\colon \mathbb{R}^n \overset{\circ}{\supset} U \to \mathbb{R}^m$ is called smooth if all higher order partial derivatives exists and are continuous.

Consequences:

- All higher order partial derivatives are differentiable. We need to assume that the partial derivatives are continuous. Indeed, one can create examples which is not even once differentiable - yet all higher order partial differentials exists.
- Notation: for $x \in U$ the differential
$$D_x f\colon \mathbb{R}^n \to \mathbb{R}^m$$
is the linear part of the unique first order approximation
$$f(x+h) = f(x) + (D_x f)(h) + \varepsilon_1(h)$$
where $\|\varepsilon_1(h)\| \leq C_1 \|h\|^2$ (in small neighborhood of x).
- In this notation the chain rule takes the form:
$$D_{f(x)} g \circ D_x f = D_x(g \circ f).$$
- The composition of smooth maps are smooth (follows inductively by chain rule).
- One can also look at the second order approximation (Taylor series), which for $m = 1$ (or coordinate wise) looks like:
$$f(x+h) = f(x) + \langle (\nabla f)_x, h \rangle + \tfrac{1}{2} h^T H_f h + \varepsilon_2(h).$$
Here $|\varepsilon_2(h)| \leq C_2 \|h\|^3$ (in small neighborhood of x) and H_f (the Hessian) is symmetric and given by
$$(H_f)_{ij} = \tfrac{\partial^2 f}{\partial x_i \partial x_j}.$$
Also the gradient $(\nabla f)_x$ at x is the vector such that $(D_x f)(h) = \langle (\nabla f)_x, h \rangle$ and thus has coordinates the first order partial derivatives of f.
- In general (but still $m = 1$ or coordinate wise) smoothness at a point x is equivalent to: for all $k \in \mathbb{N}$ there is a k^{th} degree Taylor polynomial T_k in the variables $x_1, \ldots, x_n$ such that
$$f(x+h) = T_k(h) + \varepsilon_k(h)$$
such that $|\varepsilon_k(h)| \leq C_k \|h\|^{k+1}$ (in a neighborhood of x).

- A bijective smooth map $f\colon \mathbb{R}^n \overset{\circ}{\supset} U \to V \overset{\circ}{\subset} \mathbb{R}^n$ whose inverse is also smooth is called a diffeomorphism.
- By the inverse function theorem this is equivalent to f being bijective, smooth and $D_x f$ an invertible matrix for all $x \in U$.
- The composition of two diffeomorphisms is a diffeomorphism.

Lecture 2

4. More on Manifolds

Recall definition: n-dimensional topological manifold:

- M1) Hausdorff.
- M2) Second countable.
- M3) M is locally homeomorphic to $U \overset{\circ}{\subset} \mathbb{R}^n$.

Sometimes M3 is rephrased as:
M3') locally homeomorphic to $\mathbb{R}^n$.

Lemma 4.1. M3' is equivalent to M3.

Proof. "$\Rightarrow$": Put $U = \mathbb{R}^n$.
"$\Leftarrow$": Let $x \in M$, and let $\psi\colon \mathbb{R}^n \overset{\circ}{\supset} U \to V \overset{\circ}{\subset} M$ be a chart containing x. Take an open ball $B \subset U$ around $\psi^{-1}(x)$ then $\psi\colon B \to M$ is also a chart around x. However, we see that $B \cong \mathbb{R}^n$ - so M3' follows. □

Warning: Sometimes the definition of charts differ like this (defined on all of $\mathbb{R}^n$ or just an open subset). However, we are using the most common definition.

Example 4.2. Let M be a 0-dimensional topological manifold. M3' implies that for any point $x \in M$ there is a neighborhood $V \ni x$ homeomorphic to

$$\mathbb{R}^0 = \{0\}.$$

So we see that

$$V = \{x\}.$$

So points are open and thus M has the discreet topology. So

$$M = \bigsqcup_{i \in I} \{x_i\},$$

and by $M2$ we see that I must be a countable set. Indeed, any base for the topology we must have the sets consisting of a single point $\{x_i\}$.

Definition 4.3. A closed manifold is a manifold which as a topological space is compact.

Notice that the term "compact manifold" is reserved for a different notion.

Example 4.4. A closed 0 dimensional manifold (or from now on 0-manifold) is a finite set with the discreet topology.

Example 4.5. The n-manifold S^n is a compact space - hence it is closed.

5. Smooth Manifolds.

Let M be an n-dimensional topological manifold.

Definition 5.1. Two charts $\psi_i \colon \mathbb{R}^n \mathring{\supset} U_i \to M, i = 1, 2$ are said to be *smoothly compatible* if the composition:

$$\psi_2^{-1} \circ \psi_1 \colon \psi_1^{-1}(\psi_2(U_2)) \to \psi_2^{-1}(\psi_1(U_1))$$

is a diffeomorphism.

Note that this is automatically a homeomorphism. Also note that to the left of the " : " we are suppressing the fact that the maps are restricted to subspaces.

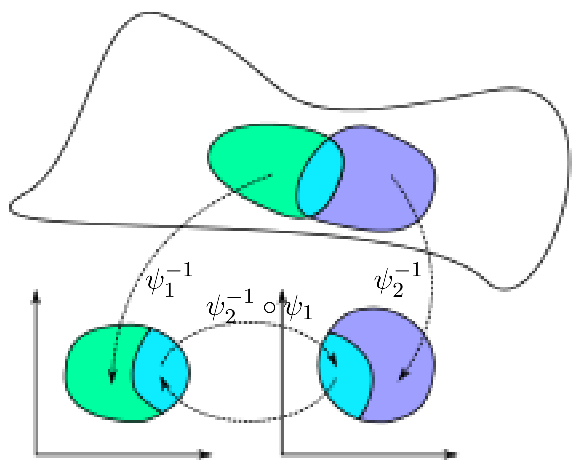


Figure 2. Two overlapping charts on a manifold (source:Wikipedia)

In the following I write "smooth atlas", where I at the lecture simply said "atlas". Indeed, we will only (unless otherwise specified) consider smooth atlases, but to compare with other literature I have included smooth in these notes.

Definition 5.2. A smooth atlas $\mathcal{A}$ on M is a collection $\mathcal{A} = \{(U_i, \psi_i)\}_{i \in I}$ of charts $\psi_i \colon \mathbb{R}^n \mathring{\supset} U_i \to M$ such that:

- all pairs are smoothly compatible and
- they cover M ($M = \cup_{i \in I} \psi_i(U_i)$).

Exercise 5.3. The cover of charts we described on S^n is a smooth atlas.

Definition 5.4. Two smooth atlases are equivalent if their union is a smooth atlas.

Lemma 5.5. This is an equivalence relation

In the following proof $x \in U_3$, which is notationally easier than what I did in the lecture (there x was in M, which is thus replaced by the point $\psi_{i_3}(x)$). Also maybe 1 and 3 was exchanged..

Proof. Assume $\mathcal{A}_j = \{(U_i, \psi_i)\}_{i \in I_j}$ for $j = 1, 2, 3$ are smooth atlases such that $\mathcal{A}_1 \sim \mathcal{A}_2$ and $\mathcal{A}_2 \sim \mathcal{A}_3$. We need to prove that for any $i_1 \in I_1$ and $i_3 \in I_3$ the composition $\psi_{i_1}^{-1} \circ \psi_{i_3}$ is a diffeomorphism on the subset of U_{i_3} where it is defined. We know it is a homeomorphism so all we need to check is that locally around a point $x \in U_3$ (where it is defined) it and its inverse are smooth.

Given such an $x \in U_3$ there is an i_2 such that $\psi_{i_3}(x) \in \psi_{i_2}(U_2)$. This implies that

$$\psi_{i_1}^{-1} \circ \psi_{i_3} = (\psi_{i_1}^{-1} \circ \psi_{i_2}) \circ (\psi_{i_2}^{-1} \circ \psi_{i_3}),$$

is well defined in a small neighborhood of x. Since the brackets a restrictions of diffeomorphisms they are smooth, and so is their composition. Similarly for the inverse. □

Definition 5.6. A smooth manifold M is a topological manifold with a choice of an equivalence class of smooth atlases.

Definition 5.7. The maximal smooth atlas for a smooth manifold M is the union of all atlases in the equivalence class.

Notice that the maximal smooth atlas is a smooth atlas. Indeed, any two charts in the maximal smooth atlas is from the union of two equivalent and smooth atlases - hence in the smooth atlas defined by their union - and hence smoothly compatible. Any chart in the maximal smooth atlas of a smooth manifold is called a smooth chart.

Example 5.8.

- $\mathbb{R}^n$ has a smooth atlas with 1 smooth chart: $\mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^n$. The maximal smooth atlas then consists of: all diffeomorphisms $\varphi : \mathbb{R}^n \overset{\circ}{\supset} U \to V \overset{\circ}{\subset} \mathbb{R}^n$. This is called the standard smooth structure on $\mathbb{R}^n$.
- $U \overset{\circ}{\subset} \mathbb{R}^n$ has a smooth atlas also with one chart $\mathrm{id}\colon U \to U$. One may think of this as restricting the standard smooth structure on $\mathbb{R}^n$ to this open subset. The maximal smooth atlas is as above but restricted to those charts with image in U.
- S^n (Exercise 5.3).

Question 5.9. Are there topological manifolds which do not have any smooth structures?

Answer: Yes! first example found by Kervaire (60).

6. Smooth maps between smooth manifolds

Let M and N be two smooth manifolds.

Definition 6.1. A map $f\colon M \to N$ is called smooth at a point $x \in M$ if; for any two smooth charts

$$\varphi\colon U \to M$$
$$\psi\colon V \to N$$

such that $x \in \varphi(U)$ and $f(x) \in \psi(V)$ the map $\psi^{-1} \circ f \circ \varphi$ is smooth at $\varphi^{-1}(x)$.

Lemma 6.2. It is enough to check smoothness at $x \in M$ for a single pair of smooth charts (φ, ψ) (as above).

Proof. If (φ', ψ') is another pair as above then knowing that $\psi^{-1} \circ f \circ \varphi$ is smooth at $\varphi^{-1}(x)$ implies

$$\psi'^{-1} \circ f \circ \varphi' = (\psi'^{-1} \circ \psi) \circ (\psi^{-1} \circ f \circ \varphi) \circ (\varphi^{-1} \circ \varphi')$$

is defined and smooth at $\varphi'^{-1}(x)$. Indeed, the first and last brackets are smooth because the charts are smooth charts and hence smoothly compatible. □

Definition 6.3. a map $f\colon M \to N$ is smooth if it is smooth at all points $x \in M$.

Note that by Lemma 6.2 it is enough to check smoothness for pairs of charts in two given atlases for M and N.

Example 6.4.

- The two notions of being smooth as a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ are the same - when both are considered with their standard smooth structure.
- Being a smooth chart is the same as; being a chart, being smooth, and having a smooth inverse.

Definition 6.5. A diffeomorphism $f\colon M \to N$ is a smooth homeomorphism whose inverse is also smooth.

Example 6.6. Any homeomorphism $h\colon \mathbb{R}^n \to \mathbb{R}^n$ (which is not a diffeomorphism with respect to the standard smooth structure) defines a different smooth structure on $\mathbb{R}^n$ in the following way:

- smooth charts are now $\psi\colon \mathbb{R}^n \overset{\circ}{\supset} U \to V \overset{\circ}{\subset} \mathbb{R}^n$ such that

$$h \circ \psi\colon U \to h(V)$$

  are smooth charts for standard $\mathbb{R}^n$.

Exercise: prove this is a smooth structure on $\mathbb{R}^n$ and that h is a diffeomorphism from $\mathbb{R}^n$ with this new smooth structure to $\mathbb{R}^n$ with the standard smooth structure.

Question 6.7. Are there different smooth structures on the same topological manifold such that the resulting smooth manifolds are not diffeomorphic?

Answer: Yes! Milnor [2]: There are 28 different smooth structures on S^7.

Also there are uncountably many smooth structures on $\mathbb{R}^4$. However, on any other $\mathbb{R}^n, n \neq 4$ there is only one!

Number of smooth structures on S^n (source Wikipedia):

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	1	1	?	1	1	28	2	8	6	992	1	3	2	16256	2	16

7. Partition of Unity

Let $g\colon \mathbb{R} \to \mathbb{R}$ be given by

$$g(t) = \begin{cases} 0 & t \leq 0 \\ e^{-\frac{1}{t}} & t > 0 \end{cases}$$

This is smooth and defining $f(t) = g(t+1)g(1-t)$ we get a smooth map $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} f(x) > 0 \qquad & \text{when} \qquad |x| < 1 \\ f(x) = 0 \qquad & \text{when} \qquad |x| \geq 1. \end{aligned}$$

This is known as a *bump* function.

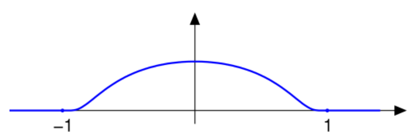


Figure 3. The bump function f (Picture:Wikipedia).

Lemma 7.1 ((Smooth) Partition of unity)**.** Let $\{U_i\}_{i \in I}$ be a covering of a (smooth) manifold M by open sets. There exists (smooth) maps

$$\rho_i\colon M \to \mathbb{R}, i \in I$$

such that:

- The image of ρ_i is contained in $[0, 1]$.
- The support
$$\operatorname{supp} \rho_i = \overline{\{x \in M \mid \rho_i(x) \neq 0\}}$$
is contained in U_i.
- For any $x \in M$ there is a neighborhood $V \ni x$ such that all but finitely many ρ_i's are the zero function when restricted to V.
- $\sum_{i \in I} \rho_i = 1$ (locally finite sum by the above bullet point).

Proof. Began proof, but will recall and finish next time. □

Lecture 3

8. Partition of Unity Continued

A more detailed formulation of the following lemma was given last time:

Lemma 8.1. Let $\{U_i\}_{i\in I}$ be a covering of a smooth manifold M by open sets. There exist smooth maps

$$\rho_i\colon\ M\to\mathbb{R},\qquad i\in I$$

such that:

- The image of ρ_i is contained in $[0,1]$.
- The support of ρ_i is contained in U_i.
- The number of non-zero ρ_i's are locally finite.
- $\sum_{i\in I}\rho_i=1$ (locally finite sum by the above bullet point).

The collection $\{\rho_i\}_{i\in I}$ is called a *partition of unity subordinate* to the open cover $\{U_i\}_{i\in I}$.

Proof. Firstly we pick a compact sequence $\{K_k\}_{k\in\mathbb{N}}$ as guaranteed in the Lemma 2.6. That is

$$K_k\subset\mathring{K}_{k+1}\subset M$$

for each $k\in\mathbb{N}$ and $M=\cup_{k=1}^{\infty}K_k$. Put $K_{-1}=K_0=\emptyset$.

Define

$$K'_k=K_k-\mathring{K}_{k-1}. \tag{8.1}$$

These are also compact and $M=\cup_{k=1}^{\infty}K'_k$. (One can heuristically visualize these as compact "rings" going off to infinity[1])

Now fix $k\in\mathbb{N}$, and notice that the set

$$V_k=M-K_{k-2}$$

is open and contains K'_k

Now, for each point $x\in K'_k$ pick a chart $\psi_x\colon\ \mathring{D}^n_2\to M$ with $x=\psi_x(0)$ such that the image is contained inside a single open set U_i and inside V_k. Indeed, one may do this by making an open ball in some smooth chart around x small enough and identifying this ball diffeomorphically (linearly even) with $\mathring{D}^n_2$ - sending the center to 0.

Define $\rho_x\colon\ M\to\mathbb{R}$ by:

$$\rho_x(y)=\begin{cases}0 & y\notin\psi_x(\mathring{D}^n_2)\\ f(\|\psi_x^{-1}(y)\|^2) & y\in\psi_x(\mathring{D}^n_2).\end{cases}$$

[1]Here infinity may be defined by 1-point compactification M^+ of M such that a neighborhood of infinity is the same as a complement to a compact set.

Here f is smooth and

$$\begin{aligned} f(x) > 0 \qquad &\text{when} \qquad |x| < 1 \\ f(x) = 0 \qquad &\text{when} \qquad |x| \geq 1. \end{aligned}$$

Notice that ρ_x is smooth (exercise: composition of smooth functions are smooth, and patching smooth functions that agree on open sets is also smooth). By compactness we may pick a finite set $\{x_{k,j}\}_{j=1}^{n_k}$ such that the open sets

$$\{\mathring{\operatorname{supp}}\, \rho_{x_{k,j}}\}_{j=1}^{n_k}$$

is a cover of $K_k - \mathring{K}_{k-1}$.

Doing this for all k we get a countable number of smooth maps $\{\rho_l\}_{l \in \mathbb{N}}$ such that

$$\{\mathring{\operatorname{supp}}\, \rho_l\}_{l \in \mathbb{N}}$$

covers all of $M = \cup_{k \in \mathbb{N}} K_k'$. Note here that $\mathring{\operatorname{supp}}\, \rho_l = \{\rho_l > 0\}$ by construction, but this is not true for any map to $[0, 1]$.

Claim: the supports of $\{\rho_l\}_{l \in \mathbb{N}}$ is locally finite. Indeed, $\mathring{K}_k$ is disjoint from V_{k+i} for all $i \geq 2$. So by construction only finitely many of the charts used to define the $\{\rho_l\}'s$ intersects each $\mathring{K}_k$ - and these cover M.

By construction we may for each $l \in N$ pick an $f(l) \in I$ such that $\operatorname{supp} \rho_l \subset U_{f(l)}$. Then define

$$\tilde{\rho}_i = \sum_{l \in f^{-1}(i)} \rho_l.$$

There may be infinitely many non-zero function terms in this sum, but it is locally finite and thus well-defined and smooth. This new collection of maps $\{\tilde{\rho}_i\}_{i \in I}$ also has the locally finite support property. Indeed, since each ρ_l is in no more than (in fact precisely) one of these sums, there are less or equally many $\tilde{\rho}_i$'s not equal to zero in some neighborhood as there are ρ_l's not equal to zero. So the $\tilde{\rho}_i$'s satisfy both the second and the third bullet point in the lemma.

Because the sets where $\rho_l > 0$ covers M we see that the sum

$$\sum_{i \in I} \tilde{\rho}_i = \sum_{j=1}^{\infty} \rho_l$$

is greater than 0 everywhere. So we are finished by defining

$$\rho_i = \frac{\tilde{\rho}_i}{\sum_{i \in I} \tilde{\rho}_i}.$$

□

Exercise 8.2. Let M be a smooth manifold. Show that for any two closed disjoint subsets $C_0 \subset M$ and $C_1 \subset M$ there exists a smooth map $f\colon M \to \mathbb{R}$ such that

- $f(x) = 1$ when $x \in C_1$ and
- $f(x) = 0$ when $x \in C_0$.

Definition 8.3. A space X is called normal if for all disjoint closed subsets $C_0, C_1 \subset X$ there exist disjoint open sets $U_0 \supset C_0$ and $U_1 \supset C_1$.

Note: this generalizes Hausdorff (when points are closed).

Corollary 8.4. A manifold is normal.

Proof. $C_0 \subset f^{-1}((-\infty, 1/2))$ and $C_1 \subset f^{-1}(1/2, \infty))$ - with f as in the exercise above. □

9. Smooth approximations and proper maps

So why is partition of unity important? An easy example of this is the following lemma.

Lemma 9.1. For any continuous function $f\colon M \to \mathbb{R}^n$ and $\varepsilon > 0$ there exists a smooth function

$$f'\colon M \to \mathbb{R}^n$$

such that

$$\|f - f'\|_\infty < \varepsilon.$$

Proof. Let $x \in M$ be any point. By continuity we can find an open set $U \ni x$ such that

$$\|f(x) - f(y)\| < \varepsilon,$$

when $y \in U$. Cover M by such open sets $U_i \ni x_i$. Now pick a partition of unity $\{\rho_i\}$ subordinate to the cover $\{U_i\}$. Then define:

$$f'(x) = \sum_i \rho_i(x) f(x_i),$$

which is locally finite and hence well-defined and smooth. We now see that

$$\|f(x) - f'(x)\| = \|\sum_i (f(x) - f(x_i))\rho_i(x)\| \leq$$
$$\leq \sum_i \|f(x) - f(x_i)\| \rho_i(x) < \sum_i \varepsilon \rho_i(x) = \varepsilon.$$

The strict inequality follows because either $\rho_i(x) = 0$ or $\|f(x) - f(x_i)\| < \varepsilon$ and this last option happens for at least some $\rho_i(x) \neq 0$. □

Note that: this can also be done for $M \to N$ (where N is an arbitrary smooth manifold), but what is distance in N then? We will postpone this question as it will be an easy consequence of something that we will see later.

A map $f\colon X \to Y$ is called *proper* if the pre-image of any compact set is compact. Note as mentioned before: Hausdorff is assumed unless otherwise stated.

Lemma 9.2. For any smooth manifold M there exists a smooth proper map $f\colon M \to \mathbb{R}$.

Note that a map $f\colon X \to Y$ induces a function $f^+\colon X^+ \to Y^+$ by

$$f^+(x) = \left\{ \begin{array}{ll} f(x) & x \in X \\ \infty & x = \infty. \end{array} \right. ,$$

and this function is continuous iff f is proper. We may thus heuristically think of a proper map - as a map that "continuously respects" infinity.

Proof. Take a compact Sequence as before $K_k \subset \mathring{K}_{k+1} \subset M$ such that $M = \cup_{k\in\mathbb{N}} K_k$. Take a partition of unity $\{\rho_k\}$ subordinate to the cover $\{\mathring{K}_k\}$. Now define:

$$f(x) = \sum_k k\rho_k(x).$$

Again this is a locally finite sum - hence well-defined. Moreover, for any compact set $K \subset \mathbb{R}$ there exists a $k \in \mathbb{N}$ such that $K \subset [-k, k]$ and we have

$$f^{-1}(K) \subset f^{-1}([0,k]) \subset \{x \in M \mid \exists j \leq k, \rho_j > 0\} \subset \bigcup_{j=1}^{k} \operatorname{supp} \rho_j \subset K_k.$$

Indeed, the first $\subset$ is because $f \geq 0$. So $f^{-1}(K)$, which is closed by continuity, is contained in a compact set. ∎

Note that one may recover the lemma producing the K_k's from such a proper function. Indeed, we see that for a proper function $f\colon M \to \mathbb{R}$ we get:

$$\cup_{k=1}^{\infty} f^{-1}([-k,k]) = M,$$

where each $f^{-1}([-k,k])$ is compact and

$$f^{-1}([-k,k]) \subset f^{-1}((-(k+1), k+1)) \subset f^{-1}([-(k+1), k+1])$$

with the middle set open hence in the interior of the last set.

10. Sub-manifolds

Definition 10.1. A k-dimensional smooth sub-manifold N of an n dimensional smooth manifold M is a subset $N \subset M$ such that: for any point $x \in N \subset M$ there exists a smooth chart $\psi\colon \mathbb{R}^n \mathring{\supset} U \to M$ such that

$$\psi(U) \cap N = \psi(U \cap \mathbb{R}^k). \tag{10.1}$$

Here $\mathbb{R}^k \subset \mathbb{R}^n$ is the standard inclusion.

Note that $n-k$ is called the co-dimension, and that because ψ is a bijection Equation (10.1) is equivalent to

$$U \cap \psi^{-1}(N) = U \cap \mathbb{R}^k.$$

Also note that for $k = n$ (i.e. codimension 0) this is equivalent to asking that $N \subset M$ is an open subset.

Example 10.2.

- $\mathbb{R}^k \subset \mathbb{R}^n$.
- any open subset of M is a sub-manifold of the same dimension.
- The graph of a smooth function $f\colon \mathbb{R}^k \to \mathbb{R}^{n-k}$ (I sketched a proof of this at the lecture, but if you don't remember: exercise).
- $S^n \subset \mathbb{R}^{n+1}$. This follows from the above two examples because the smooth charts given earlier are (modulo the reordering of coordinates) graphs on open sets in $\mathbb{R}^n$.
- $S^k \subset S^n$ (exercise).

Note that a smooth sub-manifold has an induced smooth structure by restriting the smooth charts to $\mathbb{R}^k$ (intersected with the domain). Indeed, let (ψ, U) and (φ, U') both be smooth charts as above then

$$(\varphi^{-1} \circ \psi)\colon\ \mathbb{R}^k \cap (\psi^{-1} \circ \varphi)(U') \to \mathbb{R}^k \cap (\varphi^{-1} \circ \psi)(U)$$

is a diffeomorphism. We conclude that their restrictions form an atlass (WARNING: not in general a maximal atlas!).

Question 10.3. Is a sub-manifold $N \subset M$ of a sub-manifold $M \subset M'$ a sub-manifold $N \subset M'$?

Exercise: The answer to this question is affirmative by the following lemma.

Lemma 10.4. Any smooth chart on a sub-manifold is locally the restriction of a chart as in Equation (10.1).

Proof. Let $x \in N$ be any point and let $\varphi\colon\ \mathbb{R}^k \overset{\circ}{\supset} V \to N$ be any smooth chart on N containing x. We wish to extend a restriction of φ to an open neighborhood of x to a smooth chart as in the definition above.

Let $\psi\colon\ \mathbb{R}^n \overset{\circ}{\supset} U \to M$ be a chart around $x \in N \subset M$ as in the definition above. Since we are only interested in the question locally around x we may restrict to smaller sets and assume that $\psi(U) \cap N = \varphi(V)$ (draw a sketch yourself - I did so at the lecture). We may also restrict U such that $U \subset (U \cap \mathbb{R}^k) \times \mathbb{R}^{n-k}$.

By assumptions

$$\psi^{-1} \circ \varphi$$

is a diffeomorphism from V to $U \cap \mathbb{R}^k$. This implies that

$$\Psi = (\psi^{-1} \circ \varphi) \times \mathrm{id}_{\mathbb{R}^{n-k}}$$

is (the inverse of) a diffeomorphism of U to an open subsets of $\mathbb{R}^n$. It is now easy to see that

$$\psi \circ \Psi\colon\ \Psi^{-1}(U) \to M$$

is a new smooth chart on M around x and that restricting it to $V = \Psi^{-1}(U) \cap \mathbb{R}^k$ gives φ (again it is helpfull to draw a sketch). $\square$

Next time I will try and describe (without proof) an example of a chart where the restriction to a smaller open set is nessecary. An example like this proves the warning above.

Lecture 4

- I sketched some intuition about proper maps, which has been weaved into the notes for lecture 3.
- A few of the following examples were covered in lecture 3.

11. Embeddings and Immersions.

Definition 11.1. A smooth embedding of a manifold $i\colon N \to M$ is a smooth map such that

- the image is a smooth sub-manifold and
- the map i is a diffeomorphism onto this sub-manifold.

Lemma 11.2. A map $i\colon \mathbb{R}^k \overset{\circ}{\supset} U \to \mathbb{R}^n$ is locally at $x \in U$ an embedding iff $D_x i\colon \mathbb{R}^k \to \mathbb{R}^n$ is injective.

Here "locally" means that there is a small neighborhood around x which is embedded by i. In the lecture I possibly switched the role of k and n..

Proof. "$\Leftarrow$": Pick a basis $v_1, \dots, v_{n-k}$ for the complement of the image of $D_x i$. Then define

$$\psi\colon U \times \mathbb{R}^{n-k} \to \mathbb{R}^n$$

by

$$\psi(y, t_1, \dots, t_{n-k}) = i(y) + t_1 v_1 + \cdots + t_{n-k} v_{n-k}.$$

Now the differential of ψ at $(x, 0)$ is given (in block form) by

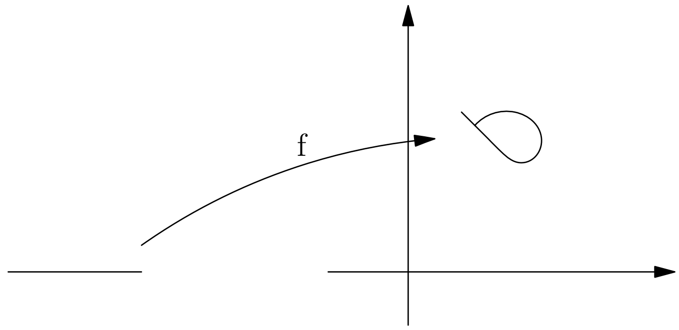
$$D_{(x,0)}\psi = (D_x i \mid v_1 \mid \cdots \mid v_{n-k})$$

which is invertible. So the inverse function theorem applies, and we get a diffeomorphism from a neighborhood $U' \times V \subset U \times \mathbb{R}^{n-k}$ of $(x, 0)$ to a neighborhood W of $\psi(x)$. The image of U' is precisely the image of i. So $i(U')$ is a sub-manifold.

"$\Rightarrow$": Let $\psi\colon U \to \mathbb{R}^n$ be a chart around $i(x)$ as in the definition of sub-manifold. Then by definition $\psi^{-1}_{|U \cap \mathbb{R}^k} \circ i$ is locally at $x \in \mathbb{R}^k$ a diffeomorphism - hence the differential of i at x is injective (chain rule). □

Definition 11.3. A map which is locally an embedding is called an immersion.

Example 11.4.



An injective immersion of an open interval into $\mathbb{R}^2$ which is not an embedding. Indeed, it is not a homeomorphism onto its own image.

Exercise 11.5. An immersion is an embedding precisely when it is a homeomorphisms onto its image.

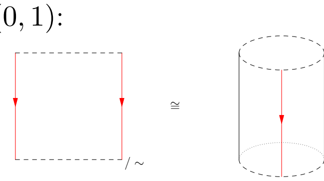
Exercise 11.6. An injective proper map is a homeomorphisms onto its image

Corollary 11.7. An injective proper immersion is an embedding.

Exercise 11.8. Let $f\colon M \to \mathbb{R}^n$ be an injective immersion and let $g\colon M \to \mathbb{R}$ be smooth and proper then $f \times g\colon M \to \mathbb{R}^n \times \mathbb{R}$ is a proper embedding.

Example 11.9. S^1: the map $\mathbb{R} \to S^1 \subset \mathbb{C}$ given by $t \mapsto e^{it}$ can be checked to be a local diffeomorphism (and is hence an immersion into $\mathbb{R}^2$). Being careful one can (with smooth structure) define S^1 as the quotient $[0, 2\pi]/(0 \sim 2\pi)$.

Example 11.10. A cylinder: $S^1 \times (0, 1)$ can be described by identifying the two red sides of $[0, 1] \times (0, 1)$:



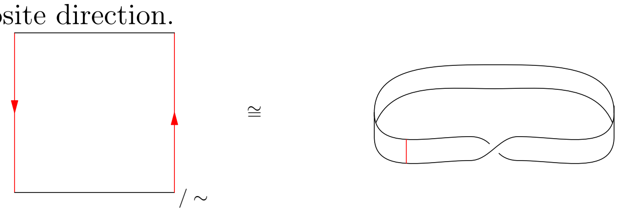
The red lines are $\{0\} \times (0, 1)$ and $\{1\} \times (0, 1)$ and we identify points in pairs: $(0, t) \sim (1, t)$. Being careful one can do this with smooth structure.

The cylinder is diffeomorphic to the open subset U of $\mathbb{R}^2$ given by:

$$U = \{x \in \mathbb{R}^2 \mid 1 < \|x\| < 2\}.$$

So it can be covered by one smooth chart.

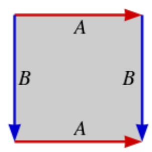
Example 11.11. The Moebius strip M: similar to the cylinder but identifying in opposite direction.



The red lines are $\{0\} \times (0,1)$ and $\{1\} \times (0,1)$ and we identify points in pairs: $(0,t) \sim (1, 1-t)$. Being careful one can do this with smooth structure.

Example 11.12. $S^1 \cong S^1 \times \{1/2\} \subset M$ (M Moebius strip as above) is a sub-manifold. So, also $S^1 \times (0,1) \subset M \times (0,1)$ is a sub-manifold. This is an example of a sub-manifold were not all charts are restrictions of charts to M! Indeed, $S^1 \times (0,1)$ can be covered with a single chart as above. However, no open neighborhood of this in $M \times (0,1)$ can be covered by one chart - exercise later when more tools have been developed

Example 11.13. Similarly we can define the torus $T^2 = S^1 \times S^1$ as the quotient of



where the red sides are identified and the blue sides are identified. This is basically the cross product of how we saw S^1 as a quotient.

Theorem 11.14. Any closed smooth manifold M embeds into $\mathbb{R}^N$ for some N large enough.

Before we prove this theorem we need the following lemma.

Lemma 11.15. Let $U \mathring{\subset} \mathbb{R}^n$ and $\rho \colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\operatorname{supp} \rho \subset U.$$

Then the map

$$f \colon \mathbb{R}^n \to \mathbb{R}^{n+1}$$

given by $f(x) = (x\rho(x), \rho(x))$ is an embedding when restricted to the set $\{\rho > 0\}$ and the complement is sent to 0.

Proof. The map $f' \colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ given by

$$f'(x) = (x, \rho(x))$$

is an embedding on all of $\mathbb{R}^n$. Indeed, it is the graph of ρ.

Composing this with the map:

$$\varphi \colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n+1}$$

given by

$$\varphi(x,t) = (xt, t)$$

gives f. The lemma now follows because φ is a diffeomorphism on the set where $t > 0$. Indeed, its differential at (x,t) is given by:

$$D_{(x,t)}\varphi = \begin{pmatrix} t & & & & x_1 \\ & t & & & x_2 \\ & & \ddots & & \vdots \\ & & & t & x_n \\ & & & & 1 \end{pmatrix},$$

where empty spots are 0. □

proof of theorem 11.14. Cover M by finite number of charts $\psi_i\colon U_i \to M, i = 1,\dots,k$ and pick a partition of unity $\{\rho_i\}$ subordinate to this. Now define

$$e(x) = (\rho_1(x)\psi_1^{-1}(x), \rho_1(x), \rho_2(x)\psi_2^{-1}(x), \rho_2(x), \dots, \rho_k(x)\psi_k^{-1}(x), \rho_k(x))$$

in $\mathbb{R}^{(n+1)k}$. This is smooth. It is also injective because for any $x \in M$ there is an i such that $x \in \psi_i(U_i)$ and $\rho_i > 0$, so by looking at the ith part of the image we can recover x. Indeed, the fact that $\rho_i > 0$ tells us that x must lie in this chart and then we can recover $\psi_i^{-1}(x)$ by the coordinates.

Similarly we see that in the chart ψ_i the function $e \circ \psi_i$ is on its ith part as in the above lemma ($x \mapsto (x\rho'(x), \rho'(x))$ where $\rho' = \rho \circ \psi_i$) - so the differential in this chart is injective. Indeed, the composition with the differential of the projection to the ith part is injective by the lemma.

We conclude that f is an injective immersion, and since it is defined on closed manifold corollary 11.7 says it is an embedding. □

12. Tangent spaces and Differentials

We see now that differentials are very important for manifolds. So, is there a good way of encoding these for any smooth manifold? The way we have used them so far they depend on chosen charts! This is ok when we only care about properties like injective, surjective, bijective. However, there is a way of describing differentials more abstractly and take the choice of charts out of the equation - and get linear maps on well-defined vector spaces - called tangent spaces.

Let M be a n-dimensional smooth manifold.

Definition 12.1. A *tangent vector* at a point $x \in M$ is an equivalence class of pairs (ψ, v) such that

- $\psi\colon U \to M$ is a smooth chart around x and
- v is a vector in $\mathbb{R}^n$.

The equivalence is defined by:

- $(\psi, v) \sim (\psi', v')$ iff $D_{\psi^{-1}(x)}(\psi'^{-1} \circ \psi)(v) = v'$.

Lemma 12.2. This is an equivalence relation.

Proof.

- Reflexive: $(\psi, v) \sim (\psi, v)$ since the differential of the identity is the identity.
- Symmetric: follows from chain rule used on $\psi \circ \psi^{-1}$.
- transitive: follows from chain rule.

□

The equivalence respects addition and scaling - so the space of tangent vectors T_xM at x is natural a vector space.

Observation 12.3. Picking a chart ψ around x provides a trivialization $T_xM \cong \mathbb{R}^n$. Indeed, $(\psi, v) \sim (\psi, v')$ only if $v = v'$ and every equivalence class has a representative of the form (ψ, v).

One may topologize T_xM using such a trivialization. This is equivalent to toplogizing it as the qoutient of

$$\bigsqcup_{\psi} \mathbb{R}^n = \{\psi \mid \psi \text{ is a smooth chart around } x\} \times \mathbb{R}^n$$

given by the equivalence relation.

Using this definition it is now possible to define a generalized differential: for any smooth function $f\colon M \to N$ we define the differential at $x \in M$ as the linear map

$$D_x f\colon T_xM \to T_{f(x)}N$$

which sends (ψ, v) to $(\psi', D_{\psi^{-1}(x)}(\psi'^{-1} \circ f \circ \psi)(v))$. This is well-defined (exercise).

We now describe the union of all the tangent spaces as a $2n$ dimensional smooth manifold. Let $\mathcal{A}$ be the set of functions

$$\psi\colon U \times \mathbb{R}^n \to \cup_{x \in M} T_xM = TM$$

given by

$$\psi(y, v) = (\phi, v) \in T_{\phi(y)}M,$$

where ϕ is any smooth chart $\phi\colon U \to M$. That is for each smooth chart ϕ we have a funtion ψ in $\mathcal{A}$.

Lemma 12.4. There is a unique structure of a smooth $2n$-manifold on TM such that $\mathcal{A}$ is an atlas.

Proof. There is a unique topological structure such that the maps in $\mathcal{A}$ are homeomorphisms onto open sets (defines same base for topology around each point). This can be checked to satisfy M1, M2 and M3 (using heavily that M and $\mathbb{R}^n$ already satisfy these).

We need to show that the charts are smoothly compatible. So let $\phi_i\colon U_i \to M, i = 1, 2$ be to smooth charts on M, and let ψ_i be the associated maps in $\mathcal{A}$. First we see that $\psi_2^{-1} \circ \psi_1$ is precisely defined on the set where $\phi_2^{-1} \circ \phi_1$ is defined times $\mathbb{R}^n$. On this set we see that

$$(\psi_2^{-1} \circ \psi_1) = ((\phi_2^{-1} \circ \phi_1), D_{-}(\phi_2^{-1} \circ \phi_1)),$$

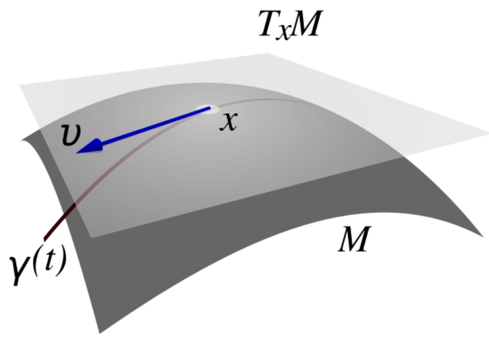
where the $-$ means: plug in the first coordinate. This is smooth. □

Observation: An immersion $f\colon N \to M$ is a smooth map where all differentials $D_x f\colon T_x M \to T_{f(x)} N$ are injective.

Observation: When a manifold U is an open subset of $\mathbb{R}^n$ (with standard smooth structure) we have a canonical identification of $T_x U$ with $\mathbb{R}^n$ - using the identity smooth chart. So assume that $e\colon M \to \mathbb{R}^N$ is an immersion at $x \in M$. Then we may identify

$$T_x M \cong \operatorname{im} D_x f \subset T_x \mathbb{R}^N = \mathbb{R}^N.$$

This motivates the name tangent space and is usually illustrated as follows:



As indicated in the picture (source:Wikipedia) one may describe the tangent vectors in $T_x M$ as derivatives of curves through x. We will explore this further in an exercise.

Lecture 5

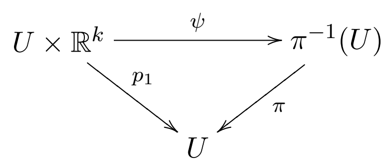
- Last time we defined a smooth $2n$ dimensional manifold TM. This is called the tangent bundle, and today we explain the word bundle.

13. Tangent Bundles

The map $\pi\colon TM \to M$ given by $\pi(x) = y$ when $x \in T_yM$ is smooth (easy to check in the charts we provided last time). As mentioned before $\pi^{-1}(y) = T_yM$ is a vector space. The tangent bundle TM is an example of a smooth vector bundle:

Definition 13.1. A smooth k-dimensional vector bundle E on a smooth manifold M is a smooth manifold E of dimension $k+n$ and a smooth surjective map $\pi\colon E \to M$ such that

- for each $x \in M$ the *fiber* over x defined as $E_x = \pi^{-1}(x) \subset E$ has the structure of a k-dimensional real vector space and
- for any $x \in M$ there exists an open neighborhood $U \ni x$ and a smooth diffeomorphism $\psi\colon U \times \mathbb{R}^k \to \pi^{-1}(U) \subset E$ such that

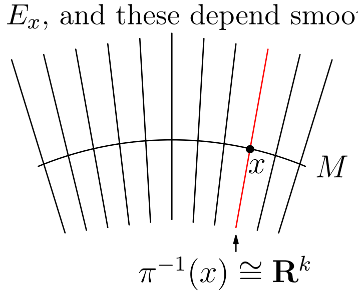
  

  commutes (here p_1 is projection to first factor), and such that restricting ψ to a point $\{y\} \times \mathbb{R}^k \cong \mathbb{R}^k$ gives a linear isomorphism:

  $$\psi\colon \mathbb{R}^k \cong \{y\} \times \mathbb{R}^k \to \pi^{-1}(y).$$

  These are called *local trivializations*.

The intuitive idea of vector bundles are: For each point $x \in M$ we have assigned a vector space E_x, and these depend smoothly on x.



Example 13.2. The tangent bundle $TM \to M$ is a vector bundle over M.

Example 13.3. The trivial bundle $\pi\colon M \times \mathbb{R}^k \to M$, where each $\pi^{-1}(x) = \{x\} \times \mathbb{R}^k$ is given the obvious vector space structure. Then the identity serves as a local trivialization, which in fact is a global trivialization.

In the following I have changed the notation as discussed in the second part of this lecture. A very common construction of vector bundles: Let $\{U_i\}$ be a cover of a manifold M, and let $\{\phi_{ij}\}$ be smooth maps

$$\phi_{ij}\colon\ U_i \cap U_j \to \mathrm{Gl}_k(\mathbb{R}) \overset{\circ}{\subset} \mathbb{R}^{k^2}$$

such that

- $\phi_{ii}(x) = I_k$ and
- $\phi_{jk}(x) \cdot \phi_{ij}(x) = \phi_{ik}(x)$ when $x \in U_i \cap U_j \cap U_k$.

Here $\cdot$ is matrix multiplication. The latter is called the *cocycle condition*, and the first condition follows from it (but to emphasize it we put it in). Now define the vector bundle E as the quotient

$$E = \left(\bigsqcup_i U_i \times \mathbb{R}^k \right) / \sim,$$

where $(x_i, v) \sim (x_j, v')$ if $x_i = x_j$ in M and $\phi_{ij}(x_i) \cdot v = v'$. It is not difficult to check that the assumptions on the ϕ_{ij}'s makes $\sim$ an equivalence relation. The projection $\pi\colon\ E \to M$ is defined by

$$\pi([x_i, v]) = x_i.$$

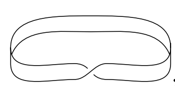
We will refer to this construction as *patching together a vector bundle by local trivializations.*

The canonical inclusions

$$U_i \times \mathbb{R}^k \to E$$

then serve both as charts in a smooth atlas and as local trivializations.

This is very similar to how we defined the tangent bundle (and its structures). The proof that the tangent bundle is a smooth manifold can be copied with very few modifications to prove that this is a smooth $k+n$ dimensional manifold - and that each *fiber* E_x is a vector space isomorphic to $\mathbb{R}^k$. Indeed, in that discussion the differentials of the chart transitions $D_{-}(\psi_1 \circ \psi_2)$ plays the role of the functions ϕ_{ij}.

Example 13.4. The Moebius strip, , can be defined as a vector bundle on S^1. Indeed, let

$$U_1 = S^1 \cap \{(x, y) \mid y > -\varepsilon\}$$
$$U_2 = S^1 \cap \{(x, y) \mid y < \varepsilon\}$$

then the intersection $U_1 \cap U_2$ is diffeomorphic to the union of two disjoint open intervals in $\mathbb{R}$, and we may define $\phi_{12}\colon\ U_1 \cap U_2 \to \mathrm{Gl}_1(\mathbb{R}) = \mathbb{R}^* = \mathbb{R} - \{0\}$ by

$$\phi_{12}(x, y) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

With ϕ_{11} and ϕ_{22} constantly equal to the identity matrix and $\phi_{21} = \phi_{12}^{-1} = \phi_{12}$ these satisfy the conditions above. We now see that taking the quotient as above we almost get the cylinder $S^1 \times \mathbb{R}$ except we have turned the $\mathbb{R}$ upside down when identifying them over the component $U_1 \cap U_2 \cap \{x < 0\}$. This describes the Moebius strip (recall that $\mathbb{R} \cong (0, 1)$). When considering this a bundle over S^1 we will refer to it as the *moebius bundle*, while the underlying manifold is called the Moebius strip.

Definition 13.5. A map between smooth vector bundles $E \xrightarrow{\pi_E} M$ and $F \xrightarrow{\pi_F} N$ over a smooth map $f\colon M \to N$ is a smooth map $\overline{f}\colon E \to F$ such that

$$\begin{array}{ccc} E & \xrightarrow{\overline{f}} & F \\ \downarrow{\scriptstyle \pi_E} & & \downarrow{\scriptstyle \pi_F} \\ M & \xrightarrow{f} & N \end{array}$$

commutes, and such that for each $x \in M$ the map $\overline{f}_x\colon E_x \to F_{f(x)}$ is linear.

Note here $\overline{f}_x$ is short for $\overline{f}_{|\pi^{-1}(x)}$.

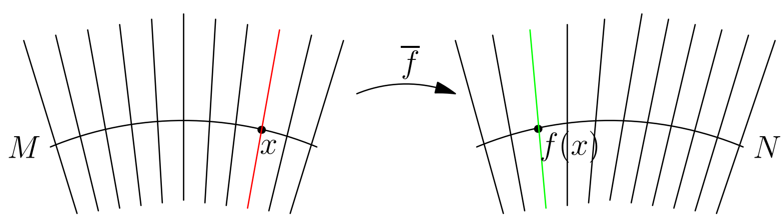


FIGURE 4. A vector bundle map. The red vector space is mapped linearly to the green vector space - by $\overline{f}_x$.

Definition 13.6. An isomorphism $\overline{f}\colon E \to F$ of smooth vector bundles is a smooth map of vector bundles which is also a diffeomorphism.

Note that this implies that the inverse is automatically a smooth map of vector bundles. Indeed the inverse to a linear map is linear (in sharp contrast to smoothness and continuity). The immediate question this introduces is: are there any vector bundles which are not isomorphic to a trivial bundle? The following lemma illustrates that the answer is yes.

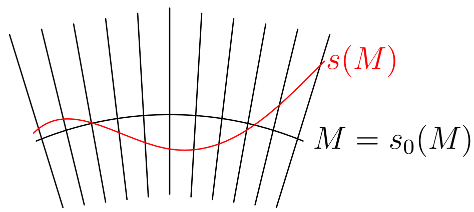
Lemma 13.7. *The Moebius bundle as a 1-dimensional vector bundle over S^1 is not isomorphic to $S^1 \times \mathbb{R}$.*

To prove this we introduce the notion of sections.

Definition 13.8. A smooth section in a vector bundle $\pi\colon E \to M$ is a smooth map $s\colon M \to E$ such that $\pi \circ s = \mathrm{id}_M$

Example 13.9. A map $M \to M \times \mathbb{R}^k$ is a section precisely when it is on the form $s(x) = (x, s'(x))$. So this is equivalent to simply having the smooth map $s' \colon M \to \mathbb{R}^k$.

So sections generalize the idea of maps. In fact one may visualize sections as graphs over M inside the vector bundle:



Example 13.10. Since 0 in a vector space is a unique point we can define the map $s_0 \colon M \to E$ by sending x to the unique $0 \in E_x$. This is known as the zero section and why the visualizations in the above pictures are very good (you see M canonically embedded inside the vector bundle as the zero-section).

Definition 13.11. If N is a sub-manifold of M then the restriction of a bundle $\pi \colon E \to M$ to N is defined as the space $E_{|N} = \pi^{-1}(N)$ and the restrictions

$$\psi \colon (U \cap N) \times \mathbb{R}^k \to \pi^{-1}(U \cap N)$$

of trivializations

$$\psi \colon U \times \mathbb{R}^k \to \pi^{-1}(U)$$

of E serves as trivializations (and atlas) for $E_{|N}$.

This definition was written slightly different in the lecture - because I decided in the break (due to questions) to include it, so it was rather improvised.

Note that that we may restrict sections when restricting a bundle to a sub-manifold, and that any isomorphism of vector bundles induces a 1-1 correspondence of their sections. This is used implicitly in the following.

Proof of Lemma 13.7: We wish to prove that all sections in the Moebius bundle intersect the zero section - shortly put; there are no non-zero sections. Indeed, this distinguishes it from $S^1 \times \mathbb{R}$ because in this bundle there is a non-zero section $s \colon S^1 \to S^1 \times \mathbb{R}$ given by e.g. $s(x) = (x, 1)$. Thought of as the section given by the constant function $s'(x) = 1$ since the bundle is trivial.

Let $s \colon S^1 \to E$ be a section in the Moebius bundle E defined in Example 13.4. Assume that this section is non-zero. By construction E came with two local trivializations:

$$\psi_1 \colon U_1 \times \mathbb{R} \to E \qquad \text{and} \qquad \psi_2 \colon U_2 \times \mathbb{R} \to E$$

defined on the cover $\{U_1, U_2\}$ of S^1. Restricting the section to U_i and using the trivialization ψ_i we get a section

$$s_i = \psi_i^{-1} \circ s_{|U_i} \colon U_i \to U_i \times \mathbb{R}$$

for each $i = 1, 2$. As above $s_i = (\mathrm{id}_{U_i}, s_i')$ can be considered as map $s_i' \colon U_i \to \mathbb{R}$. By the definition of E these satisfy $s_1'(x) = \phi_{12}(x) \cdot s_2'(x)$ for $x \in U_1 \cap U_2$. Since ϕ_{12} is 1 at $(1,0)$ and -1 at $(-1,0)$ this implies that precisely one of the maps s_1' or s_2' has opposite signs at these two points. Assume WLOG that this is s_1' - now the assumptions imply that we have a non-zero continuous function

$$s_1' \colon U_1 \to \mathbb{R}$$

which takes both positive and negative values. This is in contradiction with the intermediate value theorem and the fact that $U_1 \cong (0,1)$. □

This is a simple version of ideas which are generally used to distinguish vector bundles. Indeed, some part of the characteristic classes known as Stiefel-Whitney classes can be described by considering how many sections - which at each point are linearly independent - exist. Since having a smooth choice of a basis at each point in M is the same as having a trivialization - one sees that the isomorphism class of the trivial bundle is uniquely determined by this property.

14. Submersions and regular values

Let $f \colon M \to N$ be a smooth map. M has dimension n and N dimension k.

Definition 14.1. A critical point for f is a point $x \in M$ such that $D_x f \colon T_x M \to T_p N$ is not surjective.

Definition 14.2. f is called a submersion if $D_x f$ is surjective for all $x \in M$.

Definition 14.3. A point $p \in N$ is called a regular value if for all points $x \in M$ such that $f(x) = p$ we have that $D_x f \colon T_x M \to T_p N$ is surjective. I.e. it is not the image of a critical point.

So a map is a submersion iff all values in the codomain/image is regular.

Lemma 14.4. If $p \in N$ is a regular value then $f^{-1}(p)$ is a sub-manifold of dimension $n - k$ for any $p \in M$.

Example 14.5. projection to the z-axis of S^2 in $\mathbb{R}^3$ has regular values $\mathbb{R} - \{\pm 1\}$. The inverse image of a value in $(-1, 1)$ is a circle (the inverse image of $x > 1$ or $x < -1$ is empty, which by definition is also a 1 dimensional smooth manifold).

Proof. This is a rephrasing of implicit function theorem. This rephrasing will appear in the next proof so we omit it here. □

The following theorem is a combination of the ideas of submersions and immersions.

Theorem 14.6 (Constant rank theorem)**.** If $D_x f$ has rank r independent of $x \in M$ then $f^{-1}(p)$ is a sub-manifold of co-dimension r.

Proof. In local coordinates (such that $f(0) = 0$) the differential

$$D_{-}f\colon\ \mathbb{R}^n \to \mathbb{R}^k$$

is linear with rank r smoothly depending on $x \in U \overset{\circ}{\subset} \mathbb{R}^n$.

By a linear coordinate change (and making U smaller) we can assume that

$$f(x, y) = (f_1(x, y), f_2(x, y)) \in \mathbb{R}^r \times \mathbb{R}^{k-r}$$

is such that for fixed $(x, y) \in U \subset \mathbb{R}^r \times \mathbb{R}^{n-r}$ the differential of f_1 with respect to x is an invertible r by r matrix. Indeed, looking at standard properties for ranks of matrices one can prove that this can be done (at 0 for $D_0 f$) by simply reordering the coordinates on the domain and codomain - then it is true in a neighborhood because being invertible as a matrix is an open condition.

Now define

$$G(x, y) = (f_1(x, y), y)$$

This G is a local diffeomorphism (at 0), and thus there exists a smooth map

$$g\colon\ \mathbb{R}^r \times \mathbb{R}^{n-r} \overset{\circ}{\supset} U' \to \mathbb{R}^{k-r}$$

(locally around 0) such that

$$f(G^{-1}(x, y)) = (x, g(x, y))$$

close to 0. Since G is a diffeomorphism the map $f \circ G^{-1}$ is smooth and has constant rank r (of its differential). The differential of $f \circ G^{-1}$ at a point (in the neighborhood around 0) is given in block form by:

$$D_{x,y}(f \circ G^{-1}) = \begin{pmatrix} I_r & 0_{r,(n-r)} \\ ?_{(k-r)\times r} & A \end{pmatrix}$$

Because the rank of this is r it follows that A is 0. This implies that g does not locally depend on y. So in a smaller (connected) neighborhood around 0 we get that $g(x, y) = h(x)$. So we see that:

$$(f \circ G^{-1})(x, y) = (x, h(x)).$$

This has $h(0) = 0$ and so in the open set which we have restricted to we have $f^{-1}(0) = f_1^{-1}(0)$. This means we have reformulated the problem to a surjective differential problem (involving f_1 instead of f). Moreover the formulation of implicit function theorem in [Rudin] is precisely in this form, where $D_0 f_1$ is invertible with respect to the first r coordinates. □

The important step in this proof is realizing that g does not depend on y, and this makes the image (locally) of f an r dimensional manifold! This may be thought of as generalizing the fact that if a function has constant differential then it is constant (the image is a 0 dimensional manifold) to: The image of a map of rank r is locally an r dimensional manifold - and the constant rank theorem then follows from this and the lemma above about sub-mersions.

Lecture 6

15. More on bundles

Exercise 15.1. Show that a map which is linear on the second factor:

$$F\colon\ U \times \mathbb{R}^k \to \mathbb{R}^n$$

for all $x \in U$ is smooth if and only if the associated map

$$F^\dagger\colon\ U \to M^{n\times k}$$

given by $F^\dagger(x) \cdot v = F(x, v)$ is smooth.

Example 15.2. Any vector bundle $E \to M$ can be described (up to ismorphism) by patching together trivializations.

Indeed, cover E with local trivializations

$$\psi_i\colon\ U_i \times \mathbb{R}^k \to E.$$

Now define

$$\phi_{ij}\colon\ U_i \cap U_j \to \mathrm{Gl}_n(\mathbb{R})$$

by $\phi_{ij}(x) = ((\psi_j)_x^{-1} \circ (\psi_i)_x)^\dagger$.

These satisfy the conditions in the patching construction (from last time). Indeed,

- $\phi_{ii} = I_k$
- $\phi_{jk}(x) \cdot \phi_{ij}(x) = ((\psi_k)_x^{-1} \circ (\psi_j)_x \circ (\psi_j)_x^{-1} \circ (\psi_i)_x)^\dagger = \phi_{ji}(x)$.

It is not to difficult to check that the vector bundle constructed by patching together these local trivializations is isomorphic to E.

Definition 15.3. A sub-bundle E' of $E \xrightarrow{\pi} M$ of dimension k is a sub-manifold $E' \subset E$ such that for all $x \in M$ there exists trivializations of E

$$\psi\colon\ U \times \mathbb{R}^n \to \pi^{-1}(U)$$

such that $\psi^{-1}(E') = U \times \mathbb{R}^k$ and $x \in U$. Here $\mathbb{R}^k \subset \mathbb{R}^n$ is the standard inclusion.

This may be thought of as smoothly choosing a sub-space of constant dimension in each fiber. Note that E' is itself naturally a vector bundle.

Example 15.4. When $M \subset \mathbb{R}^n$ is an embedding then $TM \subset M \times \mathbb{R}^n$ is a sub-bundle.

Example 15.5. More generally for a sub-manifold $N \subset M$ then $TN \subset TM_{|N}$ is a sub vector bundle. Note that we need to restrict TM since they need the same base.

If we have two bundles E and F over the same manifold M we can define the fiber-wise direct sum - or in short - their direct sum $E \oplus F$ as follows. First notice that $E \times F \to M \times M$ is a vector bundle and that

$$E_x \oplus F_x \cong E_x \times F_x = (E \times F)_{(x,x)}.$$

So restricting the product to the diagonal $\Delta\colon M \to M \times M$ given by $\Delta(x) = (x, x)$ we may define

$$E \oplus F = (E \times F)_{|\Delta(M)}.$$

We think of this as a bundle over M since it is a bundle over the sub-manifold $\Delta(M)$ which is canonically diffeomorphic to M (using Δ).

It is not difficult to prove that if two sub-bundles $E'', E' \subset E$ has intersection the zero-section and their dimension sum up to the dimension of E then

$$E' \oplus E'' \cong E.$$

Indeed, the obivuous map (which fiber-wise sends (v', v'') to $v' + v''$) can be checked to be an isomorphism (exercise). When this is the case we call E'' a complement bundle to E' (and vise versa).

Question: can you for any sub-bundle $E' \subset E$ define a complement bundle? Answer: Yes, the following is a tool that is important for many things, and can be used for constructing complement bundles.

Definition 15.6. A bilinear form q on a vector bundle E is a smooth map $q\colon E \oplus E \to \mathbb{R}$ such that

$$q_x = q_{|(E \oplus E)_x}\colon E_x \oplus E_x \to \mathbb{R}$$

is bilinear.

Definition 15.7. A metric on a vector bundle E is a bilinear form q on E such that when restricted to each fiber it is an inner product. I.e. it is symmetric and positive.

Example 15.8. Any trivial bundle $M \times \mathbb{R}^k$ has

$$(M \times \mathbb{R}^k) \oplus (M \times \mathbb{R}^k) \cong M \times (\mathbb{R}^k \oplus \mathbb{R}^k)$$

and we may define a metric by using the standard inner product on $\mathbb{R}^k$ (and thus not letting it depend on $x \in M$). We call this the standard metric.

Definition 15.9. An isomorphism of vector bundles $\psi\colon E \cong F$ which are equipted with metrics q_E and q_F respectively is called an isometry if it is an isometry in each fiber. I.e. if the canonical isomorphism

$$\psi \oplus \psi\colon E \oplus E \to F \oplus F$$

satisfies

$$q_E = q_F \circ (\psi \oplus \psi).$$

Proposition 15.10. The "space" of metrics on a vector bundle E is convex and non-empty.

Usually this is formulated with convex replaced by contractible. However, this is in fact stronger. It will be clear from the proof what is meant by "convex". However, we will not specify what we mean by space. Indeed, the convexity proof works for most natural choices of space structure on the set of metrics.

Proof. Firstly, we may define metrics locally by using a trivialization

$$\psi\colon\ U \times \mathbb{R}^k \cong \pi^{-1}(U).$$

Indeed, we may define it by using the standard metric q_s on $U \times \mathbb{R}^k$ and then defining it on $\pi^{-1}(U)$ by

$$q = q_s \circ (\psi^{-1} \oplus \psi^{-1})$$

This is smooth, and ψ is by construction an isometry.

Thus, we may cover M by open sets $\{U_i\}$ and choices of local metrics q_i on $\pi^{-1}(U_i)$. Now pick a partion of unity sub-ordinate $\{\rho_i\}$ to this cover and define

$$q_x(v_1, v_2) = \sum_i \rho_i(x)(q_i)_x(v_1, v_2) \qquad \text{for } v_1, v_2 \in E_x.$$

This is for fixed $x \in M$ an inner product. Indeed we need to check:

- Symmetry - sums and scalings of symmetric functions are symmetric.
- Bilinearity - sums and scalings of bilinear maps are bilinear.
- positive - can be checked by $q_x(v_1, v_1) > 0$ for $v_1 \neq 0$ and this follows because at least one of the coefficients ρ_i are non-zero at any $x \in M$.

Convexity of the "space" of such structures follows from the fact that if q_1 and q_2 are metrics then

$$tq_1 + (1 - t)q_2$$

is a metric for all $t \in [0, 1]$. Indeed, this is for the same reasons as above. □

Lemma 15.11. Any metric vector bundle $\pi\colon\ E \to M$ (with metric q) is locally trivializable by an isometry

$$\psi\colon\ U \times \mathbb{R}^k \to \pi^{-1}(U),$$

where $U \times \mathbb{R}^k$ has the standard metric.

Proof. Let $\psi'\colon\ U \times \mathbb{R}^k \to \pi^{-1}(U)$ be any trivialization. It follows that

$$\psi' \oplus \psi'\colon\ U \times (\mathbb{R}^k \times \mathbb{R}^k) \to \pi'^{-1}(U)$$

is a trivialization of the direct sum bundle $\pi'\colon\ E \oplus E \to M$. So define the metric

$$q' = q \circ (\psi' \oplus \psi')\colon\ U \times (\mathbb{R}^k \times \mathbb{R}^k) \to \mathbb{R}.$$

This might not be the standard metric. However, by definition ψ' is now an isometry from $U \times \mathbb{R}^k$ with this non-standard metric q' to $\pi(U)$.

The standard basis in $\mathbb{R}^k$ may not be an ONB for these inner products. However, the Gram-Schmidt process is smoothly dependent on the inner products q'_x, and always produces an ONB from any basis. So we get a smooth isomorphism of vector bundles

$$\phi\colon\ U \times \mathbb{R}^k \to U \times \mathbb{R}^k$$

by (the identity on U and) applying the Gram-Schmidt process to the standard basis vectors in $\mathbb{R}^k$ using q'_x for each $x \in U$ (and extending this linearly - since we know were to send a basis this uniquely determines the linear map). This is a fiber-wise linear isomorphism because we sent a basis to a basis. In fact it is an isometry from $U \times \mathbb{R}^k$ with the standard structure to the one with q' as metric structure - indeed, it fiberwise sends an ONB to an ONB. Now defining

$$\psi = \psi' \circ \phi \colon U \times \mathbb{R}^k \to \pi^{-1}(U)$$

we see that ψ is an isometry. Indeed, it is a composition of isometries. □

A generalization of this lemma is

Lemma 15.12. Any metric vector bundle $\pi \colon E \to M$ (with metric q) with a sub-bundle $E' \subset E$ is locally trivializable by an isometry

$$\psi \colon U \times \mathbb{R}^k \to \pi^{-1}(U),$$

where

$$\psi^{-1}(E') = U \times \mathbb{R}^l$$

with $\mathbb{R}^l \subset \mathbb{R}^k$ the standard inclusion.

Proof. The proof above when applied to a trivialization as in the definition of sub-bundle automatically produces this. Indeed, the Gram-Schmidt process applied to the first l vectors of a base preserves their span. □

Definition 15.13. For any metric vector bundle $E \to M$ and sub-bundle $E' \subset E$ the orthogonal complement $E'^{\perp}$ is defined fiber-wise by

$$(E'^{\perp})_x = (E'_x)^{\perp} \subset E_x.$$

By Lemma 15.12 this defines a sub-bundle.

Note that by construction $E'^{\perp}$ is a complement bundle to E'. One could have constructed complements directly in a similar fashion by choicing complements localy and then using a partition of unity to interpolate, and indeed there is a way of viewing the set of complements as a convex set - making sense of interpolation. However, this is more tricky than it is for metrics. Indeed, metrics are much more naturally a convex set.

One could then wonder if we produce all possible choices of complement bundles in this fashion; and, in fact, we do. It is not difficult to give $E \cong E' \oplus E''$ a metric such that this splitting is orthogonal (meaning that E'' becomes the orthogonal complement of E').

Definition 15.14. Let $N \subset M$ be a sub-manifold. Since $TN \subset TM_{|N}$ we define the *normal bundle* $\nu_{N \subset M}$ as

$$\nu_N^M = TN^{\perp}$$

inside $TM_{|N}$ whenever TM is given a metric. When $N = \mathbb{R}^n$ we will simply write ν_M.

Example 15.15. When $M \subset \mathbb{R}^n$ we have $TM \subset M \times \mathbb{R}^n$ is a sub-bundle. Using the standard metric on the latter trivial bundle we get that ν_M is such that $(\nu_M)_x$ is the standard orthogonal complement to T_xM in $\mathbb{R}^n$ for each point $x \in M$.

Example 15.16. $\nu_{S^1} = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x \in S^1, y \in \text{span}(x)\}$. Indeed, the position vector for any $x \in S^1$ is also a normal vector to T_xS^1, and for dimension reasons the othogonal complement of T_xS^1 is one dimensional.

Example 15.17. $\nu_{S^n} = \{(x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid x \in S^n, y \in \text{span}(x)\}$. Similar to above. Notice that since S^n is n-dimensional inside an $n+1$ dimensional manifold the normal bundle is still 1 dimensional.

In general the normal bundle $\nu_{N \subset M}$ has the dimension of the codimension of N in M.

Lecture 7

- Note that one may define the norm in a metric vector bundle by $\|v\| = \sqrt{q_x(v,v)}$ and then an equivalent definition of isometry (from last time) is a norm preserving isomorphism of bundles.
- The lemma about being able to trivialize a vector bundle locally by isometries shows that the patching done with maps $\phi_{ij}\colon U_i \cap U_j \to \mathrm{Gl}_n(\mathbb{R})$ can in fact be done with maps to $O(n) \subset \mathrm{Gl}_n(\mathbb{R})$.
- $\mathbb{R}P^n$ is not apropri defined as a sub-manifold of $\mathbb{R}^k$. So here our embedding theorem is important!
- A Riemannian structure on a manifold M is a choice of metric on TM.

  The standard Riemannian structure on $\mathbb{R}^n$ is given by using the canonical trivialization

  $$T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$$

  and using the standard metric on each $\{x\} \times \mathbb{R}^n$. WARNING: The lemma from last time about local isometric trivializations does not say that we can find charts such that M locally looks like $U \mathring{\subset} \mathbb{R}^n$ with this standard Riemannian structure. Indeed, there is a big difference between
  - finding a local trivialization of TM as a vector bundle
  - and doing this while requiring that this local trivialization should be given by the differential of chart.

  Indeed, a big part of Riemannian geometry is about the curvature tensor, and this is an intrinsically defined object on a Riemannian manifold, which when not zero in a neighborhood of a point proves that such a trivialization does not exist.

16. Tubular neighborhoods

Notation: When $E \to M$ is a vector bundle we will write $(x, v) \in E$ to denote a point $v \in E_x$. Note that in a local trivialization this is almost standard notation since the bundle is a product. However, in a bundle $v \in E_x \subset E$ is in itself a point in E, but putting in the x in the notation is convenient.

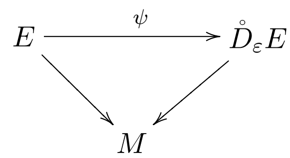
For any vector bundle $E \to M$ with metric q and any smooth function $\varepsilon\colon M \to \mathbb{R}_+$ the set

$$\mathring{D}_\varepsilon(\nu_M) = \{(x, v) \in \nu_M \mid \|v\| = \sqrt{q_x(v,v)} < \varepsilon(x)\}$$

is an open neighborhood around the zero-section $M \subset \nu_M$. We see that

$$\psi\colon E \to \mathring{D}_\varepsilon(\nu_M) \tag{16.1}$$

given by $\psi(x, v) = \varepsilon(x)\frac{\phi(\|v\|)}{\|v\|}v$ is a diffeomorphism. Here $\phi\colon \mathbb{R} \to (-1, 1)$ is a diffeomorphism such that ϕ is the identity in a neighborhood of 0. Note that ψ respects the projection to M in the sense that



commutes.

Assume that $i\colon M \subset \mathbb{R}^n$ is a smooth embedding. Let $\pi\colon \nu_M \to M$ denote the normal bundle given by this embedding.

Theorem 16.1 (Tubular neighborhood)**.** There exists an embedding

$$i'\colon \nu_M \to \mathbb{R}^n$$

such that

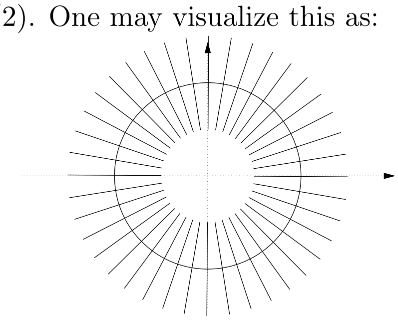
- i' extends i (recall that $M \subset \nu_M$ as the zero-section),
- i' is a diffeomorphism onto its open image, and
- for any $x \in \nu_M$ the point $i(\pi(x)) \in \mathbb{R}^n$ is the unique closest point in M to $i'(x) \in \mathbb{R}^n$.

We will build up slowly to the proof of this.

Example 16.2. The sub-manifold $S^1 \subset \mathbb{R}^2$ has such a map given by:

$$i'(x, v) = i(x) + \tfrac{\phi(\|v\|)}{2\|v\|} v.$$

Here ϕ is as described after Equation (16.1). Recall that the normal bundle of S^1 at $x \in S^1$ is the real span of x. The image of i' is all vectors in $\mathbb{R}^2$ with norm in $(1/2, 3/2)$. One may visualize this as:



In general the following lemma shows that we, in fact, always have to embed each fiber of the normal bundle onto an affine (translated linear) subspace if we want the statements in the theorem above to be true. Note there is a weaker version of the tubular neighborhood theorem which leaves out the third point in the theorem. However, in $\mathbb{R}^n$ this does not seem any easier to prove.

Lemma 16.3. If $x \in M = i(M) \subset \mathbb{R}^n$ is distance minimizing to a point $p \in \mathbb{R}^n$ then $(p - x) \in (\nu_M)_x$.

The intuitive idea is: if $x-p$ is not orthogonal to the tangent space - then moving in the direction (in M) given by projecting $p-x$ to T_xM will make the distance to p shorter to the first order. However, as is often common, intuitive ideas can look quite different when turned into a proof:

Proof. Let $\psi\colon \mathbb{R}^k \overset{\circ}{\supset} U \to M$ be a chart around x. Define the function

$$f(z) = \|\psi(z) - p\|^2 = \sum_{j=1}^{n} (\psi_j(z) - p_j)^2.$$

This is the distance from the point $\psi(z) \in M$ to p squared. By the assumptions this has a minimum at $z_0 = \psi^{-1}(x)$. The ith component of the gradient of f is given by:

$$\tfrac{\partial f}{\partial z_i}(z) = \sum_{j=1}^{n} \left(2\tfrac{\partial \psi_j}{\partial z_i}(z)(\psi_j(z) - p_j) \right) = 2\langle \tfrac{\partial \psi}{\partial z_i}(z), \psi(z) - p\rangle.$$

It follows that at the critical point z_0 the column vectors vectors $\frac{\partial \psi}{\partial z_i}(z_0)$ in the matrix $D_{z_0}\psi$ are orthogonal to $x - p = \psi(z_0) - p$. Since T_xM is the range of $D_{z_0}\psi$ it follows that $x - p$ is orthogonal to T_xM. $\square$

The lemma and the example with S^1 motivates the following map

$$j\colon \nu_M \to \mathbb{R}^n$$

by

$$j(x, v) = i(x) + v.$$

This is very far from being injective. Indeed, in the case of S^1 this would send all points of the form $(x, -x)$ to 0 - which was why we used the trick from Equation (16.1) to only use j on the part where $\|v\|$ is small. We will do this again - but not until the very end.

The lemma tells us: if $i(x)$ is the closest point in M to $p \in \mathbb{R}^n$ then p is in the image of $j_x = j(x, -)$.

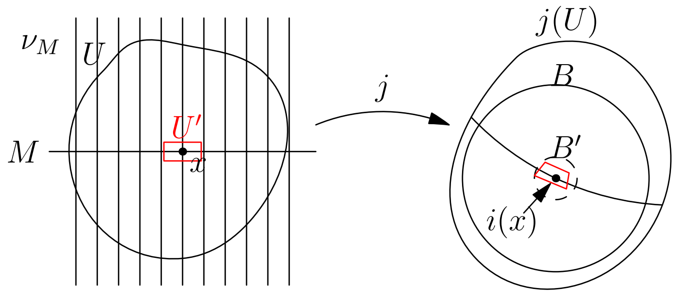
Note that as a manifold the dimension of ν_M is n, and that this has surjective differential at any point on the zero-section $M \subset \nu_M$. Indeed,

- restricting j to M yields i - so the tangent space T_xM is in the image of D_xj and
- restricting j to $(\nu_M)_x$ shows that the vectors in (ν_M) are in the image of D_xj.

So j is has invertible differential at $x \in M \subset \nu_M$ and so is a local diffeomorphism on a neighborhood of the zero-section $M \subset \nu_M$. WARNING: the use of the word "local" ín the previous sentence is NOT at this point redundant. It is not clear that a local diffeomorphism which is injective on M is a diffeomorphism on a neighborhood of M. Indeed, part of the subtlety in the following proof is to take care of this.

Proof of tubular neighborhood theorem. Let $x \in M$ be given. take a neighborhood U around $(x, 0) \in \nu_M$ such that the restriction of j is a diffeomorphism onto its open image. Now pick an $\varepsilon > 0$ such that the closed ball $B = B_{5\varepsilon}(i(x))$ is contained in the image $j(U)$. Now pick $V \ni x$ open in M and $\delta > 0$ such that

- The open set $U' = \mathring{D}_\delta(\nu_M)_{|V}$ is contained in U and
- $j(U') \subset B' = B_\varepsilon(i(x))$.



Claim: for $(z, v) \in U'$ the closest point to $j(z, v)$ in M is $i(z)$.

Proof of claim: By construction $\|i(z) - j(z, v)\| \leq 2\varepsilon$. Everything outside of B is further away than that and $B \cap M$ is compact - so there is, indeed, a distance minimizing point. Now assume that $i(z') \in i(M) \cap B$ is such a distance minimizing point. It must be as close to $j(z, v)$ as $i(z)$ is - so $\|i(z') - j(z, v)\| \leq 2\varepsilon$. This implies $\|i(z'), i(x)\| \leq 3\varepsilon$. We conclude that $j(z', v') = j(z, v)$ for some v' such that $(z', v') \in U$. Indeed, all those points for which $i(z')$ is distance minimizing is by the lemma above contained in the image of the injective affine map

$$j_{z'} \colon (\nu_M)_{z'} \to \mathbb{R}^n$$

given by

$$j_{z'}(v') = i(z') + v',$$

and those which are within 2ε of $i(z')$ lies in B and are thus mapped from U.

By injectivity of j on U we see that $z = z'$.

Covering M by such open sets W_k and associated $\delta_k > 0$ we define using a partition of unity:

$$\delta(z) = \sum_k \rho_k(z)\delta_i$$

Now it follows that $\delta(z) \leq \delta_k$ for some k where W_k contains z. Thus restricting j to the neighborhood

$$W = \mathring{D}_\delta(\nu_M)$$

we get a neighborhood as in the theorem. Indeed for any point y in the image $j(W)$ we may reconstruct $(z, v) \in W$ uniquely by defining z as the closest point in M to y and then reconstruct v as above.

Finally, by pre-composing with the diffeomorphism described in Equation (16.1) it is an embedding of all of ν_M, and the closest point condition is still satisfied because the diffeomorphism from Equation (16.1) preserves the projection to M. □

17. Milnors book and notation

Now we are moving into Milnor's book [3] (we will also use an M in lemmas to denote that they are from that book). Note that in Milnor's book the definition of smooth is defined on any subset of a $\mathbb{R}^k$ in the following way:

Definition 17.1 (M)**.** For *any* subset $X \subset \mathbb{R}^n$ a map $f\colon X \to Y \subset \mathbb{R}^k$ is called smooth if: for all $x \in X$ there exists an open neighborhood $U \ni x$ and a smooth map $F\colon U \to \mathbb{R}^k$ such that $f_{|U\cap X} = F_{|U\cap X}$.

Definition 17.2 (M)**.** A subset $M \subset \mathbb{R}^n$ is called a smooth manifold if it is locally diffeomorphic to $\mathbb{R}^k$.

Here diffeomorphism is of course generalized by the above notion of smooth, but still defined as: smooth bijective map with smooth inverse.

Exercise 17.3. Prove that this is the same as our definition of a smooth sub-manifold in $\mathbb{R}^n$.

Let M and N be smooth manifolds (abstract - not as in Milnor's book). We may combine the two definitions and define that for any subset $X \subset M$ a map $f\colon X \to Y \subset N$ is smooth if it locally is the restriction of a smooth function.

Note that when $f\colon X \to N$ is smooth and $X = \overline{\mathring{X}}$ then the differential is well-defined and smooth on X. Indeed, continuity of the differentials of any extensions tells you what the differential has to be.

Lecture 8

18. Sard's Theorem

Theorem 18.1 (Sard's)**.** Let $f\colon \mathbb{R}^n \overset{\circ}{\supset} U \to \mathbb{R}^m$ be smooth then the set of critical values in $\mathbb{R}^m$ has Lebesque measure 0.

Let's start by recalling a few definitions from measure theory. An open box I in $\mathbb{R}^m$ is a set of the form $I = \prod_{j=1}^m (a_j, b_j)$. Its volume $\mathrm{vol}(I)$ is defined to be the real number $\prod_{j=1}^m (b_j - a_j)$. The outer Lebesque-measure of a set $X \subset \mathbb{R}^m$ is defined as

$$\mu^*(X) = \inf_{X \subset \cup_i I_i} \Big(\sum_i \mathrm{vol}(I_i)\Big),$$

where each I_i is an open box. An open cube is a box with all side lengths equal. Note that when a set X is measurable its outer measure is its measure (and we think of this as volume). Recall the properties

$$\mu^*(X) = 0 \qquad \Rightarrow \qquad X \text{ is measurable and } X^c \text{ is dense}$$

and

$$\mu^*(X_i) = 0, i = 1, 2, \ldots \qquad \Rightarrow \qquad \mu^*(\cup_{i=1}^\infty X_i) = 0.$$

Corollary 18.2 (A.B.Brown)**.** The set of regular values of a smooth map $f\colon M \to N$ is everywhere dense in N.

We give a proof of this corollary here using our more general definition of manifolds. The proof for Milnor's notion of manifolds is in fact the exact same - although Milnor is somewhat short on details here.

Proof. Let $C \subset M$ be the critical points of f.

Let $\psi\colon \mathbb{R}^m \overset{\circ}{\supset} U \to N$ be any smooth chart. Then $W = f^{-1}(\psi(U))$ is open in M and can be covered by countably many charts

$$\phi_i\colon \mathbb{R}^n \overset{\circ}{\supset} V_i \to W \overset{\circ}{\subset} M, \qquad i \in \mathbb{N}.$$

Let K_i be the critical points of each map

$$f_i = \psi \circ f \circ \phi_i\colon V_i \to U.$$

Then because they cover and because formaly critical points are defined by using such charts (recall that the tangent space is defined using charts) we see that

$$\psi^{-1}(f(C)) = \bigcup_{i \in \mathbb{N}} f_i(K_i).$$

Since each f_i is as in Sard's theorem we see that $f_i(K_i)$ has measure 0, and thus so does $\psi^{-1}(f(C))$ (a countable union of measure zero sets has measure zero). This proves that the complement of $f(C)$ is dense in U, and since smooth charts cover N we have proven that the complement of $f(C)$ is locally dense - hence dense. $\square$

Corollary 18.3. All smooth maps between manifolds (except $\emptyset \to \emptyset$) have regular values.

Proof of Sard's theorem. We will prove this by induction on $n \geq 0$. For $n = 0$ we see that U is empty or $U = \mathbb{R}^0 = \{0\}$. The theorem is certainly true in this case. So assume that $n > 0$ and that we know that the theorem is true for $n - 1$.

Define the sets

$$C_i = \{x \in U \mid \text{all partial derivatives of order} \leq i \text{ are zero at } x\}$$

and as before the critical points

$$C = \{x \in U \mid D_x f \text{ is not surjective}\}.$$

So $C \supset C_1 \supset C_2 \supset \cdots$.

The proof will be divided into three steps:

(1) The image of $f(C - C_1)$ has measure 0.
(2) The image of $f(C_i - C_{i+1})$ has measure 0 for $i \geq 1$.
(3) The image of $f(C_k)$ has measure 0 for large enough k.

I like the following order better than Milnor's order:

Step 3: See [3], except note that the bound $c\|h\|^{k+1}$ comes from a Taylor approximation error term (which is given by some integration on the compact set I_δ).

Step 2: See [3].

Step 1: See [3], but remember that $V \cap (C - C_1)$ is a union of compact sets because V is a manifold (or open in $\mathbb{R}^n$) and $(C - C_1) \cap V \subset V$ is a closed subset (assuming V misses C_1 which is possible in step 1 - in fact due to the way h is constructed it is unavoidable). $\square$

19. Manifolds with Boundary

Definition 19.1. A manifold *with boundary* M is a topological space such that

M1: M is Hausdorff,
M2: M is second countable and
M3: M is locally homeomorphic to an open subset of $H^k = \{x \in \mathbb{R}^k \subset x_k \geq 0\}$.

We denote the boundary of H^k by $\partial H^k = \mathbb{R}^{k-1}$, and we define ∂M to be the union

$$\partial M = \cup_\psi \psi(\partial H^k)$$

where $\psi\colon H^k \overset{\circ}{\supset} U \to M$ is any local homeomorphism. Note this is not the boundary of M in the topological sense.

Exercise 19.2 (difficult)**.** Prove that if $x \in \psi(U \cap \partial H^k)$ and $x \in \psi'(U')$ for any two local homeomorphism ψ and ψ' then $x \in \psi'(U' \cap \partial H^k)$.

Exercise 19.3. Prove that the boundary $\partial M \subset M$ is a manifold of dimension $\mathbb{R}^{k-1}$.

Definition 19.4. A smooth atlas $\mathcal{A}$ on M is the same as before but with the extended notion of smooth. Equivalence of atlasses are similar.

Definition 19.5. A smooth structure on $(M, \partial M)$ is an equivalence class of smooth atlasses

Exercise 19.6. Prove that for M smooth $M - \partial M$ is a smooth manifold of dimension k.

Exercise 19.7. Prove that for M smooth the boundary ∂M is a smooth manifold of dimension $k-1$.

Note that the tangent space of M is defined even at the boundary. Indeed, by continuity the differentials of charts and functions are uniquely defined at the boundary. So TM is still a vector bundle over M, and any smooth map $f\colon M \to N$ for manifolds with boundary defines a map of vector bundle

$$Df\colon TM \to TN$$

over f.

Exercise 19.8. There is a natural way of defining $T\partial M \subset TM_{|\partial M}$ as a codimension 1 sub-bundle.

Observation 19.9. In light of these exercises and the fact that smooth functions extends to neighborhoods. We get that A vector $v \in T_xM - T_x\partial M$ for $x \in \partial M$ can be said to be outwards pointing or inwards pointing depending on whether or not it is so in a chart. That is whether or not the last coordinate of v in the chart is negative or postive. This does not depend on the chart since an extension of any transition functions between two charts preserve $T_x\partial M$ and sends the part with positive last coodinate to the part with positive last coordinate and similar with the negative last coorindate - at least after restricting to a smaller neighborhood.

From now on all manifolds with or without boundary will be smooth. Unless explicitly stated otherwise.

Lecture 9

Assume that $f\colon M \to N$ is smooth and $y \in N$ a regular value. The following lemma is formulated a little differently than in [3] since we have introduced vector bundles.

Lemma 19.10 (M.2.2)**.** The unions of the kernels of $D_x f\colon T_x M \to T_{f(x)} N$ for $x \in M' = f^{-1}(y)$ is precisely the sub-bundle $TM' \subset TM_{|M'}$. Hence *any* normal bundle to M' in M at $x \in M'$ is mapped isomorphically to $T_y N$ by $D_x f$.

Note that a normal bundle is a complement bundle to TM'. This is not unique but in any case the lemma is true for any choice of normal bundle.

Proof. Look at the commuting diagrams

$$\begin{array}{ccc} M' & \longrightarrow & M \\ \downarrow{\scriptstyle f_|} & & \downarrow{\scriptstyle f} \\ \{y\} & \longrightarrow & N \end{array} \qquad \begin{array}{ccc} TM' & \longrightarrow & TM \\ \downarrow{\scriptstyle D(f_|)} & & \downarrow{\scriptstyle Df} \\ T\{y\} = \{y\} \times \{0\} & \longrightarrow & TN \end{array}$$

The later commutes by chain rule and because the first commutes. The fact that the latter commutes proves that $T_x M' \subset \ker(D_x f)$ for $x \in M'$. For dimension reasons we see that they must be equal. □

Milnor does not mention the following. Because he does not concern himself with vector bundles. However, this proves that:

$$\nu_{M'}^{M} \cong M' \times T_y N \cong M' \times \mathbb{R}^{n-k},$$

which is a global trivialization of the normal bundle over M'. Not canonical though - because we needed to choose a trivialization of the vector space $T_y N \cong \mathbb{R}^{n-k}$.

Exercise 19.11 (Lemma M.2.3)**.** Let M be a smooth manifold (wo.b.) and let $f\colon M \to \mathbb{R}$ be a smooth map for which a is a regular value. Then the set of $x \in M$ where $f(x) \geq a$ is a smooth manifold with boundary equal to $f^{-1}(a)$.

Example 19.12. D^n is a smooth manifold with boundary S^{n-1}. Indeed, the function $-\|x\|^2$ has -1 as a regular value.

Milnor is not concerned about the concept transversality. However, we will look a little extra at this. So, it is convinient to consider the linear algebra behind part of lemma M.2.4. For this we look at:

Lemma 19.13. Let $V \subset \mathbb{R}^n$ be a linear sub-space and let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be linear and surjective when restricted to V. Then the orthogonal projection $\pi_V\colon \ker(L) \to V^\perp$ is surjective.

Definition 19.14. Let W be a finite dimensional real vector space. Two linear sub-spaces $V, U \subset W$ are called *transversal* if $V + U = W$.

Statements equivalent to being transverse:

- $\dim(U) + \dim(V) - \dim(U \cap V) = \dim W$. Indeed, the kernel of $c\colon U \oplus V \to W$ given by $(u, v) \mapsto u + v$ is canonically isomorphic to $U \cap V$. So the dimension of the image of c is the left hand side of the equation.
- If W has an inner product then the orthogonal projection $\pi_V\colon U \to V^\perp$ is surjective. Indeed, if $U + V = W$ then any $w \in V^\perp$ can be written as $u + v$ with $u \in U$ and $v \in V$, so

  $$w = \pi_V(w) = \pi_V(u + v) = \pi_V(u).$$

  Conversely, since $\pi_V(u) - u \in V$ we see that $\pi_V(U) = V^\perp$ implies $V^{perp} \subset V + U$. Hence V and $V^\perp$ is in $V + U$ so everything is in $V + U$.

Proof. For any $w \in \mathbb{R}^n$ then $L(w) = L(v)$ for some $v \in V$. So $v - w \in \ker(L)$. Hence $ker(L) + V = \mathbb{R}^n$. □

Let $M = (M, \partial M)$ be a smooth m-manifold with boundary and let N be a smooth n-manifold.

Lemma 19.15 (M.2.4)**.** Assume $f\colon M \to N$ is a smooth map and $y \in N$ a regular value for f and for f restricted to the boundary ∂M. Then $f^{-1}(y)$ is an $(m - n)$-manifold with boundary given by $f^{-1}(y) \cap \partial M$.

Proof. For any point in $M - \partial M$ we already know that $f^{-1}(y)$ is a smooth manifold of this dimension.

So let $x \in \partial M$ be given such that $f(x) = y$. By Taking local coordinates on M we may assume that f is defined on an open neighborhood of $0 \in H^k$, and by the definition of smooth we may extend f to be a smooth map $\tilde{f}$ on an open neighborhood $V \ni 0$ in $\mathbb{R}^m$. As before $\tilde{f}^{-1}(y)$ is in a smaller neighborhood of 0, say V' a sub-manifold. Indeed, being surjective is an open condition. We then define the smooth map

$$\pi\colon V' \to \mathbb{R}$$

by

$$\pi(x_1, \ldots, x_m) = x_m.$$

Since the tangent space of $\tilde{f}^{-1}(y)$ at 0 is the kernel of $D_0\tilde{f}$ and $D_0\tilde{f}$ is surjective when restricted to $\mathbb{R}^{m-1} \subset \mathbb{R}^m$ (indeed f is regular on ∂M) - it follows from the lemma above that $D_0\pi\colon \mathbb{R}^m \to \mathbb{R}$ is surjective on $\tilde{f}^{-1}(y)$. So 0 is a reglar value for π (in a possibly smaller neighrborhood - but again surjectivity of linear maps is an open condition) it follows from the other lemma above that $f^{-1}(y) = \tilde{f}^{-1}(y) \cap \{\pi \geq 0\}$ is a manifold with boundary, where the boundary is equal to $\tilde{f}^{-1}(y) \cap \pi^{-1}(0)$. □

20. Brower's fixed point theorem

Proved as in [3]. However, note that one can generalize smooth approximations in the following way:

Let $M \to N$ be a map of manifolds where M might have boundary. Then any continuous map can be approximated as follows:

Let $\nu_N \to \mathbb{R}^n$ be an embedding and tubular neighborhood of N then at each point $x \in M$ we may approximate the map $f\colon M \to N \subset \mathbb{R}^n$ by a smooth map into $f'\colon M \to \nu_N$ then composed with the projection $\nu_N \to N$ this "approximates" f. This will be made more precise in a later exercise.

21. Smooth Embedding Theorem

Theorem 21.1. Any closed n dimensional manifold M embedds smoothly into $\mathbb{R}^{2n+1}$

The following proof follows Guillemin and Pollack [GP].

Proof. By Theorem 11.14 there is a $k > 2n+1$ and an embedding $i\colon M \subset \mathbb{R}^k$. Let $f_1\colon M \times M \times \mathbb{R} \to \mathbb{R}^k$ be given by $f_1(x, y, t) = t[i(x) - i(y)]$. Also, let $f_2\colon TM \to \mathbb{R}^k$ be given by

$$f_2(x, v) = D_x i(v) \in \mathbb{R}^k.$$

For dimension reasons Sard's theorem implies that both maps have measure zero image. Indeed, any value in the image is critical because k is larger than the dimension of the manifolds the functions are defined on. So there is a $v \in \mathbb{R}^k$ not in the image. Claim: the projection onto the orthogonal complement of v composed with i is an embedding.

Proof of claim: since v is not in the image of f_1 the composition is an injection. Indeed, if $x \in i(M)$ and $y \in i(m)$ such that $tv + x = y$ (with $t \neq 0$) then $v = (1/t)(y - x)$. Also since v is not in the image of f_2 it is an immersion. Indeed, since v is not a tangent vector anywhere the orthogonal projection is injective on all the tangent vectors of $i(M)$.

Do this repeatedly until $k = 2n + 1$ where we no longer know that the images has measure 0. □

Corollary 21.2. If $k = 2n+1$ we may pick v such that it is not in the image of f_2 hence the composition is still an immersion. Also if it is a regular value for f_1 we see that: if $t(i(x) - i(y)) = v$ then the linear map (the differential of f_1) is

$$(D_x i, D_y i, i(x) - i(y))$$

invertible. Since the projection sends the last vector to zero. We see that the map produced by composing i with the projection has differentials at "double points" which are transverse.

Lecture 10

Theorem 21.3. Any manifold M smoothly embeds into some Euclidean space $\mathbb{R}^q$.

Proof. Let $K_k \subset \mathring{K}_{k+1} \subset M, k \in \mathbb{N}$ be a compact sequence such that

$$M = \cup_{k\in\mathbb{N}} K_k.$$

Define the compact "rings" (using $K_0 = K_{-1} = \emptyset$)

$$K'_k = K_k - \mathring{K}_{k-1}.$$

We still have $M = \cup_{k\in\mathbb{N}} K'_k$. Define the open sets

$$U_k = \mathring{K}_{k+1} - K_{k-2},$$

then $K'_k \subset U_k$.

Fix $k \in \mathbb{N}$ and pick a finite cover of charts $\{(\psi_i, V_i)\}_{i=1}^N$ of K'_k each contained in U_k. Pick a partition of unity $\{\rho_i\}_{i=1}^N \cup \rho^c$ sub-ordinate to $\psi_i(V_i)$ and $(K'_k)^c \mathring{\subset} M$. Then define $G\colon M \to \mathbb{R}^{(n+1)N}$ by

$$G(x) = (\psi_1^{-1}(x)\rho_1(x), \rho_1(x), \ldots, \psi_N^{-1}(x)\rho_N(x), \rho_N(x)).$$

As seen earlier this is an injective immersion (in fact embedding) when restricted to $G^{-1}(\mathbb{R}^{N(n+1)} - \{0\}) \supset K'_k$. Note that everything outside the support of the ρ_i's are sent to 0.

We want to patch such maps together to get an injective immersion of all of M. However, it is a problem that; as k increases the number of charts N we need may go to infinity! Hence the dimension of the Euclidean space to which we map increases.

To overcome this we use the same trick as we did in the previous theorem about closed manifolds embedding into $\mathbb{R}^{2n+1}$. Indeed define:

- $f_1\colon M \times M \times \mathbb{R} \to \mathbb{R}^{N(n+1)}$ by $f_1(x, y, t) = t(G(x) - G(y))$.
- $f_2\colon TM \to \mathbb{R}^{N(n+1)}$ by $f_2(x, v) = (D_x G)(v)$.

Event hough M is not closed we can still by Sard's theorem find $v \in \mathbb{R}^{N(n+1)}$ not in the image of these. Composing G with the orthogonal projection $\pi\colon \mathbb{R}^{N(n+1)} \to W$ to the orthogonal complement W of v still satisfies that it is an injective immersion on $(\pi \circ G)^{-1}(W - \{0\}) = G^{-1}(\mathbb{R}^{N(n+1)} - \{0\})$. Indeed

- Since $v \neq t(G(x)-G(y))$ for any $(x, y, t) \in M \times M \times \mathbb{R}$. We see that no new points in the image is identified. That is, $(\pi \circ G)(x) = (\pi \circ G)(y)$ if and only if $G(x) = G(y)$.
- Since we also avoid the image of tangent vectors it is clear that if $D_x G$ is injective then so is $D_x(\pi \circ G) = \pi \circ D_x G$. Indeed, the orthogonal projection of a linear sub-space S to the complement of a vector $v \neq 0$ is injective if and only if $v \notin S$.

Now $W \cong \mathbb{R}^{N(n+1)-1}$, and using this trick repeatedly we arrive at a map:

$$G_k\colon\ M \to \mathbb{R}^{2n+1}$$

such that when restricted to $G_k^{-1}(0^c) \supset K_k'$ it is an injective immersion. Here and in the following we abbreviate $\mathbb{R}^{2n+1} - \{0\}$ as 0^c.

We need one more trick to be able to injectively immerse M into a Euclidean space. Indeed, let ρ_k' be a partition of unity sub-ordinate to the cover $\{U_k\}_{k\in\mathbb{N}}$. Note that $U_k \cap U_{k+i} = \emptyset$ for $i \geq 3$. This implies that the proper map

$$f(x) = \sum_{k\in\mathbb{N}} k\rho_k(x)$$

"approximately" detects which U_k the point x lies in. Indeed for $x \in U_k$ we see that $\rho_l(x) = 0$ for $|k-l| > 2$ hence

$$f = (k-2)\rho_{k-2} + (k-1)\rho_{k-1} + k\rho_k + (k+1)\rho_{k+1} + (k+2)\rho_{k+2}$$

at $x \in U_k$. Since these five ρ_l's sum to 1 we see that

$$f(x) \in [k-2, k+2].$$

So that $|f(x)-k| \leq 2$ for $x \in U_k$. This implies that if we know which residue class mod 5 k is then we can recover k using $f(x)$.

We use this idea to fix an element $i \in \mathbb{Z}/5\mathbb{Z}$ and define

$$F_i\colon\ M \to \mathbb{R}^{2n+1}$$

by

$$F_i(x) = \begin{cases} G_k(x) & x \in U_k \text{ with } k = i \bmod 5 \\ 0 & \text{otherwise.} \end{cases}$$

This is smooth because it is zero outside

$$F_i^{-1}(0^c) = \bigsqcup_{k=i \bmod 5} G_k^{-1}(0^c)$$

and equal to G_k on each U_k.

Claim: the map $F = (F_0, F_1, F_2, F_3, F_4, f)\colon\ M \to \mathbb{R}^{(2n+1)5+1}$ is an injective immersion. proof of claim:

- Injective: there is always some $F_i(x) \neq 0$. So, we know that $x \in G_k^{-1}(0^c) \subset U_k$ for some $k = i \bmod 5$ (for this i). Then using $f(x)$ and $|f(x)-k| \leq 2$ on U_k we identify precisely which k. We conclude (since $F_i(x) = G_k(x) \neq 0$) that $x = G_k^{-1}(F_i(x))$. So based on the coordinates of F we have recovered x.
- Immersion: G_k and hence F_i has injective differential at x - hence so does F.

It is in fact an embedding, and to see this we simply note that

$$F^{-1}(K) \subset f^{-1}(\pi_{\mathbb{R}}(K))$$

is compact since f is proper. Here $\pi_{\mathbb{R}}$ is the projection to the last $\mathbb{R}$ factor.

So F is a proper injective immersion - hence an embedding. $\square$

This last part is rather general. Indeed, if $g\colon X \to \mathbb{R}^k$ is any map and $f\colon X \to \mathbb{R}$ is proper then $(g, f)\colon X \to \mathbb{R}^k \times \mathbb{R}$ is proper.

Corollary 21.4. Any smooth n-manifold M smoothly embeds into $\mathbb{R}^{2n+2}$

Proof. We can use the same trick with f_1 and f_2 to get from an embedding $i\colon M \to \mathbb{R}^N$ to an injective immersion $j\colon M \to \mathbb{R}^{2n+1}$ then $(j, f)\colon M \to \mathbb{R}^{2n+2}$ is a proper injective immersion - hence embedding. □

One may get down to dimension $2n + 1$ by combining the projection idea with the proper idea (see Guillemin and Pollacks book [1] if interested). One may even get the dimension down to $2n$ by combining the projection idea with the so called Whitney trick (see Wikipedia's article on Whitney embedding theorem).

22. Degree Modulo 2

We followed [3] chapter 4. But added details on the following constructions. If F is a smooth homotopy showing that $f \sim g$ and G is a smooth homotopy showing $g \sim h$ then

$$H(x,t) = \begin{cases} F(x, \psi(2t)) & 0 \le t \le 1/2 \\ G(x, \psi(2t-1)) & 1/2 \le t \le 1 \end{cases}$$

is a smooth homotopy showing $g \sim h$. Here $\psi\colon [0, 1] \to [0, 1]$ is smooth such that $\psi = 0$ in a neighborhood of 0 and $\psi = 1$ in a neighborhood of 1. Indeed this construction makes $H(x, t) = g(x)$ in a neighborhood of $M \times \{1/2\}$ - and hence smooth in this neighborhood. We will refer to this as concatenating homotopies.

In the following we assume that M and N are closed smooth manifolds of the same dimension and that N is connected.

Lemma 22.1. For any two points $z, y \in N$ there is a diffeomorphism $f\colon N \to N$ which is isotopic to the identity and such that $f(z) = y$.

Proof. Define an equivalence relation on points in N by: $y \sim z$ if the lemma is true for this y and z. This is well-defined:

- reflexive: use $f = \mathrm{id}$
- symmetric: if $f(y) = z$ as in Lemma and $F\colon N \times I \to N$ is an isotopy from id to f then $f^{-1}(z) = y$ and $F(-, t)^{-1}$ defines an isotopy from $\mathrm{id}^{-1} = \mathrm{id}$ to f^{-1}.
- transitive: if $f(y) = z$ with isotopy F and $g(z) = w$ with isotopy G then $H(y, t) = G(F(y, t), t)$ is an isotopy from $\mathrm{id} \circ \mathrm{id} = \mathrm{id}$ to $f \circ g$ (note this is slightly more direct than what I said in class, where I used the idea from above of concatenating homotopies instead of composition).

This is equivalence relation divides N into disjoint sets, and since N is connected, we are finished if we can prove that each equivalence class is open.

This last part of the proof differs from Milnor's proof. Indeed, he uses solutions of ODE's and the fact that the solutions are always diffeomorphisms.

Let $y \in N$ be given, we thus wish to prove that there is an open neighborhood U around y in N such that $y \sim z$ for all $z \in U$. Now pick a chart $\psi\colon \mathbb{R}^n \to N$ such that $\psi(0) = y$. Now let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\phi(t) = \begin{cases} 1 & t \le 0 \\ 0 & t \ge 1 \end{cases}$$

Now define $F\colon \mathbb{R}^n \times I \to \mathbb{R}^n$ by

$$F(x,t) = (x_1 + t\varepsilon\phi(\|x\|^2), x_2, \dots, x_n).$$

This has differential for fixed $t \in I$ given by:

$$D_x(F(-,t)) = \begin{pmatrix} 1+t\varepsilon? & t\varepsilon? & t\varepsilon? & \dots, & t\varepsilon? \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

The ?'s are functions with support on the compact set D_1^n. So they are globally bounded and hence if $\varepsilon > 0$ is small enough this is invertible for all $(x,t) \in \mathbb{R}^n \times I$ (use e.g. that the determinant is continuous).

Claim this is an isotopy. Proof of claim: for fixed $t \in I$ it is a local diffeomorphism so if we can see that it is a bijection we are finished. I did some fancy hand-waving in class, but there is a more direct way. Indeed, we can always recover $x_2, \dots, x_n$ as the last coordinates of $F(-,t)$. Knowing these we use that the function $\mathbb{R} \to \mathbb{R}$ given by

$$x_1 \mapsto x_1 + t\varepsilon\phi(x_1^2 + x_2^2 + \dots + x_n^2)$$

is bijective for any given $0 \le x_2^2 + \dots + x_n^2 \le 1$ and $t \in I$ when ε is small. Indeed, the differential is

$$x_1 \mapsto 1 + t\varepsilon?,$$

where ? again has a global bound (in fact this is the upper right corner of the above matrix). This is strictly positive for small ε. Hence the map is a strictly increasing function which is the identity outside $-1 \le x_1 \le 1$. So we can finally reconstruct x_1 using the inverse of this.

Since this isotopy is the identity outside D^n we may define a smooth isotopy on N by:

$$G(y,t) = \begin{cases} \psi \circ F(\psi^{-1}(y), t) & y \in \psi(\mathbb{R}^n) \\ y & \text{otherwise} \end{cases}$$

This proves that $y = \psi(0) \sim \psi(\varepsilon, 0, 0, \dots, 0)$. Using smaller ε and applying a rotation to the argument we see that in fact everything in the open neighborhood $\psi(\mathring{D}_\varepsilon)$ is equivalent to y. $\square$

We then proved the homotopy lemma precisely as Milnor does.

Lecture 11

23. degree modulo 2 - continued

As last time M and N are closed smooth manifolds of the same dimension n, and N is connected. The maps f and g are smooth maps $M \to N$.

I gave some intuitive pictures usefull when considering degree, which may appear here later.

Theorem 23.1. If $y, z \in N$ are two regular values for f then

$$\# f^{-1}(y) = \# f^{-1}(z) \qquad (\text{mod } 2).$$

Furthermore, this number mod 2 only depends on the homotopy class of f.

Proof. As in [3]. □

Milnor's examples where then discussed.

Exercise 23.2. Using $S^1 \subset \mathbb{C}$ define the map $S^1 \to S^1$ by $z \mapsto z^2$. Prove that this map has degree 0 mod 2.

24. Orientations on Vector Spaces.

Following Milnor chapter 5, but only the vector space part, and with more details.

Definition 24.1. An orientation on a finite dimensional real vector space V is a choice of *ordered* basis up to equivalence: two bases $(v_1, \ldots, v_n)$ and $(v'_1, \ldots, v'_n)$ are equivalent if the change of basis matrix, (a_{ij}), where $v_i = \sum_j a_{ij} v'_j$, has positive determinant. That is, unless $V = \{0\}$, in which case we assign a "+" or "−" as orientation

Note that there are precisely two orientations on any f.d.r.v.s V. Indeed, if $(v_1, \ldots, v_n)$ represents an orientation then the other orientation is given by those ordered bases $(v'_1, \ldots, v'_n)$ such that the change of basis matrix has negative determinant. This motivates the following; when a vector bundle is equipted with an orientation we say that an ordered basis $(v_1, \ldots, v_n)$ is either positive or negative depending on whether or not it defines the same orientation or the opposite (other) orientation. We also refer to this as the sign of the basis. This makes sense in the 0-dimensional case because here there is one unqiue ordered basis given by the empty set and so the orientation "+" tells us that this is a positive basis, and "−" that it is negative.

Example 24.2.

- If V has dimension 0 we think of this as a point and an orientation is a sign + or −.
- If V has dimension 1 we think of this as a line and an orientation is picking a direction on the line.
- If V has dimension 2 we think of this as a plane and an orientation is picking a positive rotation in the plane.

Example 24.3. Interchanging two vectors in any basis changes the sign of orientation. Indeed, the determinant of any base change matrix where the basis' order is simply permuted is the sign of the permutation.

Example 24.4. The standard orientation on $\mathbb{R}^n$ is given by the standard ordered basis:

$$e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1).$$

A linear isomorphism $L\colon V \cong W$ between oriented vector spaces V and W is called orientation preserving if it sends positive bases to positive bases. (in the case of 0 dimensions the sign has to be the same).

Definition 24.5. If V and W are oriented vector spaces we define the induced orientation on $V \oplus W$ as given by appending oriented bases. Indeed, if $(v_1, \dots, v_n)$ and $(w_1, \dots, w_k)$ are positive bases for V and W respectively then $((v_1, 0), \dots, (v_n, 0), (0, w_1), \dots, (0, w_k))$ defines the orientation on $V \oplus W$.

Note that this means that a basis for V with sign $s_1 = \pm 1$ appended with a basis for W with sign $s_2 = \pm 1$ has sign $s_1 \cdot s_2$. So, e.g. for $\dim(V) = 0$ and V oriented with a $-$ this corresponds to saying that the canonical isomorphism $V \oplus \{0\} \to V$ is in fact orientation *reversing*. Indeed, we are appending a basis for V with the emptyset basis for $\{0\}$, but the latter has sign -1.

WARNING: the canonical linear isomorphism $V \oplus W$ to $W \oplus V$ is only orientation preserving if one of the vector spaces are even dimensional!

Example 24.6. Two out of three property. Assume two of $V, W, V \oplus W$ are oriented then there is a unique orientation on the last one such that the orientation on $V \oplus W$ is the canonical one defined above.

25. Orientations on Vector Bundles

This is the vector bundle version of chapter 5 [3]. Using vector bundles makes the discussion slightly more general.

Definition 25.1. An *orientation* on a smooth vector bundle $\pi\colon E \to M$ is a choice of orientation for each vector space E_x such that; for each $x \in M$ there is a local trivialization

$$\phi\colon \pi^{-1}(U) \cong U \times \mathbb{R}^k \qquad (x \in U) \tag{25.1}$$

where $\phi_y\colon E_y \cong \mathbb{R}^k$ is orientation preserving for each $y \in U$. A bundle is said to be *orientable* if there exists an orientation of it.

Lemma 25.2. If M is connected there are precisely two or no orientations on any vector bundle $E \to M$.

Proof. Fix the vector bundle E. For $x, y \in M$ we define: $x \sim y$ if; for any choice of orientation on E the orientation on E_x determines the orientation on E_y - and vice versa (i.e. if you know one you know the other).

It is easy to check that this is an equivalence relation.

equivalences classes are open: exercise! Uses continuity of determinants.

It follows since M is connected that any orientation on E is fully determined by the orientation on E_x. □

Definition 25.3. If E and F are oriented then $E \oplus F$ is given the orientation defined by $(E \oplus F)_x = E_x \oplus F_x$.

The two of three property from Example 24.6 still holds. Indeed, if two of E, F and $E \oplus F$ are oriented then there is a canonical orientation on the last compatible with the above definition. In particular if two of three are orientable then so is the third.

WARNING: as for vector spaces the induced orientation is not compatible with the canonical isomorphism of bundles $E \oplus F \cong F \oplus E$ unless one of the bundles have even dimension.

Example 25.4. A 1-dimensional oriented vector bundle is trivial. Hence the bundle $M \to S^1$ with total space the mobious band from Example 13.4 is by Lemma 13.7 not orientable.

Note that this is not the same as saying that M as a manifold is not orientable (although, indeed, it is not)

Example 25.5. A trivial bundle is orientable.

Definition 25.6. An orientation on a manifold M is an orientation of the vector bundle $TM \to M$.

Example 25.7. The manifold $\mathbb{R}^k$ has the standard orientation using the canonical identification $T\mathbb{R}^k = \mathbb{R}^k \times \mathbb{R}^k$.

Definition 25.8. A diffeomorphism is said to be orientation preserving if its differential is point-wise orientation preserving.

Note that one may cover any oriented manifold by orientation preserving charts. Indeed, if a chart is not orientation preserving one may pre-compose it with an orientation reversing linear isomorphism to get a new chart which is.

Exercise 25.9. The composition of two orientation reversing diffeomorphisms is an orientation preserving diffeomorphism.

In [3] Milnor discusses induced orientations on the boundary of an oriented manifold M. He defines outward pointing vectors $v_1 \in T_xM$ for $x \in \partial M$ (in the obvious way since his manifolds are sub-spaces in $\mathbb{R}^k$). Then he says that a basis $(v_2, \ldots, v_n)$ for $T_x(\partial M)$ is positive if $(v_1, v_2, \ldots, v_n)$ is a positive basis for T_xM and v_1 is outwards pointing.

In our vector bundle Language we may pick a metric on the bundle

$$TM \to M$$

and define the 1 dimensional vector bundle

$$(T\partial M)^{\perp} \subset TM_{|\partial M},$$

which we saw in Section 15 is a complement bundle. We thus have a canonical isomorphism

$$(T\partial M)^{\perp} \oplus T\partial M \cong TM_{|\partial M}, \tag{25.2}$$

of vector bundles over the identity on ∂M. The 1 dimensional bundle $(T\partial M)^{\perp}$ is cannonically oriented by making a non-zero vector a positive basis if it points outwards. This is well-defined by the following exercise.

Exercise 25.10. Recall the exercise proving that ∂M coincides with ∂H^k in any chart. Use this to prove that defining "outwards" vectors in a chart is independent of charts.

Now we use the two of three property to define the orientation on $T\partial M$ such that Equation (25.2) is orientation preserving. Note that as in the warning above the order of the direct sum terms in Equation (25.2) is important! However, only if M is even dimensional.

Lecture 12

26. The Brouwer Degree

For $f\colon M^n \to N^n$ a map between oriented manifolds we define

$$\deg(f;y) = \sum_{x\in f^{-1}(y)} \operatorname{sign}(D_x f)$$

for $y \in N$ a regular value, and

$$\operatorname{sign}(D_x f) = \begin{cases} +1 & \text{if } D_x f \text{ preserves orientation} \\ -1 & \text{if } D_x f \text{ reverses orientation} \end{cases} \tag{26.1}$$

In general we may define a canonical orientation on a sub-manifold which is given by the pre-image of a critical value of a map between oriented manifolds. Indeed, if $F\colon X \to N$ is a smooth map $y \in N$ regular, X and N oriented, then we may orient $Q = F^{-1}(y)$ by:

- First, choising the orientation on ν_Q^X such that $D_x F\colon (\nu_Q^X)_x \cong T_y N$ is orientation preserving for all $x \in Q$.
- Then, choising the orientation on $T_x Q$ such that the canonical isomorphism $T_x Q \oplus (\nu_Q^X)_x \cong T_x X$ is orientation preserving when the direct sum is given the induced orientation.

In this language the signs in Equation (26.1) are precisely the canonical orientation of the points in $f^{-1}(y)$ (a 0 dimensional manifold).

Exercise 26.1. The orientations induced on ∂M (defined last time) and the orientation on $Q = F^{-1}(y)$ does not depend on the choice of normal bundle.

In the following X^{n+1} is a compact oriented manifold with boundary. Note that compact is important.

Lemma 26.2. If $F\colon X \to N^n$ is smooth and $y \in N$ a regular value for both F and $f = F_{|\partial X}$ then $\deg(f,y) = 0$.

Proof. By Lemma 19.15 (also M.2.4) we have that $Q = F^{-1}(y)$ is an 1-manifold with boundary equal to $\partial X \cap F^{-1}(y)$. Because X is compact then so is Q. By the above discussion Q is canonically oriented. Let $x \in \partial Q$ be given. Recall that the tangent space $T_x Q$ and the boundary $T_x \partial X$ are transversal (this was covered in these notes when proving Milnor's Lemma 2.4). Implying for dimension reasons that

$$T_x Q \cap T_x \partial M = \{0\}.$$

Now pick the metric on TX (Riemannian structure) such that:

$$T_x Q \perp T_x \partial X.$$

This implies that $(\nu_Q^X)_x = (T_x Q)^\perp = T_x \partial X$ and $(T_x \partial X)^\perp = T_x Q$. We now have two canonical orientations on both of these vector spaces.

(1) Recall that we orient ∂X by asking that

$$T_x X \cong (T_x \partial X)^{\perp} \oplus T_x \partial X = T_x Q \oplus T_x \partial X \tag{26.2}$$

is orientation preserving, where $(T_x\partial X)^{\perp} = T_x Q$ is given the orientation corresponding to being outwards pointing.

(2) We oriented Q by asking that

$$T_x X \cong T_x Q \oplus (\nu_Q^X)_x = T_x Q \oplus T_x \partial X$$

and $\nu_Q^X \cong Q \times T_y N$ using the differential of F.

Since changing the orientation on one factor of $V \oplus W$ changes the induced orientation on $V \oplus W$ we see that either the two orientations we have on $T_x Q$ and $T_x \partial M$ both agree or both disagree. the consequence is:

- The differential $D_x F\colon T_x \partial M \to T_y N$ is orientation preserving precisely when $T_x Q$ is oriented in (2) such that the positive vector points out.

The lemma now follows because Q (any oriented 1-dimensional compact manifold) has that each component is either an S^1 without boundary or a closed interval such that the orientation points out at one end and in at the other end. Hence using Q we see that the terms in the sum for the degree cancel in pairs. □

Note that in the proof we used the classification of 1-manifolds which can be found in [3].

Theorem 26.3. The degree $\deg(f, y)$ does not depend on y and only depends on the homotoypy class of f.

Proof. Since M is oriented we get an orientation on $M \times I$ by

$$T_{(x,t)}(M \times I) \cong T_x M \times T_t I = T_x M \oplus \mathbb{R}.$$

The boundary of $M \times I$ is now the disjoint union of two copies of M. Indeed,

$$\partial(M \times I) = M \times \{0, 1\}.$$

The inclusions

$$i_j\colon\ M \to M \times \{j\}$$

have opposite orientations. I.e. one preserves the orientation the other reverses it. Indeed, a positive basis $(v_1, \dots, v_n)$ for $T_{(x,0)}\partial M$ is negative as a basis for $T_{(x,1)}\partial M$ - since the inwards direction on the last factor $\mathbb{R}$ changes sign.

Now we use Lemma 26.2 to see that for any homotopy $F\colon M \times I \to \mathbb{R}$ such that $y \in N$ is regular for F and $F_{|\partial(M\times I)}$ we have that

$$\deg(F_{|M\times\{0\}}) = -\deg(F_{|M\times\{1\}})$$

using the induced orientation on the boundaries. Changing the orientation on a manifold changes the degree by a sign. So we see that for $f \sim g$ we have

$$\deg(f, y) = \deg(g, y)$$

when the homotopy has y as a regular value.

The rest of the proof is as the proof of independence of degree mod 2. $\square$

I went through the examples in Milnors book, but added that the non-zero vector field on S^{2n-1} is given by complex multiplication on $S^{2n-1} \subset \mathbb{C}^n$, and that the homotopy constructed from the identity to the antipodal map is really then multiplication with $e^{i\pi t}$.

Example 26.4. Define $f_k \colon S^n \to S^n$ by

$$f_k(x + iy, z) = \frac{((x + iy)^k, z)}{\sqrt{|x + iy|^{2k} + \|z\|^2}}$$

for $(x + iy, z) \in \mathbb{C} \times \mathbb{R}^{n-1} \cong \mathbb{R}^{n+1}$. Exercise: $\deg(f_k) = k$. Hint: compute degree at $y \in S^n$ where $y = (1, 0, \dots, 0)$ and reduce to dimension $n = 1$.

Corollary 26.5. There are infinitely many homotopy classes of smooth maps $S^n \to S^n$.

We will see later that in fact the maps in the above example are all maps up to homotopy. So essentially the degree detects the homotopy type of a map $S^n \to S^n$.

27. Transversality

We now consider a setup where we have two smooth maps $f_1 \colon M \to N_1$ and $f_2 \colon M \to N_2$. Let $y_1 \in N_1$ and $y_2 \in N_2$ be regular values for f_1 and f_2 respectively. Define $Q_1 = f_1^{-1}(y_1)$ and $Q_2 = f_2^{-1}(y_2)$. We may ask: is there a "good" condition assuring that $Q_1 \cap Q_2$ is a manifold? To answer we simply note that

$$Q_1 \cap Q_2 = (f_1, f_2)^{-1}(y_1, y_2), \tag{27.1}$$

and so we could simply require that (y_1, y_2) is a regular value for $(f_1, f_2) \colon M \to N_1 \times N_2$. For $x \in Q_1 \cap Q_2$ this amounts to saying that

$$T_x M \xrightarrow{(D_x f_1, D_x f_2)} T_{y_1} N_1 \oplus T_{y_2} N_2 \cong T_{(y_1, y_2)}(N_1 \times N_2)$$

is surjective. So we arrive at the question: when is $(L_1, L_2) \colon W \to V_1 \oplus V_2$ surjective assuming each L_i is surjective.

Exercise 27.1. This happens precisely when $\ker(L_1)$ and $\ker(L_2)$ are transversal. I.e. $\ker(L_1) + \ker(L_2) = W$

Recall that $T_x Q_i = \ker(D_x f_i)$. This idea leads to the very important definition:

Definition 27.2. Smooth submanifolds $M_1, M_2 \subset N$ are called transversal if for all $x \in M_1 \cap M_2$ the linear sub-spaces $T_x M_1, T_x M_2 \subset T_x N$ are transversal.

We will see that this leads to $M_1 \cap M_2$ being a manifold, but we don't really need both of these to be submanifolds, and the following generalization is very useful later.

Definition 27.3. A map $f\colon M_1 \to N$ is said to be transversal to a submanifold $M_2 \subset N$ if for all $y \in M_2$ we have that for each $x \in f^{-1}(y)$ the image of $D_x f$ is transversal to $T_y M_2$ in $T_y N$.

Note this is the same definition as above if f is the inclusion of M_1.

Theorem 27.4. If $f\colon M_1^{k_1} \to N^n$ is transversal to $M_2^{k_2} \subset N$ then $Q = f^{-1}(M_2)$ is a sub-manifold of dimension $k_1 + k_2 - n$.

Proof. Let $y \in M_2$ be given. Pick a submanifold coordinate chart

$$\psi\colon \mathbb{R}^n \overset{\circ}{\supset} U \to N$$

such that $y \in V_2 = \psi(\mathbb{R}^{k_2} \cap U) \overset{\circ}{\subset} M_2$.

Define $V = \psi(U)$ and define $F\colon V \to \mathbb{R}^{n-k_2}$ by

$$F(z) = \mathrm{proj}^{\perp}(\psi_2^{-1}(z)),$$

where $\mathrm{proj}^{\perp}\colon \mathbb{R}^n \to \mathbb{R}^{n-k_2}$ is the projection onto the last $n - k_2$ factors in $\mathbb{R}^n$. By construction this is a submersions and 0 is a regular value such that $V_2 = F^{-1}(0)$. Claim: there is a neighborhood W of $f^{-1}(y)$ such that the composition of $f\colon W \to N$ and F is defined and has surjective differential. Indeed, let $x \in f^{-1}(y)$ be given then the map

$$\phi\colon T_x M_1 \oplus T_y M_2 \to T_y N$$

given by $\phi(v_1, v_2) = (D_x f(v_1) + v_2)$ is surjective hence composing with $D_y F$ is still surjective. However, $T_y M_2 \subset T_y N$ is the kernel of $D_y F$ - hence $D_y F \circ D_x f$ is surjective. Since surjective is an open condition we get W as the open set in the open set $f^{-1}(V)$ where this is surjective.

It now follows that $F \circ f$ restricted to W has $0 \in \mathbb{R}^{n-k_2}$ as a regular value and thus by lemma 14.4 we get that

$$f^{-1}(V_2) = f^{-1}(F^{-1}(0)) = (F \circ f)^{-1}(0)$$

is a submanifold of dimension $k_1 - (n - k_2) = k_1 + k_2 - n$. $\square$

Lecture 13

28. Transversality continued.

After the long break I will start by recalling some stuff from last time. Let M_1, M_2 and N be smooth manifolds without boundary.

Definition 28.1. If $M_1 \subset N$ and $M_2 \subset N$ are sub-manifolds then they are called transversal if for all $x \in M_1 \cap M_2$ the linear sub-spaces $T_xM_1, T_xM_2 \subset T_xN$ are transversal.

Recall that transversal for linear sub-spaces means that

$$T_xM_1 + T_xM_2 = T_xN.$$

We saw last time that this definition leads to $M_1 \cap M_2$ naturally being a manifold, in fact the proof of the theorem below (from last time) almost proves the following intuitive mental picture.

Example 28.2. Transversal sub-manifolds locally looks like linear sub-spaces. That is there is a chart $\phi\colon \mathbb{R}^n \to N$ around $x \in M_1 \cap M_2$ in which M_1, M_2, and $M_1 \cap M_2$ looks like linear subspaces in $\mathbb{R}^n$. Here (by the transversality assumption) the linear subspaces corresponding to M_1 and M_2 sum to everything in $\mathbb{R}^n$, and so the dimensions must satisfy

$$\dim(M_1 \cap M_2) = \dim(M_1) + \dim(M_2) - n.$$

We generalized these notions to.

Definition 28.3. A map $f\colon M_1 \to N$ is said to be transversal to a sub-manifold $M_2 \subset N$ if for all $x \in f^{-1}(M_2)$ we have that the image of D_xf is transversal to T_yM_2 in T_yN (with $y = f(x)$).

This recovers the definition above if f is the inclusion of M_1. We then proved the following theorem.

Theorem 28.4. If $f\colon M_1^{k_1} \to N^n$ is transversal to $M_2^{k_2} \subset N$ then $Q = f^{-1}(M_2)$ is a sub-manifold in M_1 of dimension $k_1 + k_2 - n$.

Now assume that M_1, M_2 and N are all oriented. We give $Q = f^{-1}(M_2)$ the orientation which is compatible with the following sequence of ideas:

- Let $\nu_Q^{M_1}$ be a normal bundle to $TQ \subset TM_{1|Q}$ then since $D_xf\colon T_xQ \to T_yM_2$ we have that D_xf restricted to $(\nu_Q^{M_1})_x$ still has transversal image to T_yM_2. This implies for dimension reasons that this restriction of D_x is injective and $D_x((\nu_Q^{M_1})_x)$ is a complement to T_yM_2 in T_yN.
- Now orient $(\nu_Q^{M_1})_x$ such that $T_yM_2 \oplus D_x((\nu_Q^{M_1})_x) \cong T_yN$ is orientation preserving.
- Then orient T_xQ such that $T_xQ \oplus (\nu_Q^{M_1})_x \cong T_xM_1$.

Exercise 28.5. Prove that this does not depend on the choice of complement $\nu_Q^{M_1}$.

Example 28.6. Assume that M_1 and N have dimension n and M_2 is a point and regular value for $f\colon M_1 \to N$. We can recover the notion of degree. Indeed, let M_2 have $+$ as orientation. Then we get orientation on the zero dimensional manifold $f^{-1}(M_2)$. Summing up the orientations $+$ and $-$ gives the degree. Indeed the normal bundle to points $x \in M_1$ and $y \in N$ are the entire tangent bundle so each point is oriented $+$ if the differential agrees with the given orientations and $-$ if it does not.

In the following M_2 and N still has no boundary. Also S is a manifold with no boundary, but M_1 may have a boundary. Recall that the boundary of $M_1 \times S$ is identified with $(\partial M_1) \times S$.

Definition 28.7. A map $f\colon M_1 \to N$ where M_1 has boundary is transversal to $M_2 \subset N$ if $D_x f$ has transversal image to $T_y M_2$ for all $x \in f^{-1}(y)$ and if $f_{|\partial M_1}$ is transversal to M_2.

Example 28.8. So a function $f\colon M_1 \to N$ is transversal to a point $y \in N$ iff it is regular for both f and f restricted to the boundary.

Lemma 28.9. the set $Q = f^{-1}(M_2)$ is a manifold with boundary and $\partial Q = Q \cap \partial M_1$.

We leave out the proof since it is a not so difficult combiniation of the proof of the above theorem and the proof of Lemma 2.4 in Milnor. However, note that (as when proving Lemma 2.4) the fact that both f and $f_{|\partial M_1}$ is transversal to M_2 means that the tangent space of Q (for $x \in f^{-1}(M_2)$) is transversal to ∂M_1. It thus makes sense to say (and is as very important intuitive picture) that Q is transversal to the boundary ∂M_1.

Theorem 28.10 (transversality theorem)**.** If $F\colon M_1 \times S \to N$ is transversal to $M_2 \subset N$ then $f_s = F(-,s)\colon M_1 \to N$ is transversal to M_2 for almost all $s \in S$.

Proof. Since F is transversal to M_2 we have that $Q = F^{-1}(M_2)$ is a manifold with boundary given by

$$\partial Q = Q \cap (\partial M_1 \times S).$$

Let $\pi\colon M_1 \times S \to S$ be the projection.

It is enough to show that; when $s \in S$ is regular for the projections $\pi_{|Q}\colon Q \to S$ and $\pi_{|\partial Q}\colon \partial Q \to S$ then $f_s\colon M_1 \to N$ is transversal to M_2. Indeed, by Sard's theorem $s \in S$ is regular for both for almost all $s \in S$.

So assume $s \in S$ is regular for the first projection $\pi_{|Q}$. Let $x' \in f_s^{-1}(M_2)$ be given. Then by definition of f_s we have $x = (x', s) \in \pi^{-1}(s) \cap Q$. Denote $y = F(x) \in M_2 \subset N$. Transversality of F is equivalent to: for any $w \in T_y N$ there is a vector

$$(v, e) \in T_{x'} M_1 \times T_s S \cong T_x(M_1 \times S)$$

such that $D_x F(v, e) - w \in T_y M_2$. We wish to show that this can be picked such that $e = 0$. Indeed, this would show that $D_{x'} f_s(v) = D_x F(v, 0)$ satisfies this same condition.

The assumption that $s \in S$ is regular means that

$$D_x \pi_{|Q} \colon T_x Q \to T_s S$$

is surjective. This implies that for our $e \in T_s S$ from above there is a $(v', e) \in T_x Q$. Now notice that since the image of $F_{|Q}$ is in M_2 we have

$$D_x F(T_x Q) \subset T_y M_2.$$

So we conclude that

$$\begin{aligned} D_x F((v - v', 0)) - w =& D_x F((v, e) - (v', e)) - w = \\ =& (D_x F(v, e) - w) - D_x F(v', e) \in T_y M_2. \end{aligned}$$

So, indeed, we could initially have picked $e = 0$. Showing that the image of $D_{x'} f_s$ is transversal to $T_y M_2$.

The proof for the boundary ∂Q is the exact same argument restricted to $\partial Q \subset \partial M_1 \times S$, and assuming that $\pi_{|\partial Q}$ has $s \in S$ as a regular value. □

Corollary 28.11. Let $M_2 \subset N$ be a given submanifold. Any map $f \colon M_1 \to N$ is homotopic to a map $g \colon M_1 \to N$ transversal to $M_2 \subset N$.

Furthermore, if M_2 is proper (the embedding is proper) and f restricted to a closed subset $C \subset M_1$ is already transversal we may keep the homotopy constant on C.

Note that the closed set C can be somewhat arbitrary. So it may intersect the boundary in a very general way.

Proof. Assume first that $C = \emptyset$. Then we properly embed N into $\mathbb{R}^k$ and let $\nu \supset N$ be a tubular neighborhood. Let $\varepsilon \colon N \to \mathbb{R}_+$ be such that everything in $\mathbb{R}^k$ with distance less than $\varepsilon(x)$ to $x \in N$ is in ν. Then we may extend $f \colon M_1 \to N \subset \nu \subset \mathbb{R}^k$ to a map

$$F \colon M_1 \times \mathring{D}^k \to \nu$$

given by $F(x, s) = x + \varepsilon(x) s$. This is a submersion because

$$\operatorname{im} D_{(x,s)} F \supset \operatorname{im} D_s F(x, \cdot) = \operatorname{im}(\varepsilon(x) I) = \mathbb{R}^k.$$

So composing it with the submersion $\pi \colon \nu \to N$ given by nearest point is a submersion. It thus follows that $\pi \circ F \colon M_1 \times \mathring{D}^k \to N$ is transversal to M_2. So by the above theorem there is an $s \in \mathring{D}^k$ such that $f_s = (\pi \circ F)(-, s)$ is transversal to M_2, and since $f_0 = f$ we get that $f_{ts}, t \in I$ is a homotopy between them.

Now if $C \neq \emptyset$ we use that transversality to a proper sub-manifold at a point is an open condition. Indeed, transversality is characterised by:

- either $x \notin f^{-1}(M_2)$ which is an open condition because M_2 is proper and hence a closed subset of N,
- or $\operatorname{im}(D_x f) + T_{f(x)} M_2 = T_{f(x)} N$, which is a statement about surjectivity of linear maps ($D_x f$ and the inclusion of $T_{f(x)} M_2$). Surjectivity of linear maps is an open condition, and so in an open neighborhood of such a point either $x \notin f^{-1}(x)$ or this is satisfied.

So, there is an open set $U \subset M_1$ containing C on which f is transversal. Now pick a function $\phi\colon N \to [0,1]$ which is 0 on C but 1 on the complement of U. Then we claim that replacing the function ε with $\varepsilon\phi$ in the above argument proves the corollary. Indeed, $\pi \circ F$ is transversal at points where $\phi > 0$ by the same argument as above, and it is transversal on $\phi = 0$ because here $F(x,s) = f(x)$ and f is already transversal at these points. Now the f_s we get from the above is equal to f on C. Indeed, even the homotopy f_{st} from above is constant on C. □

Corollary 28.12. Given closed compact submanifolds $M_1, M_2 \subset N$ such that $\dim(M_1)+\dim(M_2) < \dim(N)$ we may homotopy one of the embeddings $f\colon M_1 \to N$ such that its image becomes disjoint from M_2.

Proof. Being transversal when the dimensions add up to less than the dimension of N means that $f^{-1}(M_2)$ is empty. □

Lecture 14

29. Intersection Theory

Let $M_2 \subset N$ be a proper submanifold. Let M_1 be compact possibly with boundary such that $\dim(M_1) + \dim(M_2) = \dim(N)$, and let $f_1 \colon M_1 \to N$ be a map such that $f(\partial M_1) \cap M_2 = \emptyset$ (equivalent to $f_{|\partial M_1}$ being transversal to M_2).

If f is transversal to M_2 we define

$$I_2(f, M_2) = \text{ the number of points in } f^{-1}(M_2) \bmod 2.$$

If all manifolds are oriented we define

$$I(f, M_2) = \sum_{x \in f^{-1}(M_2)} \operatorname{sign}(x).$$

Here $\operatorname{sign}(x)$ denotes $+1$ or -1 depending on the orientation sign of $x \in f^{-1}(M_2)$.

Definition 29.1. A homotopy $F \colon M_1 \times I \to N$ is said to be relative to $C \subset M_1$ if $F(x,t)$ does not depend on $t \in I$ for $x \in C$.

Two maps f and g are said to be homotopic relative to C if there is a homotopy between them relative to C.

Note that homotopic relative C thus implies that $f_{|C} = g_{|C}$. We can now fomulate a rather general intersection theory lemma.

Lemma 29.2. The number $I_2(f, M_2)$ (and when defined $I(f, M_2)$) only depends on the homotopy type of f relative to the boundary ∂M_1.

The ideas in this proof are very similar to previous ideas. Because of this lemma we extend the definitions of I_2 and I to include all maps which are transversal on ∂M_1 (empty intesection with M_2).

Proof. Assume we have a homotopy $F \colon M_1 \times I \to N$ such that $F_{|\partial M_1 \times I}$ is constantly (in t) equal to $f_{|\partial M_1}$ and such that both F_0 and F_1 are transversal to M_2.

Since both manifolds in the product $M_1 \times I$ has boundary this is strictly speaking not a manifold with boundary. However, we do know what smooth means, and we do know that $F_t(\partial M_1) \cap M_2 = \emptyset$ for all t. It follows from compactness of M_1, properness of M_2 and some topology that there is a closed set $C \subset M_1$ containing ∂M_1 in its interior and such that $F(C \times I) \cap M_2 = \emptyset$. Letting $\mathring{M}_1 = M_1 - \partial M_1$ we now simply consider the homotopy

$$F_| \colon \mathring{M}_1 \times I \to N,$$

which is defined on a manifold with boundary and which we now know to be transversal on the closed set:

$$C' = ((C - \partial M_1) \times I) \cup (\mathring{M}_1 \times \{0, 1\}).$$

So using the corollary above we may change the homotopy yet keeping it fixed on C' to a transversal homotopy:

$$F'\colon \mathring{M}_1 \times I \to N.$$

Since we did not change F' in a neighborhood of $\partial M_1 \times I$ we see that $F'^{-1}(M_2)$ does not intersect $(C - \partial M_1) \times I$, and hence is contained in the compact set $(M_1 - \mathring{C}) \times I$. It is also closed because M_2 is proper, and it is hence compact. The boundary is by previous result contained in

$$\mathring{M}_1 \times \{0, 1\}.$$

It now follows that $I_2(F_0, M_1) = I_2(F_1, M_1)$. Indeed, the sum

$$I_2(F_0, M_1) + I_2(F_1, M_1)$$

is the number of boundary points in the compact 1-manifold with boundary $F'^{-1}(M_2)$, which is thus even.

In the oriented case we see that F'^{-1} is an oriented 1-manifold and so the signs of the boundary sums up to 0 in $\mathbb{Z}$. So

$$I(F_0, M_1) - I(F_1, M_1) = 0.$$

Recall that the minus is there because the given orientation of $\mathring{M}_1$ agrees at one end of $\mathring{M}_1 \times I$ with the induced boundary orientation, but disagrees at the other end. For this argument it is not important at what end it agrees and at what end it disagrees. □

Example 29.3. The orientation of a point $x \in M_1 \cap M_2$ for transversal sub-manifolds M_1 and M_2 in N can be traced back to the sign of the basis

$$(v_1, \ldots, v_{k_1}, w_1, \ldots, w_{k_2})$$

for $T_x N$. Here $(v_1, \ldots, v_{k_1})$ is a positive basis for $T_x M_1$ and $(w_1, \ldots, w_{k_2})$ is a positive basis for $T_x M_2$.

Example 29.4. Mapping S^1 into the torus $T^2 = S^1 \times S^1$ by the two maps

$$f_1(z) = (z, 1) \qquad \text{and} \qquad f_2(z) = (1, z)$$

Letting $M_2 = (f_2(S^1))$ and $M_1 = (f_1(S^1))$ we see that

$$I(f_1, M_2) = -I(f_2, M_1).$$

Indeed, M_1 and M_2 intersects transversally in a single point $(1, 1)$. The orientations differs in the two cases because the two bases (v_1, w_1) and (w_1, v_1) differs in sign no matter what the orientation on T^2 is.

Example 29.5. In general we see that if M_1 and M_2 are both odd-dimensional and N oriented then

$$I(f_1, M_2) = -I(f_2, M_1)$$

where $f_i\colon M_i \to N$ is the inclusion.

Definition 29.6. For an oriented manifold $M^n \subset N^{2n}$ (with N oriented) we define the self intersection number by

$$I(f, M)$$

where f is the inclusion of M. Note that to actually calculate this we have to "push" M of itself and count transversel intersection points.

Example 29.7. The self intersection of an odd dimensional oriented manifold in an oriented manifold is 0. Indeed, this follows from example 29.5.

The following example is very closely related to the idea used earlier to prove that the Mobius bundle $M \to S^1$ (with total space the Mobious strip) is not trivial (as a bundle over S^1).

Example 29.8. Let

$$f\colon\ S^1 = [0,1]_{/0\sim 1} \to M = [0,1] \times (-1,1)_{/(0,u)\sim(1,-u)}$$

be the inclusion which is zero in the second factor. We may smoothly push S^1 of itself by the formula

$$f_t(s) = (s, \tfrac{1}{2}t\cos(\pi s)) \qquad t \in I.$$

For $t > 0$ this is transversel to f and this shows that $I_2(f, S^1) = 1$. We thus conclude since S^1 is orientable that M is not (as a manifold).

Ie the Moebius strip is an unorientable manifold.

Lecture 15

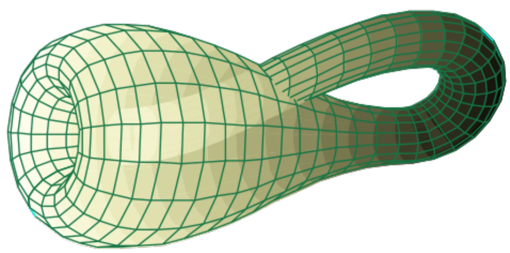
30. Jordan-Brouwer's Separation Theorem

Theorem 30.1 (Jordan-Brouwer separation theorem - (smooth))**.** Let $M \subset \mathbb{R}^n$ be a codimension 1 connected proper sub-manifold. Then the complement $\mathbb{R}^n - M$ has precisely two path components.

Notice in particular that this makes M oriented (see the proof below). Also, notice that for M closed there is an unbounded component and a bounded component. This we will call the outside and inside of $M \subset \mathbb{R}^n$ respectively.

Corollary 30.2. There are no proper smooth embedding of the mobious strip into $\mathbb{R}^3$.

Definition 30.3. The Klein bottle K is defined as $[0,1] \times [0,1]_{/\sim}$ where $\sim$ identifies $[x,0] \sim [x,1]$ and $[1,y] \sim [0,1-y]$. This can be done in a smooth way, and an immersion of K into $\mathbb{R}^3$ can be visuallized as:



Picture: Wikipedia

Corollary 30.4. There is no smooth embedding of K into $\mathbb{R}^3$.

Proof. If there was an embedding K would be oriented. This can't be since K contains a copy of the Mobious band: $M = [0,1] \times (1/4, 3/4)/\sim$. □

Corollary 30.5. There is no smooth embedding of $\mathbb{R}P^2$ into $\mathbb{R}^3$.

Proof. Again we want to prove that $\mathbb{R}P^2$ contains a copy of the Mobious band. For this consider the standard embedding of $S^2 \subset \mathbb{R}^3$

The cylinder $S^1 \times (-\varepsilon, \varepsilon) \subset \mathbb{R}^3$ can be embedded (projected) into S^2 by

$$f(z,t) = \frac{(z,t)}{\|(z,t)\|}.$$

This is an open set in S^2, which is saturated for the quotient $q\colon S^2 \to \mathbb{R}P^2$. So, it defines an open set U in the image.

Restricting the quotient to $V = (S^1 \cap \mathbb{R} \times \mathbb{R}_{\geq 0}) \times (-\varepsilon, \varepsilon)$. We still have a surjection $q_{|V}\colon V \to U$ which precisely identifies $(-1,0,t)$ with $(1,0,-t)$. So we recognize U as a copy of the Mobious band. □

Proof of separation theorem. For any two points $x, y \in \mathbb{R}^n - M$ consider smooth paths $\gamma\colon [0,1] \to \mathbb{R}^n$ such that $\gamma(0) = x$ and $\gamma(1) = y$. All such smooth paths are homotopic relative the boundary $\partial I = \{0,1\}$. Indeed, $\mathbb{R}^n$ is convex so

$$\gamma_t(s) = t\gamma(s) + (1-t)(sx + (1-s)y)$$

is a homotopy relative $\{0,1\}$ to the path $s \mapsto (sx + (1-s)y)$ (straight line from x to y). We use this to define an equivalence relation $\sim$ on $\mathbb{R}^n - M$ by

- $x \sim y$ if $I_2(\gamma, M) = 0$ for any smooth $\gamma\colon I \to \mathbb{R}^n$ with $\gamma(0) = x$ and $\gamma(1) = y$.

This is well-defined because

- It is reflexive because the constant path at x does not intersect M.
- It is symmetric because a path γ which is transversal (to M) from x to y intersects M the same number of times as the path $s \mapsto \gamma(1-s)$ going backwards from y to x (which is also transversal to M).
- It is transitive because concatenation of transversal paths simply adds the intersection numbers. Indeed, assume that γ_1 is a transversal path from x to y and that γ_2 is a transversal path from y to z. Then as we concatenated homotopies earlier in Section 22 we can define a path γ first following γ_1 from x to y and then following γ_2 from y to z such that it is transversal and the intersection numbers satisfy

$$I_2(\gamma, M) = I_2(\gamma_1, M) + I_2(\gamma_2, M). \tag{30.1}$$

Claim 1: This equivalence relation divides $\mathbb{R}^n - M$ into precisely two non-empty equivalence classes.

Claim 2: The equivalence classes are open.

Claim 3: The equivalence classes are path connected.

Proof of claim 1: Fix $x \in \mathbb{R}^n - M$ define the two sets: $U = \{y \in \mathbb{R}^n \mid x \sim y\}$ and $U^c = \mathbb{R}^n - M - U$. Now assume we have $y_1, y_2 \in U^c$ then by definition we have

$$I_2(\gamma_1, M) = 1 \qquad \text{and} \qquad I_2(\gamma_2, M) = 1$$

where γ_i is a path from x to y_i. The construction above leading to Equation (30.1) shows that if γ is the path from y_1 to y_2 given by first going back along γ_1 and then following γ_2 we have

$$I_2(\gamma, M) = I_2(\gamma_1, M) + I_2(\gamma_2, M) = 1 + 1 = 0.$$

So we conclude that $y_1 \sim y_2$. We, however, still need to argue that U^c is non-empty. So, pick a small sub-manifold chart $\phi\colon \mathbb{R}^n \overset{\circ}{\supset} V \to \mathbb{R}^n$ around some point $p \in M$. As always $\phi(\mathbb{R}^{n-1} \cap V) = M \cap \phi(V)$ and we may assume that $0 \in V$. Pick a small $\varepsilon > 0$ so that the path $\gamma(s) = (0, \varepsilon(s - 1/2)) \in \mathbb{R}^{n-1} \times \mathbb{R}$ lies in V. This path intersects $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ transversely in precisely one point. Hence the path $(\phi \circ \gamma)\colon I \to \mathbb{R}^n$ intersects M transversely in 1 point. So, we conclude that $\gamma(0)$ is not in the same equivalence class as $\gamma(1)$.

Proof of claim 2: this is easy since each point in $\mathbb{R}^n - M$ has a ball neighborhood disjoint from M. So the direct path between the points shows they are in the same equivalence class.

Proof of claim 3: Firstly we notice that the proof of claim 1 and 2 actually shows that $M \cup U$ is a manifold with boundary equal to M. It follows that M is orientable by giving it the induced boundary orientation. It also follows that the normal bundle ν_M of M is 1-dimensional and trivial. So pick a trivialization and a tubular neighborhood embedding:

$$i\colon M \times \mathbb{R} \cong \nu_M \subset \mathbb{R}^n.$$

We may assume that i maps $M \times \mathbb{R}_{\geq 0}$ into an open neighborhood of M in $U \cup M$. Now let $x \in U$ be given. Then there is a path $\gamma\colon I \to \mathbb{R}^n - M$ from x to some x' in the image $i(M \times \mathbb{R}_+) \subset U$. Indeed, stop any path going to U^c moments before its first intersection point with M. So any point in U is path connected to a point in the image $i(M \times \mathbb{R}_+) \subset U$ (with a path in U), and this image is path connected. So U is path connected. Similar for U^c. ☐

Corollary 30.6. *M as in the above theorem is always orientable.*

Let M be an $n-1$ manifold and $f\colon M \to \mathbb{R}^n$ a smooth map. For $x \in \mathbb{R}^n - f(M)$ we define the mod 2 winding number

$$W_2(x, f) = \deg_2(g_x)$$

where $g_x\colon M \to S^{n-1}$ is defined by:

$$g_x(z) = \frac{f(z) - x}{\|f(z) - x\|}$$

If M is oriented we may define the winding number in $\mathbb{Z}$ as the degree of this map.

Example 30.7. for $f\colon S^1 \to \mathbb{R}^2$ we recover usual 2-dimensinoal winding number.

When M is embedded g_x may be written as

$$g_x(z) = \frac{z - x}{\|z - x\|} \tag{30.2}$$

for $z \in M \subset \mathbb{R}^n$.

Corollary 30.8. *If $M^{n-1} \subset \mathbb{R}^n$ is closed (compact) then the mod 2 winding number determines the inside and outside of M.*

Proof. Let $x \in \mathbb{R}^n - M$ be given. Let $y \in S^{n-1}$ be regular for g as in Equation (30.2). This is the same as $T_z M$ for $z \in g^{-1}(y)$ being a linear complement to $y \in \mathbb{R}^n$. Indeed, one may calculate that the differential of g_x is

$$D_z(g_x) = \frac{1}{\|z - x\|}(I - \mathrm{proj}^{\perp}_{z-x}) = \frac{1}{\|z - x\|}\,\mathrm{proj}^{\perp}_{(z-x)^{\perp}}\,.$$

Here $\mathrm{proj}_v^\perp$ is the orthogonal projection onto the vector v and $\mathrm{proj}_{v^\perp}^\perp$ is the orthogonal projection onto the orthogonal complement of v.

The value $y = \frac{z-x}{\|z-x\|}$ being regular means that the projection along y to the unit sphere centered at x takes T_zM surjectively (and hence isomorphically) to the orthogonal complement of v. It follows that the half line $\gamma\colon [0,\infty) \to \mathbb{R}^n$ given by

$$\gamma(t) = x + ty$$

intersects M transversally, and that the mod 2 winding number is the same as the mod 2 intersection of this half line with M. We hence conclude that $W_2(x, f)$ is 1 on the inside of M and 0 on the out-side. Indeed, a point far away from M has a half-line going to ∞ not intersecting M. □

Let M be a smooth manifold, pick a so-called *base-point* $x_0 \in M$. Define $s_0 = (1, 0, \dots, 0) \in S^n$ to be the canonical base-point in S^n. A space with a choice of base-point is called a *based* space. A map between spaces with base-points is said to be *based* if it send the base-point to the base-point.

Definition 30.9.

$$\pi_i(M, x_0) = \text{homotopy classes of based maps } f\colon S^i \to M \text{ relative } s_0.$$

Example 30.10. Since $S^0 = \{-1, 1\}$ we get that a map $f\colon S^0 \to M$ with $f(1) = x_0$ is precisely determined by the point $f(-1)$. A homotopy relative to $s_0 = 1$ is then simply a path. So

$$\#\pi_0(M) = \text{ number of path componenents of } M$$

Independent of base-point $x_0 \in M$. Indeed, two points $f_0(-1)$ and $f_1(-1)$ are equivalent in M if there is a path between them.

So for $M = M^{n-1} \subset \mathbb{R}^n$ connected. Jordan-Brouwer's fixed point theorem then states:

$$\#\pi_0(\mathbb{R}^n - M) = 2.$$

Proposition 30.11. For $M^k \subset \mathbb{R}^n$ a properly embedded oriented manifold with $k < n - 1$ we have

$$\#\pi_i(\mathbb{R}^n - M) = \begin{cases} 1 & i < n-k-1 \\ \infty & i = n-k-1 \end{cases}$$

independent of base-point $x_0 \in \mathbb{R}^n - M$.

Sketch of proof. Let $i < n - k - 1$ and look at any smooth map $f\colon S^i \to \mathbb{R}^n - M$ with $f(s_0) = x_0$. We define a homotopy

$$F\colon S^n \times I \to \mathbb{R}^n$$

by

$$F(s,t) = f(s)t + x_0(1-t).$$

This is a homomotopy from f to the constant map to $x_0 \in \mathbb{R}^n - M$. Using the lemma about homotoping maps to be transversal relative to any closed set

C where it is already transversal. We get with $C = \partial(S^n \times I) = S^n \times \{0,1\}$ a new transversal (to M) homotopy

$$F' \colon S^n \times I \to \mathbb{R}^n$$

with $F'_0 = f$ and F'_1 still constant at x_0. Now, transversal to M means for dimension reasons: does not intersect. Hence

$$F' \colon S^n \times I \to \mathbb{R}^n - M$$

is a homotopy showing that f is homotopic in $\mathbb{R}^n - M$ to the constant map to x_0.

To prove the last part when $i = n - k - 1 \geq 1$ we use intersection theory. Indeed, for any map

$$f \colon S^i \to \mathbb{R}^n$$

we define its linking with M denoted $\mathrm{Link}(f, M)$ by

$$\mathrm{Link}(f, M) = I(F, M),$$

where F is the homotopy defined above

claim 1: The linking with M is the same for maps f homotopic to g in $\mathbb{R}^n - M$. claim 2: All linking numbers can be attained.

proof of claim 1: By previous results the linking number does not depend on the choice of F as long as $F_0 = f$ and $F_1 = x_0$ (all homotopies in $\mathbb{R}^n$ are homotopic relative anything by convexity). So we can use any one we want to compute the linking number. Let $G \colon S^n \times I \to \mathbb{R}^n - M$ be a homotopy between two maps f and g, and let $F \colon S^n \times I \to \mathbb{R}^n$ be a homotopy from f to x_0. By concatenating these homotopies we get a homotopy from g to x_0. However, all intersection with M takes place in the last part given by F so we conclude:

$$\mathrm{Link}(g) = \mathrm{Link}(f).$$

Proof of claim 2: This is a local construction. So look at $\mathbb{R}^k \subset \mathbb{R}^n$ then the map $f \colon S^i \to \mathbb{R}^n$ given by

$$f_i(z, x_1, \ldots, x_{i-1}) = \frac{(0, \ldots, 0, z^n, x_1, \ldots, x_{i-1})}{\|(0, \ldots, 0, z^l, x_1, \ldots, x_{i-1})\|}$$

has linking l with $\mathbb{R}^k$. It is not too dificult to prove that this last part is also independent of base-point, but we omit it here. □

Corollary 30.12. For M unoriented we have

$$\#\pi_i(\mathbb{R}^n - M) = \begin{cases} 1 & i < n - k - 1 \\ \geq 2 & i = n - k - 1 \end{cases}$$

Proof. The proof above works the exact same. However, only linking mod 2 is defined, but we can get both 0 and 1 linking with a sphere as above. □

Lecture 16

31. Vector Fields and The Euler Number

Recall that a vector field on a Manifold M is a section in $TM \to M$. Ie a vector field is smooth map $v\colon\, M \to TM$ assigning a tangent vector $v(x) \in T_xM$ for each $x \in M$.

We wish to analyze the zero's of such vector fields. We start by defining the Euler number of a compact manifold M possibly with boundary. Milnor does not do this. However, because we developed intersection theory we start here. We will, however, not prove that this is completely well-defined until we reach the end of this section.

Notice first that any diffeomorphism $f\colon\, M \to N$ defines a correspondence between vector fields. Indeed, if $v\colon\, M \to TM$ is a vector field then

$$Df \circ v \circ f^{-1}\colon\, N \to TN$$

is the corresponding vector field on N.

Lemma 31.1. Any vector field $v\colon\, M \to TM$ is homotopic through vector fields to one $v'\colon\, M \to TM$ which is transversal to the zero-section.

Here “through vector fields” means: a homotopy $v_t\colon\, M \to TM, t \in I$ where each v_t is a vector field. This is a slight generalization of the lemma that any map is homotopic to a transversal map. Indeed, we need the homotopy and the end result to be sections. However, we can almost use the same proof.

Proof. Let $M \subset \mathbb{R}^n$ be any embedding. Then $T_xM \subset \mathbb{R}^n$ and we may consider $TM \subset M \times \mathbb{R}^n$ a sub-bundle in the trivial bundle. Let $\pi_x\colon\, \mathbb{R}^n \to T_xM$ be the orthogonal projection. These assemble to a smooth bundle map

$$\pi\colon\, M \times \mathbb{R}^n \to TM$$

which is a submersion (exercise). Now the map

$$V\colon\, M \times \mathbb{R}^n \to TM$$

define by $V(x, s) = v(x) + \pi_x(s)$ is also a submersion, and all maps $V_s = V(-, s)\colon\, M \to TM$ are sections. So using the transversality theorem we can get a section V_s homotopic through sections $V_{st}, t \in I$ to v, which is transversal to any given sub-manifold - in particular the zero section. □

Example 31.2. The zero section $\mathbb{R} \to T\mathbb{R}$ shows that any intersection number between the zero-section and itself can (in general) not really be well-defined and independent when homotoping a section. Indeed, one can move it completely of itself to have no intersection or one can tilt it to have a single transversal intersection at 0.

Because of this example we want to somehow restrict what happens at “infinite”. For this purpose we simply assume in the following that M is compact with boundary, and define the following notion.

An outward pointing vector field (o.p.v.f.) v is a vector field $v\colon M \to TM$ such that $v(x)$ points out at each $x \in \partial M$. Formally outward pointing might depend on charts, but observation 19.9 describes how it does not. Also, a standard partition of unity argument proves that o.p.v.f.'s exists.

Lemma 31.3. Any o.p.v.f $v\colon M \to TM$ is homotopic through o.p.v.f.'s to one which is transversal to the zero-section in the interior of M.

Note that an o.p.v.f. is automatically "transversal" at the boundary because it is non-zero.

Proof. This follows as the above lemma (transversality theorem only used on the interior of M). However, it is important to note that for $s \in \mathbb{R}^k$ small enough adding the part $\pi(s)$ at the boundary of M does not change the fact that it points outward. □

This proof used compactness of the boundary - however one can get around this.

We now describe how TM as a manifold is canonically oriented. Indeed, if $\psi\colon U \to M$ is a chart on M we get a chart on TM by

$$D\psi\colon U \times \mathbb{R}^k \to TM.$$

We have a canonical orientation on $U \times \mathbb{R}^k$ given by declaring that

$$((b_1, 0), \dots, (b_n, 0), (0, b_1), \dots, (0, b_n)) \tag{31.1}$$

is positively oriented for *any* basis $b_1, \dots, b_n$. This induces an orientation on the image of $D\psi$, which does not dependent on the chart. Indeed, if ψ' is another chart the charts on TM differ on the overlap by

$$D\phi\colon U \times \mathbb{R}^k \cong U' \times \mathbb{R}^k$$

with $\phi = \psi \circ \psi'^{-1}$. The differential of this is

$$D(\phi)_{(x,v)} = \begin{pmatrix} D_x\phi & 0 \\ ? & D_v(D_x\phi) = D_x\phi \end{pmatrix}$$

Indeed, $D\phi(x, v) = (\phi(x), D_x\phi(v))$ has first n coordinates not depending on v and $D_x\phi(v)$ is linear in v. Now we see that the determinant of this is always positive. The general idea here being that the same basis or map is repeated twice hence if ϕ is not orientation preserving we still get -1 is squared to a $+1$.

We can now in the oriented case define the Euler number.

Definition 31.4. The Euler number $\chi(M)$ of an oriented compact manifold with boundary is defined as the oriented intersection number $I(s, M)$ of an outward pointing vector field $s\colon M \to TM$ with the zero-section $M \subset TM$. Here TM is given the canonical orientation.

We could proceed as we have done a couple of times before and prove that this only depends on the homotopy type of such o.p.v.f.'s. Then because the set of o.p.v.f.'s is convex we see that it only depends on M (and possibly the non-canonical orientation on M, which we will see it does not). However, since we will partially follow Milnors book after this initial digression this will follow from that anyway.

If M is not oriented the important thing to notice is that: we can in fact still define a $\mathbb{Z}$ valued intersection number of the zero-section and a transversal v.f. v. Indeed, notice that if $x \in M$ is a zero for $v\colon M \to TM$ then in the oriented case we oriented this intersection point by: let $(b_1, \ldots, b_n)$ be an oriented basis for T_xM and let $i\colon M \to TM$ be the inclusion of the zero-section. Then the intersection point gets a plus if the basis

$$(D_x i(b_1), \ldots, D_x i(b_n), D_x v(b_1), \ldots, D_x v(b_n)) \tag{31.2}$$

is positively oriented for $T_{(x,0)}(TM)$. We immediately notice that if we change the sign of $b_1, \ldots, b_n$ this does not change sign. Indeed, again we are multiplying -1 with -1 to get $+1$.

More concisely, in a chart (and with $v'\colon U \to \mathbb{R}^k$ a map representing a v.f.) the ordered basis from Equation (31.2) becomes:

$$((b_1, 0), \cdots, (b_n, 0), (b_1, D_x v'(b_n)), \ldots, (b_n, D_x v'(b_n)). \tag{31.3}$$

We now see that the sign coming from Equation (31.2) does not depend on whether or not $(b_1, \ldots, b_n)$ is positive or not. Indeed, changing the sign on the basis also changes the sign in the second factor. In fact the sign of this basis is precisely the sign of the determinant of $D_x v'$. Indeed,

$$\det \begin{pmatrix} I & I \\ 0 & D_x v' \end{pmatrix} = \det(D_x v')$$

and the basis in Equation (31.3) is this matrix applied to the oriented basis in Equation (31.1).

We conclude that we can define the intersection number between the zero-section and a transversal section in $\mathbb{Z}$ even if M is not orientable. We will see another proof of why this does not depend on any orientations when proving Milnors Lemma M.6.1.

Definition 31.5. Define the Euler number $\chi(M)$ of any compact manifold with boundary to be the $\mathbb{Z}$ valued intersection number (defined as above) of a o.p.v.f. with the zero-section.

Here we could also continue (in a fashion similar to earlier proofs) by proving the that this does not depend on the section up to homotopy of sections that keeps the section outwards pointing at the boundary. However, again this will follow from the proofs we will do from Milnor's book. However, given that it is well defined the following is immediate.

Corollary 31.6. For any closed odd dimensional M we have $\chi(M) = 0$.

Proof. The formula for this intersection sign is such that taking minus the same section gives in the odd dimensional case that all intersection points changes signs. Indeed, in Equation (31.2) changing the sign on v changes the sign on the last n vectors - hence the orientation is changed. So we see that

$$\chi(M) = -\chi(M).$$

Notice that $-v$ will not be outwards pointing if there is a boundary so this only works when the boundary is empty. □

We then move to Euclidean space and follow Milnor chapter 6, and the following are some extra details to that. So assume that we have a vector field v on $U \mathring{\subset} \mathbb{R}^n$. Since TU is canonically identified with $U \times \mathbb{R}^n$ this is the same as having a smooth map

$$v' \colon U \to \mathbb{R}^n.$$

Further assume that $z \in U$ is an isolated 0. Pick a ball $B_\varepsilon(z)$ so small that z is the only zero on it. The boundary of this small ball has the induced boundary orientation. Define the *index* of v at z as the degree of the map:

$$w \colon \partial B_\varepsilon(x) \to S^{n-1}$$

defined by

$$w(x) = \frac{v'(x)}{\|v'(x)\|}.$$

So this is related to the winding number. Indeed, it is the winding number of v' around 0 of a small sphere centered at $z \in U$.

Lemma 31.7. As long as $B_\varepsilon(x)$ contains no other zeros this definition does not depend on $\varepsilon > 0$.

Proof. For small ε define the oriented diffeomorphism

$$f_\varepsilon \colon S^{n-1} \cong \partial B_\varepsilon(x)$$

given by

$$f_\varepsilon(z) = x + \varepsilon z.$$

Define $v_\varepsilon = v \circ f_\varepsilon$, and $w_\varepsilon(x) = \frac{v_\varepsilon(x)}{\|v_\varepsilon(x)\|}$. Then the degree of w as above for a fixed ε is the degree of the map

$$w_\varepsilon \colon S^{n-1} \to S^{n-1}$$

given by

$$w_\varepsilon(z) = \frac{v(x + \varepsilon z)}{\|v(x + \varepsilon z)\|}.$$

Degree is a homotopy invariant and this continuously depends on ε. So the homotopy

$$t \mapsto w_{t\varepsilon + (1-t)\varepsilon'}$$

proves the independence on ε as long as v has no zeros on

$$B_\varepsilon - \mathring{B}_{\varepsilon'}$$

when $\varepsilon > \varepsilon'$. $\square$

We continued in Milnors book, but use in the proof of lemma [M.6.2] that

$$\begin{aligned} f(x) &= \int_0^1 \tfrac{\partial f(tx_1,\ldots,tx_n)}{\partial t} dt = \\ &= \int_0^1 \sum_{i=1}^n x_i \tfrac{\partial f}{\partial x_i}(tx_1,\ldots,tx_n) dt = \\ &= \sum_{i=1}^n x_i \left(\int_0^1 \tfrac{\partial f}{\partial x_i}(tx_1,\ldots,tx_n) dt \right) = \\ &= \sum_{i=1}^n x_i g_i(x). \end{aligned}$$

Also Milnor uses in this proof that any orientation preserving linear map is smoothly isotopic to the identity. This is true because of the following lemma

Lemma 31.8. $\mathrm{Gl}_n(\mathbb{R}) \mathring{\subset} \mathbb{R}^{n^2}$ has precisely two path components.

Note that since the determinant is continuous these are (using intermediate value theorem) precisely detected by the sign of the determinant - hence the two path components corresponds to the orientation preserving matrices and orientation reversing matrices respectively.

Proof. We will prove that for any matrix there is a path to the identity I or the identity with the last 1-entry changed to -1§. Denote the later R (R for reflection in coordinate x_n). We start by proving this in the special case where $E \in \mathrm{Gl}_n(\mathbb{R})$ is any of the matrixes:

(1) An elementary matrix corresponding to adding a multiple of one row to another.
(2) An elementary matrix exchanging two adjacent rows.
(3) An elementary matrix multiplying a row with a constant.

With $E = I + \alpha e_{ij}$ as in (1) we have a path from I to E given by:

$$E_t = I + t\alpha e_{ij}.$$

Here e_{ij} is the matrix with one non-zero entry at (i,j) equal to 1.

With E as in (2) we start by using that multiplying with the 2 by 2 matrix given by rotation with angle $t\pi/2$ provides a path from

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Using this we can in an n by n matrix start by moving the two adjacent misplaced 1's into the diagonal creating the same $-$ sign on the lower entry.

Now, we can move this minus sign down to the right corner by iterating the path from

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{to} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

given by rotating $t\pi$. So we get a path to R.

In the case of (3) we use a path scaling the constant to be ± 1 (depending on the sign of the constant since we do not wish to cross 0). If it is $+1$ we are finished. If it is -1 we do as the second half of case (2) to get a path to R.

The general lemma now follows from the fact that all matrices can be written as a product of such elementary matrices. So the product of the paths defined above defines a path from A to a product of I's and R's which is again an I or an R. $\square$

Lecture 17

32. Vector Fields and Euler Number continued.

Last time we defined the index of an isolated zero of a vector field $v\colon U \to TU$. We may extend the definition of index to any isolated 0 of vector field $v\colon M \to TM$ by using charts, and corresponding vector fields.

Lemma 32.1 (M.6.1 reformulated)**.** The index of a vector field is independent of the choice of chart.

Proof. To prove independence of charts we must prove independence for any diffeomorphism $\psi\colon \mathbb{R}^n \mathring{\supset} U \to U' \mathring{\subset} \mathbb{R}^n$. Indeed, if $\phi_i\colon U_i \to M$ are two charts around a 0 for a v.f. then

$$v_1 = (D\phi_1)^{-1} \circ v \circ \phi_1,$$

which corresponds to v using ϕ_1, also corresponds under $\phi_2^{-1} \circ \phi_1$ to

$$v_2 = (D\phi_2)^{-1} \circ v \circ \phi_2$$

in the overlap between the two charts. Now the proof proceeds as the proof in Milnor. □

Definition 32.2. A zero z for a vector field $v\colon \mathbb{R}^n \mathring{\supset} U \to \mathbb{R}^n$ is called non-degenerate if $D_z v\colon \mathbb{R}^n \to \mathbb{R}^n$ is nonsingular.

We already saw that this is equivalent to v being transversal to the zero section, and we thus know that this does not depend on the chart. Now we proved lemma M.6.4 as in the book. Comparing with last time we immediately see:

Corollary 32.3. The index of a non-degenerate zero equals the signed intersection of the zero with the zero-section.

We also knew from last time that this sign of the determinant does not depend on charts chosen. So in this case we have proven Lemma M.6.1 twice.

We now start by restricting to the case where $X \subset \mathbb{R}^n$ is compact submanifold with boundary of dimension n.

Then we proved Lemma M.6.3 as in the book, and thus gets the following corollary.

Corollary 32.4. In this special case the Euler number $\chi(X)$ is well-defined, and for any vector field $v\colon X \to TX$ with isolated zero's the sum of all the indices equals $\chi(X)$.

Proof. If $v\colon X \to TX$ is non-degenerate (transversal) and o.p. then by Lemma M.6.3 and the corollary above we have that the intersection number of v with the zero section is equal to the $\sum \iota$ which equals the degree of the Gauss map. So the Euler number does not depend on v.

For any v.f. v with isolated zero's we can change the vector field slightly to be transversal to the zero section without changing the sum of the indices.

Indeed, we can change v to be transversal with out introducing new zeros on X minus the balls from Lemma M.6.3. □

Note that this proof provides a very good intuitive picture. Indeed, a very small homotopy of the section will only have zero's very close to the old zero's. Hence each of the zero's "breaks up" (inside each ε ball) into a number of non-degenerate (transversal) zero's, and the index of the original zero is thus the sum of the ± 1 associated to each non-degenerate zero.

Corollary 32.5. The degree of the Gauss map N does not depend on the choice of embedding $X \subset \mathbb{R}^n$.

Now we continue in the case of any closed n dimensional manifold M *without boundary*. Instead of proving Lemma 5 and using Milnors proof of Theorem 1 we will do something more general. Indeed, the geometric intuition is the same, and the tools we have developed makes this approach possible. However, first a few lemmas that we will not explicitly use, however, the ideas of constructions in them is essential for the arguments later.

Lemma 32.6. Let $v\colon M \to TM$ be any vector field on a sub-manifold $M^m \subset N^n$. Then we can extend v to a vector field on N.

Note that here extension refers to the fact that $v\colon M \to TM \subset TN$ already looks like the restriction of a vector field on N to M, which happens to map to the sub-vector bundle given by the tangent space of TM.

Proof. Let $\psi\colon \mathbb{R}^m \times \mathbb{R}^{n-m} \overset{\circ}{\supset} U \to N$ be a sub-manifold chart around a point in M. Since $D_{(x,0)}\psi$ maps $\mathbb{R}^m$ isomorphically to $T_{\psi(x,0)}M$ the vector field v' on $U \cap \mathbb{R}^m$ corresponding to v under ψ is

$$v'\colon U \cap \mathbb{R}^m \to \mathbb{R}^m \subset \mathbb{R}^m \times \mathbb{R}^n.$$

Here we have written v' as a map and not a section. This is easy to extend to a neighborhood by the formula:

$$v^e(x,y) = v'(x).$$

This corresponds (using ϕ^{-1}) to a vector field on $\psi(U)$ which extends v.

If we locally have extensions $v_1, \dots, v_k\colon N \overset{\circ}{\supset} U \to TU$ of v then for any smooth maps $f_1, \dots, f_n\colon \mathrm{U} \to \mathbb{R}$ such that $\sum_i f_i = 1$ the vector field defined by

$$(\sum_i f_i v_i)(x) = \sum_i f_i(x) \cdot v_i(x)$$

(using that T_xU is a vector space) is again an extension of v. Indeed,

$$(\sum_i f_i v_i)(x) = \sum_i f_i(x) v_i(x) = \sum_i f_i(x) v(x) = v(x) \qquad \text{for } x \in M.$$

We thus see that these local extensions can be patched together using a partition of unity on N. Here one may use that *any* vector field is an "extension" on the compliment $N - M$. □

Lecture 18

33. Vector Fields and the Euler Numbers Continued II

Let $\pi\colon E \to M$ be any k dimensional vector bundle.

Lemma 33.1. Let $v\colon M \to TM$ be a vector field, then there exists an extension vector field $v^e\colon E \to TE$ such that

$$\begin{array}{ccc} E & \xrightarrow{v^e} & TE \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle D\pi} \\ M & \xrightarrow{v} & TM \end{array}$$

commutes.

Recall that $M \subset E$ is the zero-section. So, the word extension makes sense. A map v^e such that the diagram commutes is sometimes called a lift.

Proof. As the previous lemma (last time) except using charts given by locally trivializing E (see the proof of Lemma 33.3 below for more details) and the fact that

$$(D_{(x,v)}\pi)(\sum_i f_i(x,v)v_i(x,v)) = \sum_i f_i D_{(x,v)}(v_i(x,v)) =$$
$$= \sum_i f_i(x,v)v(x) = v(x).$$

So a convex combination of lifts is a lift. Precisely as a convex combination of extensions is an extension. □

Definition 33.2. The disc bundle of E is defined as the space

$$DE = \{(x,v) \in E \mid \|v\| \leq 1\}.$$

Lemma 33.3. The disc bundle DE is naturally a smooth manifold with boundary equal to

$$SE = \{(x,v) \in E \mid \|v\| = 1\}.$$

Furthermore, any v.f. $v\colon M \to TM$ with isolated zeros extends to an o.p.v.f. on DE with the same zero's having the same indices.

Note that when we eventually prove that the Euler number is well-defined for these manifolds then the latter part implies $\chi(DE) = \chi(M)$.

Proof. Let

$$\psi\colon \pi^{-1}(U) \cong U \times \mathbb{R}^k$$

be a local isometric trivialization (which we proved to exist in Lemma 15.11). We see that

$$\psi(DE) = U \times D^k$$

is a manifold with boundary

$$\psi(SE) = U \times S^{k-1}.$$

So locally we can use charts of these to define charts of DE as a manifold with boundary.

Let $z_1, \ldots, z_q$ be the zeros of v. Pick disjoint charts $\phi_i \colon U_i \to M$ around each $z_i = \psi_i(0)$. Fix an i and denote $\phi = \phi_i$ and $U = U_i$. Shrink this chart (around z_i) and pick isometric trivializations

$$\psi \colon \pi^{-1}(\phi(U)) \cong \phi(U) \times \mathbb{R}^k.$$

together we get coordinates on $\pi^{-1}(\phi(U)) \subset E$ by

$$(\phi^{-1} \times \mathrm{id}_{\mathbb{R}^k}) \circ \psi \colon \pi^{-1}(\phi(U)) \to U \times \mathbb{R}^k \mathring{\subset} \mathbb{R}^n \times \mathbb{R}^k.$$

The vector field v (which is only defined on M) corresponds in this chart to a map

$$w \colon U \times \{0\} \to \mathbb{R}^n \subset \mathbb{R}^{n+k}.$$

We define the extension of w (in the chart) by

$$w^e(x, b) = (w(x), b) \in T_x U \times T_b \mathbb{R}^k. \tag{33.1}$$

This is outwards pointing since $(w(x), b) \cdot (0, b) = \|b\|^2$ at the boundary of $U \times S^{k-1}$, and $(0, b)$ is the unit normal vector pointing out at (x, b) for any $(x, b) \in U \times S^{n-1}$.

Also $(w(x), b) = 0 \Rightarrow (x, b) = 0$. Indeed, w only had a 0 at 0.

The index of w at 0 (and hence v at z_i) is the degree of the map

$$g \colon S^{n-1}_\varepsilon \to S^{n-1}$$

defined by

$$g(x) = \frac{w(x)}{\|w(x)\|}.$$

Similarly the degree of w^e is the degree of the map

$$g^e \colon S^{n+k-1}_\varepsilon \to S^{n+k-1}$$

defined by

$$g^e(x, b) = \frac{(w(x), b)}{\|(w(x), b)\|}.$$

Now let

$$S = \{x_1, \ldots, x_p\} = g^{-1}(y)$$

for some regular $y \in S^{n-1}$. Then by construction we see

$$S \times \{0\} = \{(x_1, 0), \ldots, (x_p, 0)\} = (g^e)^{-1}(y, 0).$$

The differential of g^e (viewed locally as a function from $\mathbb{R}^{n+k}$ to $\mathbb{R}^{n+k}$ defined by the same formula above) at such a point is given by

$$D_{(x,0)}g^e = \left(\begin{array}{c|c} D_x g & 0 \\ \hline 0 & \frac{1}{\|w(x)\|}I \end{array}\right)$$

(where g is viewed locally as a function from $\mathbb{R}^k$ to $\mathbb{R}^k$ defined by the same formula above). Indeed:

- The first columns follows from putting $b = 0$ and differentiating with respect to x_j.
- The top left because the derivative of $\frac{w_i(x)}{\|(w(x),b)\|}$ with respect to b_j at $b = 0$ is zero. Indeed, using chain rule we get:
$$\frac{\partial}{\partial b_j}\frac{w_i(x)}{\|(w(x),b)\|} = -\frac{b_j w_i(x)}{\|(w(x),b)\|}.$$
- The lower left because the derivative of $\frac{b_i}{\|(w(x),b)\|}$ with respect to b_j is equal to $\frac{\delta_{ij}}{\|(w(x))\|}$ at $b = 0$.

The tangent spaces of the spheres at the relevant points are:

$$T_{(x,0)}S_\varepsilon^{n+k-1} = T_x S_\varepsilon^{n-1} \times \mathbb{R}^k \qquad \text{and} \qquad T_{(y,0)}S^{n+k-1} = T_y S^{n-1} \times \mathbb{R}^k$$

It follows that the difference between $D_x g\colon T_x S_\varepsilon^{n-1} \to T_y S^{n-1}$ and

$$D_{(x,0)}g^e\colon T_{(x,0)}S_\varepsilon^{n+k-1} \to T_{(y,0)}S^{n+k-1}$$

is given by multiplication in the new trivial factor $\mathbb{R}^k$ with $\frac{1}{\|w(x)\|}$. Hence we conclude that $D_{(x,0)}g^e$ is invertible as a map between the tangent spaces of the spheres (because $D_x g$ is) and it is orientation preserving if and only if $D_x g$ is. We hence conclude that $(y, 0)$ is a regular value for g^e and that

$$\deg(g^e) = \sum_{x\in S} \operatorname{sign}(D_{(x,0)}g^e) = \sum_{x\in S} \operatorname{sign}(D_x g) = \deg(g).$$

Here $\operatorname{sign}(L)$ is $+1$ or -1 depending on whether L is orientation preserving on not.

So we have a local construction of the extensions as stipulated in the lemma. It is easier to do this away from the zeros in the same way. Indeed, following the exact same procedure to construct w^e as in Equation (33.1) we see that $w^e(x, b) = 0$ does not happen because $w(e) \neq 0$. So we do not have to check any degree. Picking a partition of unity *on* M and interpolating we get a vector field v^e which satisfies:

- It is outwards pointing because each local construction was and this is preserved under point-wise convex interpolation.
- It has no other zeros because we constructed it as a lift (in the sense of the above lemma) so we see that: $v^e(x, b) = 0 \Rightarrow D_{(x,b)}\pi(v^e(x, b) = 0) \Rightarrow v(x) = 0$, and we know that $v(x) = 0$ only happens at $z_1, \ldots, z_q$ where we made sure (in our charts that no other zeros was constructed).

□

Theorem 33.4 (M.6.1 slightly more general)**.** The Euler number $\chi(M)$ is well-defined and any vector field with isolated zeros have index sum equal to it.

Proof of Theorem M.6.1. Pick any embedding of M into $\mathbb{R}^n$ and a tubular neighborhood embedding

$$\nu_M \to \mathbb{R}^n.$$

This embeds $D\nu_M$ into $\mathbb{R}^n$ and then Lemma M.6.3. proves that $\chi(D\nu_M)$ is unique and the sum of indices of o.p.v.f on it is always equal to it. Then the above lemma implies the same for M. □

Now Milnor has a 3 step canon to prove the general theorem.

- Step 1: prove that $\chi(M) = \sum_{i=0}^{n}(-1)^i \operatorname{rank}(H_i(M))$. This we will not do. However, we are happy that we actually have a definition of $\chi(M)$.
- Step 2: we already proved this.
- Step 3: we will sketch a proof of this in the following.

So assume that M has a boundary, but is still compact. Also $\pi\colon E \to M$ is still a metric vector bundle defined on M. The trouble is that technically $D\nu_M \to M$ is not a manifold with boundary. It has "corners". However, we will get around this by smoothening. However, since we do not wish to prove that all the choices of the smoothenings are essentially equivalent we prove the following convenient lemma.

The only thing we really need is to prove that any two vector fields with isolated zeros has the same index sum. Indeed, then it follows that $\chi(M)$ is well-defined and that all v.f. with isolated zero's has this as their sum. So the essential lemma is:

Lemma 33.5. For any two o.p.v.f v_1 and v_2 on M there is compact submanifold $W \subset \mathring{\nu}_M$ with boundary such that:

- v_i extends to an o.p.v.f on W with the same zero's having the same indices, for both $i = 1$ and $i = 2$.

Sketch of proof. Pick a collar neighborhood $U \subset M$ with

$$\psi\colon U \cong \partial M \times [0, 1).$$

By shrinking we can assume that the closure of U does not contain any of the zero's of either v_1 or v_2. Indeed, these cannot be on the boundary since v_i is outwards pointing.

By further shrinking we can assume that the vector field v_i satisfies

$$(v_i)_{(x,t)} \in T_x\partial M \times T_t[0, 1)$$

(here written in the collar neighborhood product) has second coordinate negative. Indeed, outwards pointing at ∂M means that this second coordinate is negative on $\partial M \times \{0\}$ hence by continuity such a shrinking exists.

Now define $h\colon \mathbb{R} \to \mathbb{R}$ by

$$h(t) = \begin{cases} e^{-\frac{1}{1/2-t}} & t \leq 1/2 \\ 0 & t \geq 1/2 \end{cases}$$

This is smooth. Define $f\colon \nu_M \to \mathbb{R}$ by

$$f(x', v) = \begin{cases} \|v\|^2 & x' \notin U \\ h(t) + \|v\|^2 & x' = (x,t) \in U \end{cases}.$$

This is thus also smooth. Moreover it has 1 as a regular value (exercise) so we define

$$W = f^{-1}([-1,1]) \subset D\nu_M$$

which by Lemma M.2.3 is a smooth manifold with boundary given by

$$\partial W = f^{-1}(1).$$

Indeed, notice that f does not take negative values. We know how to pick local extensions around points in $M - (\partial M \times [0,1/2])$. Indeed, this is what we constructed in Lemma 33.3. Now let $(x_0, t_0) \in \partial M \times [0,1)$ be given.

Pick a chart $\psi\colon \mathbb{R}^{n-1} \overset{\circ}{\supset} U \to \partial M$ with $\psi(0) = x_0$. Shrink this chart such that we can isometrically trivialize ν_M on $(\psi(U) \times (a,a')) \cap H^k$ where $t_0 \in (a,a')$. Then use this chart times the identity on (a,a') and such a trivialization to define the extensions w^e as in the above proof (for either v_1 or v_2). That, is in the chart we have

$$w^e(x,t,b) = (w(x,t), b) \tag{33.2}$$

with w corresponding to either v_1 or v_2. Claim: This is outwards pointing.

Proof of claim: In this chart W is identified with the sup-space

$$W' = \{(x,t,b) \in U \times (a,a') \times \mathbb{R}^k \mid \|b\|^2 + h(t) \leq 1\} \subset \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^k.$$

The outwards pointing normal vector at the boundary point $(x,t,b) \in \partial W'$ is given by

$$(0, h'(t), b).$$

Indeed, the normal vector in $\mathbb{R}^{n+k}$ to an equation $f(z) \leq a \in \mathbb{R}$ is the gradient of f.

We now see that the inner product for this normal vector with the vector field w^e is given by

$$w^e(x,t,b) \cdot (0, -h'(t), b) = \tfrac{\partial}{\partial t} w(x,t) h'(t) + \|b\|^2,$$

which is positive since:

- $\frac{\partial}{\partial t} w(x,t) < 0$ and $h'(t) \leq 0$,
- $\|b\| \geq 0$
- $h'(t)$ and $\|b\|$ are not zero at the same time. Indeed, $h'(t) = 0 \Rightarrow h(t) = 0$ where $\|b\| = 1$ at W' boundary points.

□

Lecture 19

34. Lefschetz Fixed Point Theory

Let M be a closed oriented manifold. Let $f\colon M \to M$ be a smooth map. Let $\Delta\colon M \to M \times M$ be the diagonal map defined by $\Delta(x) = (x, x)$. This is a smooth embedding of M into $M \times M$.

Definition 34.1. The Lefschetz number $L(f)$ is defined by

$$L(f) = I(\mathrm{id}_M \times f, \Delta(M)).$$

A function is called Lefschetz if $\mathrm{id}_M \times f$ is transversal to $\Delta(M)$.

Example 34.2. The identity $\mathrm{id}_M\colon M \to M$ is not Lefschetz - except if dimension of M is 0. Indeed, we have infinitely many fixed points, which corresponds to intersections of $\mathrm{id}_M \times f$ with $\Delta(M)$.

Lemma 34.3. Any map $f\colon M \to M$ is homotopic to a Lefschetz map.

Proof. Let $M \subset \mathbb{R}^n$ be an embedding. Extend this to a tubular neighborhood embedding $\nu_M \mathring{\subset} \mathbb{R}^n$ with projection $\pi\colon \nu_M \to M$.

Let $\varepsilon > 0$ be such that $B_\varepsilon(x) \subset \nu_M$ for all $x \in M$. Define

$$F\colon M \times \mathring{D}^n \to M \times M$$

by

$$F(x, b) = (x, \pi(f(x) + \varepsilon b)).$$

This is a submersion. Indeed, for any $v \in T_xM$ we get that

$$D_{(x,b)}F(v, 0) = (v, ?)$$
$$D_{(x,b)}F(0, w) = (0, D_{f(x)+\varepsilon b}\pi(\varepsilon w)).$$

The first shows that $D_{(x,b)}F(T_xM, 0)$ is a complement to $\{0\} \times T_xM$, and the second (plus the fact that π is a submersion) proves that $\{0\} \times T_xM$ is also in the image of $D_{(x,b)}F$. Hence, we conclude that $D_{(x,b)}F$ is surjective.

Now we pick $s \in D^n$ such that $F_s = F(-, s)$ is transversal to $\Delta(M)$. This is the identity in the first factor, and hence the second factor is a Lefschetz function with a homotopy to f given by the second factor of $F_{st}, t \in I$. □

If f is a Lefschetz function we can define the local Lefschetz number at a fixed point $x \in M$ by

$$L_x(f) = \pm 1$$

depending on the intersection sign of $\mathrm{id} \times f$ and $\Delta(M)$ at (x, x). So, we get that the sum of all the local Lefschetz numbers is the “global” Lefschetz number.

35. Morse Functions

Recall that for a function $f\colon \mathbb{R}^n \overset{\circ}{\supset} U \to \mathbb{R}^n$ we have that

$$D_x f = \nabla f$$

Does this generalize? It uses the canonical identification of $\mathbb{R}^n$ with its linear dual $(\mathbb{R}^n)^*$. This canonical identification uses the canonical basis of $\mathbb{R}^n$. However, one can notice that it in fact only depends on the standard inner product on $\mathbb{R}^n$, because the isomorphism is given by:

$$x \mapsto \langle x, - \rangle.$$

Let M be a manifold. Assume $f\colon M \to \mathbb{R}$ is a smooth function, and that M has a Riemannian structure q. Since

$$D_x f\colon T_x M \to T_{f(x)}\mathbb{R} \cong \mathbb{R}$$

is linear and the Riemannian structure provides an inner product on $T_x M$ we can find a unique vector $(\nabla f)_x$ in $T_x M$ such that

$$D_x f(w) = q_x(\nabla f, w) \qquad (= q_x^*(\nabla f)).$$

This is by the following exercise a vector field known as the gradient of f and depends *very much* on the Riemannian structure. Note that $\nabla f = 0$ precisely at critical points (which is thus independent of the Riemannian structure). Also, note that given f then $(\nabla f)_x$ only depends on q_x not all of q.

Exercise 35.1. Prove that ∇f is a smooth section in $TM \to M$.

Example 35.2. The vector fields on S^{2n+1} without zeros can not be gradients of any function in any Riemannian structure. Indeed, any real valued function on S^{2n+1} has a maximum and a minimum - hence the associated gradient has at least two zero's.

This example shows that there is a big difference in considering gradients of functions compared to general vector fields on a manifold. There are fewer gradients than vector fields, and in fact gradients have more structure. In local coordinates we know this because gradients are rotation free. However, it is not as easy to describe what this corresponds to on general manifolds (for the interested: look up closed/exact differential 1-forms).

Definition 35.3. A critical point x for $f\colon M \to \mathbb{R}$ is called non-degenerate if ∇f is transversal to the zero-section at $x \in M$.

This was the same definition as for general vector fields. So, one may think of this as restricting the old definition to the case of gradients.

Definition 35.4. The function $f\colon M \to \mathbb{R}$ is called a *Morse function* if all critical points are non-degenerate. Hence if ∇f is transversal to the zero-section.

We wish to prove that both of these definition does not depend on the Riemannian structure. However, we need a little linear algebra for this. So, assume V is a finite-dimensional real vector space. Also assume that $b\colon V \times V \to \mathbb{R}$ is a non-degenerate bilinear form. Then we get an isomorphism

$$b^*\colon V \cong V^*$$

by

$$x \mapsto b(x, -).$$

If $V = \mathbb{R}^n$ we have the standard identification of $\mathbb{R}^n$ with its dual, and we thus get an invertible matrix B associated to the isomorphism b^*. This matrix B is given by:

$$b(x, y) = y^T Bx = \langle Bx, y\rangle.$$

Indeed, the map $b(x, -) = Bx$ is a non-zero vector for all x precisely when B is invertible. We also see that the entries of B are given by $B_{ij} = b(e_i, e_j)$.

Now assume we have a smooth map $f\colon \mathbb{R}^n \overset{\circ}{\supset} U \to \mathbb{R}$. Also assume we have a Riemannian structure q on $U \overset{\circ}{\subset} \mathbb{R}^n$. Then for each $x \in U$ the bilinear form q_x on $\mathbb{R}^n$ is non-degenerate and thus as above defines an invertible matrix. This defines a smooth map $Q\colon U \to \mathrm{Gl}_n(\mathbb{R})$ (which in fact lands in symmetric positive definite matrices), and the difference between the gradient $\nabla^q f$ defined using q and the standard gradient ∇f at a point $x \in U$ is given by

- $(\nabla f)_x = I(D_x f) = D_x f$
- $(\nabla^q f)_x = q_x^*(D_x f) = Q_x(\nabla f)$.

Hence the diagram

$$\begin{array}{ccc} U \times \mathbb{R}^n & \xrightarrow{\nabla f} & U \times \mathbb{R}^n \\ & \searrow^{\nabla^q f} & \downarrow Q \quad \begin{smallmatrix}(x,b)\\ \mapsto \\ (x,Q_x b)\end{smallmatrix} \\ & & U \times \mathbb{R}^n \end{array} \qquad (35.1)$$

commutes. Notice that Q is identified with a bundle map (over the identity).

Lemma 35.5. The definition of non-degenerate critical point does not depend on the Riemannian structure. Hence neither does the definition of Morse function.

Proof. Firstly, because ∇f is transversal to the zero-section it follows that zeros are isolated. Now pick any chart $\phi\colon \mathbb{R}^n \overset{\circ}{\subset} U \to M$ around a zero $\phi(0) = x_0 \in M$. Then (as seen before) we get an induced Riemannian metric q' on U by using

$$q'\colon T_x U \times T_x U \xrightarrow{D_x\phi \times D_x\phi} T_{\phi(x)}M \times T_{\phi(x)}M \xrightarrow{q} \mathbb{R}.$$

This can also be written as

$$q'_x(z, y) = q_{\phi(x)}(D_x\phi(z), D_x\phi(y)) \qquad \text{for } z, y \in T_x U.$$

This is not equal to the standard Riemannian structure on U and we claim that the corresponding vector field to ∇f under ϕ is given by the gradient of $f \circ \phi$ using q'. Indeed, we see that

$$\begin{aligned} q'_x((D(\phi^{-1}) \circ \nabla f \circ \phi)_x, w) =& q'_x((D_{\phi(x)}(\phi^{-1})((\nabla f)_{\phi(x)}), w) = \\ =& q_{\phi(x)}((\nabla f)_{\phi(x)}, D_x\phi(w)) = \\ =& D_{\phi(x)} f(D_x \phi(w)) = D_x(f \circ \phi)(w). \end{aligned}$$

So, indeed the vector field $D(\phi^{-1}) \circ \nabla f \circ \phi$ satisfies the equation defining the gradient - hence it is the gradient. This fact should not be surprising since we are simply "transporting" *every* structure in sight from M to U using the diffeomorphism ϕ, and had the gradient changed it would have indicated that our definitions would not have been natural.

We may now check in local coordinates whether or not the definition of non-degenerate depends on the Riemannian structure. Here we use the smooth map

$$Q' \colon U \to \mathrm{Gl}_n(\mathbb{R})$$

defined as above (using q'), and the corresponding bundle map

$$Q' \colon U \times \mathbb{R}^n \to U \times \mathbb{R}^n$$

defined by $Q'(x, b) = (x, Q'_x b)$. By Equation (35.1) this takes the gradient of $f \circ \psi$ in the structure q' to the gradient defined by the usual structure. Also, it is a diffeomorphism. Indeed, its differential at $(x, 0)$ is given by

$$D_{(x,0)} Q' = \begin{pmatrix} I & 0 \\ 0 & Q'_x \end{pmatrix}.$$

We conclude that the gradient of $f \circ \psi$ is transversal when defined using q' if and only if it is so using the standard structure on U. Indeed, transversality is preserved under diffeomorphisms. $\square$

Lecture 20

36. More on Morse Functions.

Recall that we defined ∇f of a function $f\colon M \to \mathbb{R}$ on a Riemannian manifold M. Recall also that we defined what a non-degenerate critical points was (same as before considering that ∇f is a vector field). Also if a function had only non-degenerate critical points (same as ∇f being transversal to the zero section). We also proved that even though the gradient ∇f depends on the Riemannian structure the notions of non-degenerate and Morse does not.

When $M \subset \mathbb{R}^n$ is given the restriction Riemannian structure, Ie restricting the inner product on $\mathbb{R}^n$ to the tangent spaces $T_xM \subset \mathbb{R}^n$, then we have the following.

Lemma 36.1. If $f\colon \mathbb{R}^n \to \mathbb{R}$ is a smooth map then

$$(\nabla(f_{|M}))_x = \pi_x(\nabla f)_x.$$

Here $\pi_x\colon \mathbb{R}^n \to T_xM$ is the orthogonal projection.

Proof. We simply check that it satisfies the criteria for the gradient: so let $w \in T_xM$ be given then

$$\langle \pi_x(\nabla f)_x, w\rangle = \langle(\nabla f)_x, w\rangle = D_xf(w) = D_xf_{|M}(w).$$

□

Lemma 36.2. Any function $f\colon M \to \mathbb{R}$ can be homotoped (perturbed if you prefer) to a Morse function.

Proof. Assume that $M \subset \mathbb{R}^n$ is an embedding. We give M the Riemannian structure as above. When proving that any vector field could be homotoped to a transversal one we added the vector field

$$\pi_-(s)\colon\ M \to TM \tag{36.1}$$

to it for some $s \in \mathbb{R}^n$ and $\pi_x\colon \mathbb{R}^n \to T_xM$ the orthogonal projection (see proof of Lemma 31.1). This is in fact the gradient of the function $g\colon \mathbb{R}^n \to \mathbb{R}$ given by

$$g(x) = \langle x, s\rangle = x_1s_1 + x_2s_2 + \cdots + x_ns_n$$

restricted to $M \subset \mathbb{R}^n$. Indeed the gradient of this function on $\mathbb{R}^n$ is constantly equal to s and then the Lemma above states that the restricted gradient is the projection in Equation (36.1).

Since $D_x(f+g) = D_xf + D_xg$ we get that for any Riemannian structure that $\nabla(f+g) = \nabla f + \nabla g$. It follows that for almost all $s \in \mathbb{R}^n$ adding the function g to f provides a Morse function $f+g$. □

Let $0 \in U \mathring{\subset} \mathbb{R}^n$. Let H_f^0 denote the Hessian of $f\colon U \to \mathbb{R}$ at 0. Concisely we have

$$(H_f^0)_{ij} = \left[\tfrac{\partial^2}{\partial x_i \partial x_j} f\right]_{|x=0}.$$

Definition 36.3. For a non-degenerate critical point $x \in M$ of $f\colon M \to \mathbb{R}$ we define the Morse index as the number of negative eigenvalues counted with multiplicity of the Hessian $H^0_{(f\circ\phi)}$ of

$$f \circ \phi\colon U \to \mathbb{R}$$

at 0, where $\phi\colon U \to M$ is any chart around $\phi(0) = x$.

Notice that it follows from:

- the definition of non-degenerate zero of a vector field and
- the fact (proven last time) that this does not depend on Riemannian structure

that:

- non-degenerate critical point $\Leftrightarrow$ this Hessian is non-degenerate.

Indeed, for the standard Riemannian structure on U we have

$$H^0_{(f\circ\phi)} = D_0(\nabla(f \circ \phi)).$$

Lemma 36.4. The Morse index is well-defined.

Proof. To prove this we only need to consider transition functions between two different charts. Ie. as usual we consider a diffeomorphism $\phi\colon \mathbb{R}^n \mathring{\supset} U \to U' \mathring{\subset} \mathbb{R}^n$ with $\phi(0) = 0$, and we let $f\colon U \to \mathbb{R}$ be any function with $f(0)$ a critical point. This means that $(\frac{\partial}{\partial x_k} f)_{|x=0} = 0$ for all $k = 1, \dots, n$. Using this we get:

$$\begin{aligned} H^0_{f\circ\phi} &= \left[\frac{\partial^2}{\partial x_i \partial x_j}(f \circ \phi)\right]_{|x=0} = \\ &= \left[\frac{\partial}{\partial x_i}\left(\sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \cdot \frac{\partial \phi_k}{\partial x_j}\right)\right]_{x=0} = \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} \frac{\partial^2 f}{\partial x_k \partial x_l}(0) \cdot \frac{\partial \phi_k}{\partial x_j}(0) \cdot \frac{\partial \phi_l}{\partial x_i}(0) = \\ &= \left((D_0\phi)^T H^0_f (D_0\phi)\right)_{ij} \end{aligned}$$

The lemma now follows from the following linear algebra lemma. $\square$

Lemma 36.5. If H is any symmetric matrix and V is an invertible matrix of the same dimension. Then the number of

- negative eigenvalues,
- zero eigenvalues, and
- positive eigenvalues

counted with multiplicity are the same for $V^T H V$ and H respectively.

Proof. Let $v_1, \dots, v_n$ be orthogonal eigenvectors for H with eigenvalues

$$\lambda_1, \dots, \lambda_n.$$

Then we have for $x = b_1v_1 + \cdots + b_nv_n$ that

$$\langle Hx, x\rangle = \sum_{i=1}^{n} \lambda_i b_i^2.$$

It follows that the number of positive eigenvalues is equal to the maximal dimension of linear sub-spaces $W_+ \subset \mathbb{R}^n$ such that

$$\langle Hx, x\rangle > 0 \qquad \text{for all} \qquad x \in W_+ - \{0\}.$$

Indeed, putting W_+ equal to the direct some of the positive eigenspaces proves that this maximal dimension is at least the number of positive eigenvalues, and if W_+ had a greater dimension it would intersect the direct sum of the non-positive eigenspaces in a non-trivial way - hence have a non-zero vector x in it where $\langle Hx, x\rangle \leq 0$.

Similarly for the number of negative eigenvalues and the maximum possible dimension of $W_- \subset \mathbb{R}^n$ such that

$$\langle Hx, x\rangle < 0 \qquad \text{for all} \qquad x \in W_- - \{0\}.$$

Since $\langle V^THVx, x\rangle = \langle HVx, Vx\rangle$ it follows that $V^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ takes such a $W_\pm$ to a similar space $V^{-1}(W_\pm)$ for V^THV, and hence the number of positive and negative eigenvalues must both be larger for V^THV. Using the exact same argument for $H = (V^{-1})^TV^THVV^{-1}$ and V^THV we get the inequality in the other direction.

Now the number of zero eigenvalues follows from the fact that the total number of eigenvalue multiplicity must equal n. □

Example 36.6. You may have seen the following description of maxima, minima and saddle points of a function f:

- If $\nabla f = 0$ at 0 and $\det(H_f^0) \neq 0$ at x then
  - If H_f^0 has two positive eigenvalues then x is a minimum.
  - If H_f^0 has precisely one positive eigenvalue then x is a saddle point.
  - If H_f^0 has no positive eigenvalues then x is a maximum.

This is precisely possible for $\det(H_f) \neq 0$ because this makes the critical point non-degenerate, and hence if you try and move it slightly it does not change its appearance. The critical point is "stable" (defined by an open transversality condition).

The above example explains how the number of eigenvalues says something about the type of critical point. The following example generalizes this to higher dimensions, and describes different types of saddle points (with different Morse index). Furthermore, the Morse lemma (further down) proves that these are up to the choice of chart around the critical point the *only* types of non-degenerate critical points.

Example 36.7.

$$f(x) = -x_1^2 - x_2^2 - \cdots - x_m^2 + x_{m+1}^2 + \cdots x_n^2$$

Has gradient

$$(\nabla f)_x = (-2x_1, \dots, -2x_m, 2x_{m+1}, \dots, 2x_n)$$

and thus Hessian at 0 given by

$$H_f^0 = \begin{pmatrix} -2I_m & 0 \\ 0 & 2I_{n-m} \end{pmatrix}.$$

We conclude that the Morse index is m.

Observation 36.8. Let $q_t, t \in I$ be a homotopy of Riemannian structures on $U \mathring{\subset} \mathbb{R}^n$. Let $f\colon U \to \mathbb{R}^n$ be a function with non-degenerate critical point at $0 \in U$. Since the fact that $\nabla^{q_t} f$ is transversal at 0 is independent of $t \in I$ it follows that the determinant of the matrix

$$D_0(\nabla^{q_t} f)$$

is either always positive or negative. Indeed, this is the determinant we used to define the sign of a zero of a non-degenerate vector field (such as $\nabla^{q_t} f$ for fixed t).

Since the space of Riemannian structures is path-connected (convex - see Lemma 15.10) it follows that this sign does not depend on the Riemannian structure, and for the standard Riemannian structure on U it equals

$$\operatorname{sign}(\det(H_f^0)).$$

We also know that the determinant for a symmetric invertible matrix A satisfies

$$\operatorname{sign}(\det(A)) = (-1)^{\text{number of negative eigenvalues of } A}.$$

It follows that for x a non-degenerate zero for ∇f (in any Riemannian structure) we have

$$\operatorname{index}(x, \nabla f) = (-1)^{\text{Morse index of } f \text{ at } x}.$$

Example 36.9. $\mathbb{R}P^2$ is non-orientable. However we can define the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = x^2 + 2y^2 + 3z^2.$$

This is preserved under the ± 1 action, and thus defines a smooth map on $\mathbb{R}P^2$. Its restriction to S^2 has critical points given by the points were

$$\nabla f = (2x, 4y, 6z) \text{ is parallel to } (x, y, z).$$

This happens when two of the three coordinates are equal to 0. So we conclude that the function defined on $\mathbb{R}P^2$ has three critical points. By using the charts we defined in the beginning of the course we see that in a neighborhood of $x = y = 0$ we have

$$\begin{aligned} F_3(x, y) =& (f \circ \phi_3)(x, y) = f(x, y, \sqrt{1 - y^2 - x^2}) = \\ =& x^2 + 2y^2 + 3(1 - y^2 - x^2) = 3 - 2x^2 - y^2. \end{aligned}$$

Hence

$$H^0_{F_3} = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix},$$

and the Morse index is 2. Similarly we get for

$$F_2(x,z) = x^2 + 2(1 - x^2 - z^2) + 3z^2 = 2 - x^2 + z^2,$$

with Hessian

$$H^0_{F_2} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

So that the Morse index is 1. Finally,

$$F_1(y,z) = (1 - y^2 - z^2) + 2y^2 + 3z^2 = 1 + y^2 + 2z^2,$$

with Hessian

$$H^0_{F_1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

which is the global minimum with Morse index 0.

It follows that since ∇f is a transversal vector field on M (in any Riemannian structure) that

$$\begin{aligned} \chi(\mathbb{R}P^2) &= \sum_{x \text{ a crit. pt. for } f} \operatorname{index}(x, \nabla f) = \\ &= \sum_{x \text{ a crit. pt. for } f} (-1)^{\text{Morse index of } x} = 1 - 1 + 1 = 1. \end{aligned}$$

Lemma 36.10 (Lemma of Morse)**.** If $z \in M$ is a non-degenerate critical point of Morse index i for the function $f\colon M \to \mathbb{R}$. Then there exists a chart $\psi\colon \mathbb{R}^n \overset{\circ}{\supset} U \to M$ with $\psi(0) = z$ such that

$$(f \circ \psi)(x) = -x_1^2 - x_2^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2 + f(z). \tag{36.2}$$

Proof. First by picking a chart we may assume that f is defined on $U \overset{\circ}{\subset} \mathbb{R}^n$ with the critical point at 0, and the goal is to pre-compose it with a local diffeomorphism at 0 such that it takes the form in Equation (36.2). We will only care about points close to 0 so we will sometimes have to shrink U to make the statements true, but will not always mention this. Since we can always add a constant to f it does not matter what $f(z)$ is. Hence we can assume that $f(z) = 0$.

Now as we saw in the proof of Lemma M.6.2 (we gave details of this in Section 31) we can always write

$$f(x) = \sum_{i=1}^{n} x_i g_i(x),$$

for some functions $g_i\colon U \to \mathbb{R}$. By differentiating it follows that $g_i(0) = \frac{\partial f}{\partial x_i}(0) = 0$ and we can thus write each g_i similarly and get

$$\begin{aligned} f(x) &= \sum_{j=1}^{n}\sum_{i=0}^{n} x_i x_j \overline{h}_{ij}(x) = \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \frac{\overline{h}_{ij}(x) + \overline{h}_{ji}(x)}{2} = \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j h_{ij}(x). \end{aligned}$$

Here by the last part we got that $h_{ij} = h_{ji}$ as functions. It follows by differentiating twice that

$$h_{ij}(0) = 2(H_f^0)_{ij}$$

and so the matrix defined by $h_{ij}(x)$ is symmetric and invertible at 0.

Now for induction assume that we have been able to pre-compose f with a diffeomorphism (sending 0 to 0) to get it on the form:

$$f(x_1, \ldots, x_n) = \pm x_1^2 \pm \cdots \pm x_{k-1}^2 + \sum_{i,j \geq k} x_i x_j h_{ij}(x), \tag{36.3}$$

Here - by symmetrizing as above we can assume that $h_{ij}(x)$ is symmetric in i and j. By abuse of notation we have denoted the function in the new coordinates also by f - although it is obviously a new function.

Since the non-degeneracy of the Hessian is independent on charts it follows that the sub matrix $h_{ij}(0)$ for $i, j \geq k$ is invertible, and we may pre-compose with a rotation of the last $n-k$ coordinates to get that $h_{kk}(0) \neq 0$. Now define the new coordinates:

$$\phi(x) = (\phi_1(x), \ldots, \phi_n(x))$$

by

$$\begin{aligned} \phi_i(x) &= x_i \qquad \text{when} \qquad i \neq k \\ \phi_k(x) &= \sqrt{|h_{kk}(x)|}x_k \pm \sum_{i>k} \frac{x_i h_{ik}(x)}{\sqrt{|h_{kk}(x)|}}. \end{aligned}$$

The differential is an upper triangular matrix with non-zero diagonal entries, and hence this is a local diffeomorphism. The $\pm$ is the sign of $h_{kk}(0)$. We may write f in these new coordinates as

$$f(x_1, \ldots, x_n) = \pm \phi_1(x)^2 \pm \cdots \pm \phi_k(x)^2 + \sum_{i,j>k} x_i x_j h'_{ij}(x),$$

for new symmetric functions h'_{ij}. Indeed, all the terms in Equation (36.3) involving an x_k is contained in the new single term $\pm\phi_k(x)^2$. So we see that the function $f \circ \phi^{-1}$ satisfies the induction hypothesis for one larger k.

We end up when $k = n$ with a function f on the form

$$\sum_i \pm x_i^2,$$

which by the invariance of Morse index must have precisely m minuses. Hence we can pre-compose with a coordinate permutation to get the minuses in front as in Equation (36.2). □

Lecture 21

37. Classification and Euler Characteristic of Surfaces

Recall from last time that $\chi(\mathbb{R}P^2) = 1$ and that we showed this by constructing a Morse function with 1 maximum, 1 minimum, and one saddle point (automatically Morse index 1 since we are in two dimensions). Today and next time we sketch a classification of all closed surfaces up to diffeomorphism. We start by constructing a lot of surfaces and calculating their Euler characteristic.

Consider the torus

$$\Sigma_1 = T^2 = S^1 \times S^1 = \mathbb{R}^2 \times \mathbb{R}^2_{/2\pi\mathbb{Z}\times 2\pi\mathbb{Z}} \qquad \text{(group quotient)}.$$

The quotient map $q\colon \mathbb{R}^2 \times \mathbb{R}^2 \to S^1 \times S^1$ is given by

$$[s,t] = q(s,t) = (e^{is}, e^{it}) \in \mathbb{C} \times \mathbb{C},$$

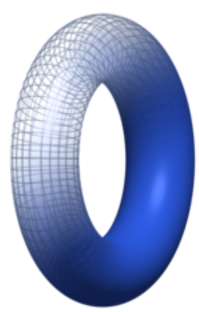
and is a local diffeomorphism. Define the function

$$f\colon \Sigma_1 \to \mathbb{R}$$

by

$$f([s,t]) = (\sin(s) + 3)\sin(t).$$

This is in fact the height of an embedding:



$\subset \mathbb{R}^3$

It has gradient:

$$(\nabla f)_{[s,t]} = (\cos(s)\sin(t), (\sin(s)+3)\cos(t)),$$

which is zero precisely at the four points $(\pm\pi/2, \pm\pi/2)$. These corresponds to the 4 horizontal points in the embedding above - ie the intersection of the torus with the vertical line of symmetry. The Hessian is given by

$$H_f = \begin{pmatrix} -\sin(s)\sin(t) & \cos(s)\cos(t) \\ \cos(s)\cos(t) & -(\sin(s)+3)\sin(t) \end{pmatrix}.$$

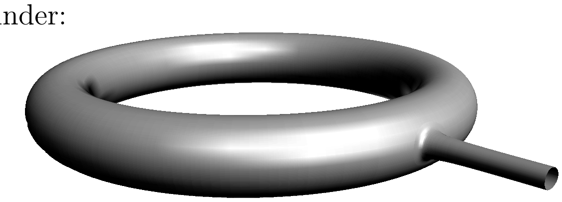
So the four Hessians in question are:

$$\begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}.$$

It follows that the critical points are non-degenerate and that the Morse indices are $0, 1, 1, 2$ hence we conclude that

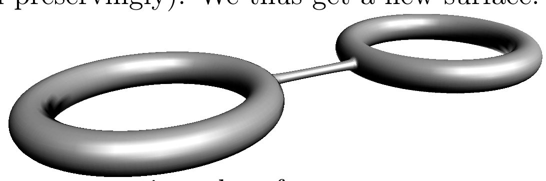
$$\chi(\Sigma_1) = 1 - 1 - 1 + 1 = 0.$$

The connect sum of the torus Σ_1 with itself is heuristically described as: remove a small disc from Σ_1 and "stretch the neck" of the disc to get a protruding cylinder:



We then take the two copies of this and identify the cylinders symmetrically (and orientation preservingly). We thus get a new surface:

$\Sigma_2 =$



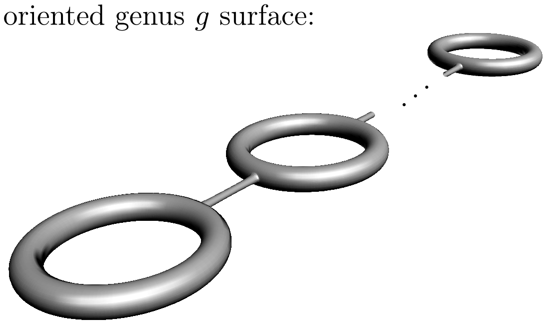
This is called the genus two oriented surface.

We may iterate the above connect sum construction and define Σ_g, the g-fold connect sum of Σ_1 with itself. Ie

$$\Sigma_g = \underbrace{\Sigma_1 \# \Sigma_1 \# \cdots \# \Sigma_1}_{g}$$

This is known as the oriented genus g surface:

$\Sigma_g =$



Embedding this standing up (as we did with the torus) we see that the critical points of the height function are now given by:

- 1 minima,
- $2g$ saddle points, and
- 1 maxima.

Indeed, we have removed the maxima and minima for each "connection cylinder". It follows that

$$\chi(\Sigma_g) = 2g - 2.$$

We will make this more precise in the following. Notice that this Formula works also for $\Sigma_0 = S^2$.

Continuing like this we may also choose to add copies of $\mathbb{R}P^2$ to the chain. This is, however, not as easily visualized as the above. However, we can generally perform the connect sum of two surfaces (in fact it generalizes rather easily to higher dimensions) M and N as follows:

Let $\psi\colon \mathring{D}^2 \to M$ be a chart and $f\colon M \to \mathbb{R}$ be a Morse function such that $(f \circ \phi)(x, y) = -x^2 - y^2$. This can always be constructed by scaling a chart as in the Morse lemma and scaling and adding constants to a Morse function.

Similarly we may let $h\colon N \to \mathbb{R}$ be a Morse function and

$$\phi\colon \mathring{D}^2 \to N$$

be such that $(h \circ \phi)(x, y) = x^2 + y^2$.

Locally in these charts we may visualize the graphs of these Morse functions as two paraboloids touching in $(0, 0, 0)$:

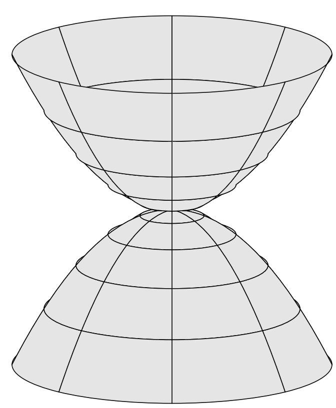
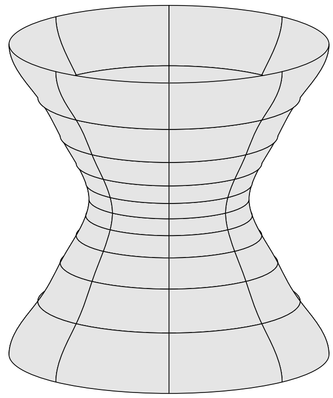


Figure 5. Paraboloids touching

These solve the equation: $x^2 + y^2 = |z|$. This is singular. However replacing the non-smooth function $|z|$ with some positive smooth function $g(z)$ (almost equal to $|z|$) equal to it away from 0 we get a smooth revolution surfaces as solution:



Defined by $x^2 + y^2 = g(z)$. Replacing the two original discs in $M \sqcup N$ with this surface defines $M \# N$, but moreover we may define

$$f \# g \colon M \# N \to \mathbb{R}$$

as f or g outside this "cylinder", and as the height function on this part. We have thus defined a new Morse function which has the same critical points as f and g except we removed a maximum and a minimum. Hence we get the formula:

$$\chi(M \# N) = \chi(M) + \chi(N) - 2.$$

The interested reader may notice that this formula also works in higher *even* dimensions.

Proposition 37.1. The Surfaces:

$$\Sigma_g, \qquad \Sigma'_g = \Sigma_g \# \mathbb{R}P^2 \qquad \text{and} \qquad \Sigma''_g = (\Sigma_g \# \mathbb{R}P^2) \# \mathbb{R}P^2$$

are all non-diffeomorphic (even for different g).

Proof. The Euler characteristic calculation above gives:

$$\begin{aligned} \chi(\Sigma_g) & & = 2g - 2 \\ \chi(\Sigma'_g) &= 2g - 2 + \chi(\mathbb{R}P^2) - 2 & = 2g - 1 \\ \chi(\Sigma''_g) &= 2g - 1 + \chi(\mathbb{R}P^2) - 2 & = 2g. \end{aligned}$$

So the only possible surfaces that could be the same are Σ_g and Σ''_{g-1}. However, since we can embed a Moebius strip in Σ''_{g-1} it is non-orientable - and hence not the same as Σ_g. Indeed, the complement of the chart used in defining the last connect sum attachment of $\mathbb{R}P^2$ is a Moebius band (one may inspect this directly for the previous constructed Morse function on $\mathbb{R}P^2$). ◻

Theorem 37.2 (Classification of Surfaces)**.** Any connected surface is diffeomorphic to either Σ_g, Σ'_g or Σ''_g for some $g \in \mathbb{N}_0$.

We will need the following lemma.

Lemma 37.3. Let $f\colon M \to \mathbb{R}$ be a smooth map. The manifold with boundary $M_a = f^{-1}((-\infty, a])$ for different $a \in (b, c)$ are diffeomorphic provided that (b, c) contains no critical values of f.

Proof. Let $a \in (b, c)$ be given as in the lemma. Then $Q = \partial M_a = f^{-1}(a)$ is a sub-manifold in M. As when proving the collar neighborhood theorem we extend the identity on this to a small neighborhood $Q \subset V \mathring{\subset} M$. This provides a map

$$\pi\colon V \to Q$$

Then we notice that the map $G = (\pi, f)\colon V \to Q \times \mathbb{R}$ has surjective (hence injective) differential at all points in $Q \subset V$. It is also injective on Q. Hence we conclude that it we may shrink V to a smaller set to make it a diffeomorphism onto its image. Indeed, $G^{-1}(Q \times (a - 1/n, a + 1/n))$ becomes smaller and smaller around Q and if G restricted to all of these are non-injective we get sequences $x_n \neq y_n$ with $G(x_n) = G(y_n)$. Now, using convergent subsequences and continuity we may assume that these limit to the same point in Q. However, since G is locally injective close to Q this is impossible.

It follows that locally $f^{-1}((a - \varepsilon, a + \varepsilon)) \cong Q \times (a - \varepsilon, a + \varepsilon)$, where the last factor is the value of f. By picking diffeomorphism of $(-\infty, a) \cong (-\infty, a + \delta)$ for $|\delta| < \varepsilon$ which is the identity for values less than $a - \varepsilon$ we get the wanted diffeomorphisms for small variations of a. However, a standard compactness argument then shows the lemma to be true for all compact intervals $[a_0, a_1] \subset (b, c)$. □

We need a few more constructions to sketch the proof of the classification theorem.

Let $\psi\colon \mathring{D}^2_\varepsilon \to M$ be a chart around a critical point $z \in M$ such that

$$f'(x, y) = (f \circ \psi)(x, y) = -x^2 + y^2 + f(z).$$

Assume also that the critical value $f(z)$ is 0 and not taken by any other critical points of f and that the values of f' are all regular except 0. Let $\phi\colon \mathbb{R} \to [0, 1]$ be the standard bump function

$$\phi(t) = e^{-1/(1-t^2)}$$

for $t \in (-1, 1)$ and $\phi(t) = 0$ when $|t| \geq 1$. This defines

$$\phi_2(x, y) = \phi(x^2 + y^2),$$

which has some global bound C on its Hessian (indeed it has support on the compact set D^2). It follows that for small $\delta > 0$ the function

$$g'(x, y) = f'(x, y) + \delta\phi_2(2\varepsilon^{-1}x, 2\varepsilon^{-1}y)$$

is Morse, with 1 critical point at $(0, 0)$ with critical value

$$g'(0, 0) = \tfrac{\delta}{e}$$

It equals f' on the complement of $D^2_{\varepsilon/2}$. So we may define a function

$$g\colon M \to \mathbb{R}$$

by defining $g(z) = f(z)$ except when $z \in \phi(D^2_{\varepsilon/2})$ where we use $g' \circ \psi^{-1}$. Using $g \geq f$ we thus see that:

$$f^{-1}((-\infty, \delta/(2e)]) = g^{-1}((-\infty, \delta/(2e)]) \cup A.$$

Here $A = f^{-1}((-\infty, \delta/(2e)]) - g^{-1}((-\infty, \delta/(2e)])$ the pictured set in Figure 6.

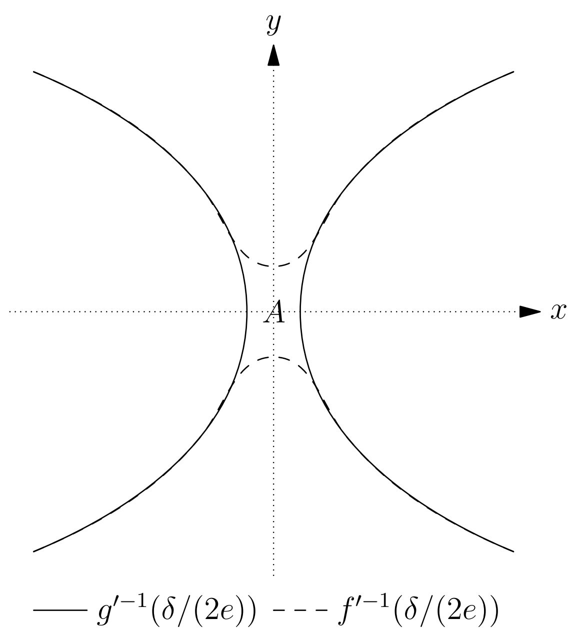


Figure 6. Handle attachment.

We may in general define: "attaching a strip" to a surface Σ with boundary as the process of selecting two distinct points $z_0, z_1 \in \partial\Sigma$ and picking two diffeomorphism

$$\phi_i\colon \mathbb{R}^2 \supset g'^{-1}(\delta/2e) \cap \{(-1)^i x < 0\} \to M, \qquad i = 0, 1$$

with disjoint images and such that $\phi_i(?, 0) = z_i$. We may thus add the strip A described above.

Notice that since

$$g^{-1}((-\infty, \delta/(2e)]) \cong g^{-1}((-\infty, \varepsilon^2/4) = f^{-1}((-\infty, \varepsilon^2/4)$$

such a strip attachment precisely describes the change of the manifold $f^{-1}((-\infty, a])$ when a passes the critical value 0. This is the main point of this construction and what we will use in the classification of surfaces.

Lecture 22

38. More on the Classification of Surfaces

In general we may consider a surface Σ with boundary $\partial\Sigma$ diffeomorphic to a disjoint number of copies of S^1. Note that one should be careful to remember that such a Σ does not come equipped with a canonical diffeomorphism of the boundary to copies of S^1. This is a subtle, but very important, fact. Indeed, this part is often what makes things different in higher dimension as we will explain below.

Last time we defined what it meant to add a strip A to Σ (see figure 6). We also argued that this is what happens to the "level-set"

$$f^{-1}((-\infty, a])$$

when a passes the value of a single critical point with Morse index 1.

Similarly to the strip attachment we can identify a neighborhood of a boundary component as $S^1 \times [0, 1)$. We can "attach a disc" by identifying this with

$$\{(x, y) \in \mathbb{R}^2 \mid 1 \leq \|(x, y)\| < 2\},$$

and simply adding the middle part. This removes a boundary component and we may inductively "cap off" all boundary components and get a closed surface.

It is important for the classification of surfaces the both types of attachments does not really depend on the choices made - except for the strip where there is an orientation issue.

Lemma 38.1. Attaching a disc as above to each boundary component defines a unique closed surface up to diffeomorphism.

Attaching a strip to a surface with boundary Σ only depends (up to diffeomorphism of the resulting surface with boundary) on which boundary components z_0 and z_1 lie - except that we can possibly get another surface by pre-composing one of the ϕ_i with the orientation reversing diffeomorphism $(x, y) \mapsto (x, -y)$. These are the only two possible diffeomorphism types for fixed z_0 and z_1.

Pictorially we may depict the other way of attaching the strip at z_0 and z_1 as in figure 7.

Sketch of proof. Assume we have a smooth family, depending on $t \in I$, of the data needed to attach a strip or a disc. Note, that a smooth family of maps is a homotopy. Then one can show that the attachments (all at once) define a smooth 3-manifold with a submersion to $\mathbb{R}$. It follows (as in the proof above, but also considering boundary) that the resulting manifolds are diffeomorphic. We omit the details of this argument.

The lemma for discs now follows by checking the following:

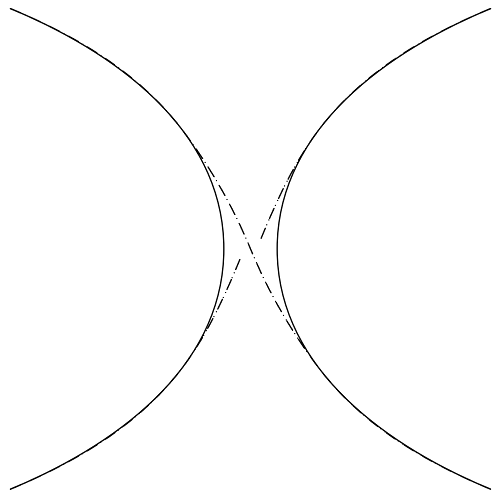


FIGURE 7. The “other” strip attachment. Here the strip is twisted as in the difference between a Moebius strip and the cylinder.

- Any orientation preserving diffeomorphism of S^1 to itself is isotopic to the identity. Note that this needs a little thinking - indeed, this goes wrong for S^6, and is why exotic spheres in dimension 7 exists.
- The orientation reversing diffeomorphism $(x, y) \mapsto (-x, y)$ constructs the same surface, and all orientation reversing diffeomorphism are homotopic to this.

The lemma for strips now follow by checking that:

- Pre-composing both with the orientation reversing diffeomorphism produces the same manifold. This is simply because the mirror image of the attachment is the same attachment.
- Any two choices of ϕ_i with the same orientation at z_0 are homotopic through charts. This can be proved using a variation on the proof we saw in Milnor's book chapter 6, which can be thought of as “locally charts are isotopic” except if they reverse the orientation.
- We may move z_0 slightly - hence as far as we want to on the same boundary component.

□

Sketch of proof of the classification theorem. Let M be a closed connected surface. Pick a Morse function $f\colon M \to (-1, 1)$. Change it slightly to have distinct critical values at its critical points (can be done as above by locally adding a function close to 0).

Now each minimum looks locally as the top part of the graph in Figure 5. So, by making it steeper close to zero, we can change f to have as small a value as we want on this local minimum - without changing the fact that f is Morse and what its critical points are.

We can similarly change the maxima to take greater values, and hence assume that all local minima takes the value $\lambda_0 = -1$, and that all local maxima takes the value 1, and that all critical points of Morse index 1 takes distinct values

$$\lambda_1, \dots, \lambda_k \in (-1, 1).$$

We now reconstruct the surface using induction: $f^{-1}((-\infty, -1+\varepsilon])$ is for small ε contained in Morse charts around the minima and hence diffeomorphic to a number of disjoint copies of D^2.

Induction assumption: the set $f^{-1}((-\infty, \lambda_i + \varepsilon])$ is (for small ε) a surface with boundary and each component is diffeomorphic to a standard surface as in Figure 8. The first type of strips (of which there are p_1) we refer to as

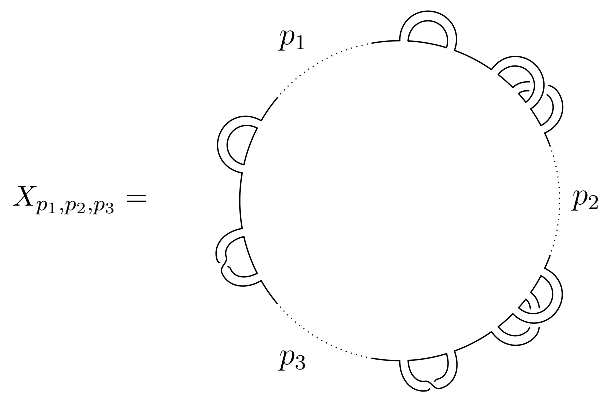


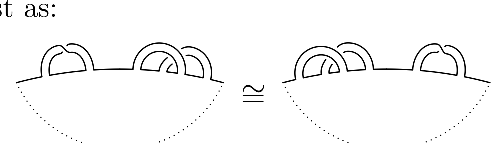
FIGURE 8. Standard surfaces

"standard strips", the second type as "pringles", and the third type as "twists". If $p_1 = p_2 = p_3 = 0$ we have none of these attachments, and $X_{0,0,0} = D^2$.

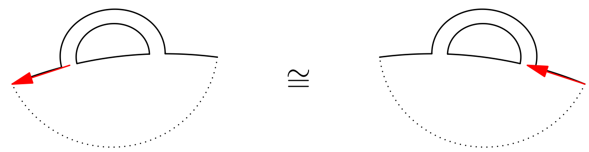
We know that $f^{-1}((-\infty, \lambda_i + \varepsilon]) \cong f^{-1}((-\infty, \lambda_{i+1} - \varepsilon])$ and passing through λ_{i+1} simply adds a strip up to diffeomorphism. So all we need to check is that adding a new strip to a surface with its components on this form is on this form. The above lemma implies the following observations:

O1: The number of boundary components of X_{p_1,p_2,p_3} is equal to $1 + p_1$. Indeed, the pringle and twist does not add to the number of boundary components. However, the standard strip does add a new small component. For the surfaces X_{p_1,p_2,p_3} we will refer to the large boundary component as the "outer boundary component".

O2: By moving the attaching points we can move the three types of strips freely past each other. We may Visualize this in the case of a pringle and a twist as:

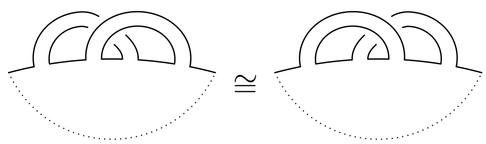


O3: We may move one of the attaching points of a standard strip all the way around the outer boundary:



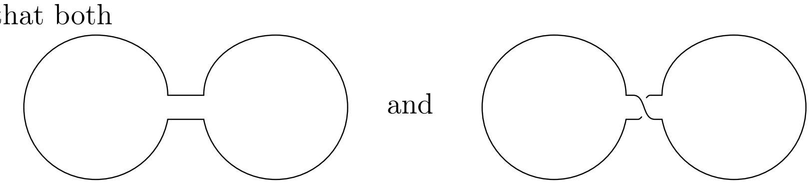
This changes the identification of the outer boundary component. Indeed, the boundary component which was the outer boundary before is now the small inside of the new standard strip.

O4: The pringle does not depend on which part is drawn on top:



We now divide the induction argument into cases:

Case 1: The strip attaches to two different components identified with X_{p_1,p_2,p_3} and $X_{p'_1,p'_2,p'_3}$ respectively. By applying move O3 above we may assume that the strip attaches to the outer component of both. By moving the attaching points around the outer components we may assume that the point of attachment is a standard choice. Now using that both

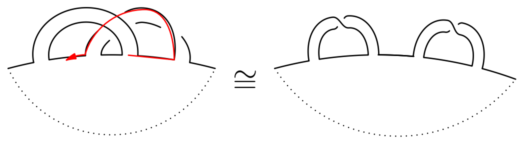


are diffeomorphic to D^2, and using O2 and O4 repeatedly we see that the new resulting component is diffeomorphic to

$$X_{p_1+p'_1,p_2+p'_2,p_3+p'_3}.$$

Case 2: The two attaching points are on the same boundary component. Here we may use O3 to move that boundary out as the outer boundary, and then move the two attaching points close to get a new standard strip or a new twist strip.

case 3: The two attaching points are on two different boundary components of the same component identified with X_{p_1,p_2,p_3}. We may assume that one of the boundary components is the outer (move that component out using O3). Now move the attaching point on the outer boundary close to the standard strip in which the other end point attaches. We end up with either a pringle or a pringle like configuration



where one of the strips is twisted. However, moving one attaching point as indicated in the figure shows that this is diffeomorphic to simply having two twists next to each other.

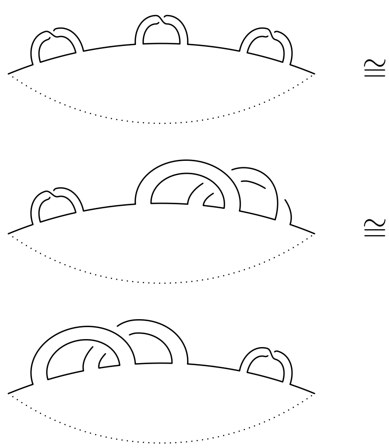
It follows that $M_{2-\varepsilon} = f^{-1}((-\infty, 2-\varepsilon])$ is of the induction assumption form. We recover all of M by attaching a disc to each boundary component of $M_{2-\varepsilon}$. Indeed, this is the precise picture we see in each Morse chart around the maxima for small ε. Since we are not connecting up different components of $M_{2-\varepsilon}$ in this process it follows that since M is connected so is $M_{2-\varepsilon}$. We conclude that

$$M_{2-\varepsilon} \cong X_{p_1,p_2,p_3}$$

for some $p_1, p_2, p_3 \geq 0$. Since attaching a disc to the inner part of a standard strip simply removes that standard strip we can assume that

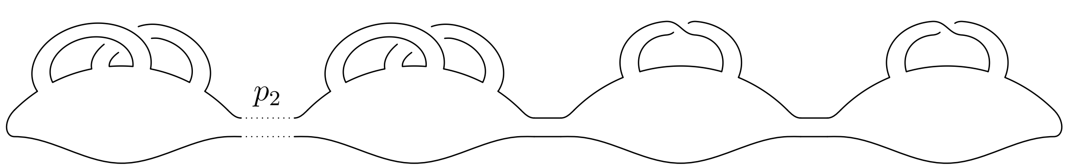
$$M_{2-\varepsilon} \cong X_{0,p_2,p_3}$$

We may also turn 3 twist into a pringle and a twist. Indeed, by moving some attaching points around one realizes that:

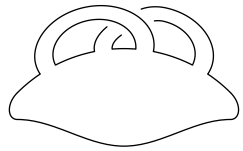


Thus we may assume that $0 \leq p_3 \leq 2$. That, is we have precisely 0,1 or 2 twists.

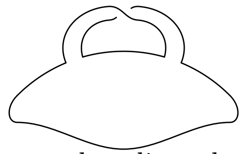
Then we pick a diffeomorphism from this standard surface to:



Here we have only pictured the case $p_3 = 2$, but the others are similar. By arguing backwards from this picture it follows that M can be identified with an iterated connect sum of the surfaces we get from

$\Sigma_1 \cong$  $\cup$ disc

and

$\mathbb{R}P^2 \cong$  $\cup$ disc

Here "$\cup$ disc" means that we attach a disc a long the single boundary component. □

Lecture 23

39. Hopf's Degree Theorem

Let M^n be an oriented closed manifold.

Theorem 39.1 (Hopf)**.** Two maps $M^n \to S^n$ are homotopic iff they have the same degree.

Before starting the proof of this we will introduce a few notions, and prove a few lemmas. Since we will start by constructing some homotopies locally we will need a compactness assumption to make sure that we can glue them together later. Indeed, let

$$f\colon N^n \to S^n$$

be a smooth map. We will define the support of f as

$$\operatorname{supp}(f) = \overline{f^{-1}(S^n - \{s_0\})}$$

where s_0 is the base-point

$$s_0 = (1, 0, \dots, 0) \in S^n.$$

We can thus make sense of the term *compactly supported* if everything outside a compact set is mapped s_0.

It will be convienient to fix some bump functions. Indeed, for the rest of this section we let $\phi_1\colon \mathbb{R}^n \to I$ be a bump such that

$$\phi_1(x) = \begin{cases} 1 & x \in D^n_{1/2} \\ 0 & x \notin D^n \end{cases}$$

Then define $\phi_\varepsilon(x) = \phi_1(\varepsilon^{-1}x)$ as a similar bump function with support in D^n_ε and equal to 1 on $D^n_{\varepsilon/2}$. We will use these bump functions to "modify" maps $f\colon \mathbb{R}^n \to S^n$. In particular we will use them to force maps to be compactly supported. However, we will describe this for more general functions than bump functions.

Let $f\colon N \to S^n$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ be maps such that: for any point $x \in N$ with $f(x) = -s_0 = (-1, 0 \dots, 0) \in S^n$ we have $g(x) = 1$. We may define a map $h\colon N \to \mathbb{R}^{n+1}$ by

$$h(x) = g(x)f(x) + (1 - g(x))s_0. \tag{39.1}$$

The point $h(x)$ is a point on the line from $f(x)$ to s_0. So by the assumptions $h(x) \neq 0$. Hence we may define $f^g = \frac{h}{\|h\|}\colon N \to S^n$ by

$$f^g(x) = \frac{h(x)}{\|h(x)\|} \tag{39.2}$$

If g maps to $[0, 1]$ then this construction "pulls" on everything mapped to $S^n - \{-s_0\}$ "down" along S^n towards the point s_0 on the opposite side (how far depends on $g(x)$). Furthermore we see that $f^{1+(1-t)g}$ is a homotopy from $f = f^1$ to f^g. Also, if either g or f is compactly supported (only one needs be) then f^g will be compactly supported. So, one can heuristically think of this as a sort of product function (ie if one is zero the product is zero - although in one case s_0 is playing the role of 0).

Lemma 39.2. If a compactly supported map $f\colon \mathbb{R}^n \to S^n$ does not hit the point $(-s_0) \in S^n$ it is homotopic through compactly supported maps to the constant map at s_0.

Proof. The construction above works for $g = 0$ since no point is mapped to $-s_0$. Hence we conclude that f^t is a compactly supported homotopy from f to the constant map. ◻

We call a compactly supported map $f\colon \mathbb{R}^n \to S^n$ a *plus map* if:

- the pre-image of $-s_0$ is a single point $x \in \mathbb{R}^n$ at which $D_x f$ is orientation preserving.

Lemma 39.3. All plus maps are homotopic through compactly supported maps.

Note that the construction in the following proof actually produces a homotopy through plus maps. However, this is slightly more difficult to argue and we wont directly need it.

Proof. Let f be a plus map. By pre-composing f with a translation homotopy $((x,t) \mapsto (x + tv))$ we may assume that the unique point sent to $-s_0$ is 0. Indeed, precomposing with a diffeomorphism on f preserves the fact that it is a plus map.

Now consider the homotopy $g_t\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ given by

$$g_t(x) = (1-t)f(x) + t(D_0 f(x) - s_0) \tag{39.3}$$

This interpolates between f and its first order approximation at 0. This is *not* compactly supported, it does does not even map to S^n - and it may even hit $0 \in \mathbb{R}^{n+1}$. However, we see that g_t is transversal at $x = 0$ to the line $\mathrm{span}_{\mathbb{R}}(-s_0)$. Indeed, its differential at 0 is given by

$$D_0(g_t) = (1-t)D_0 f + tD_0 f = D_0 f$$

which is transversal to $T_{-s_0}S^n$. We conclude that for $\varepsilon > 0$ small enough the map

$$g'_t(x) = \phi_\varepsilon(x) g_t(x) + (1 - \phi_\varepsilon(x)) s_0$$

does not hit 0 (by compactness we can make this true for all $t \in I$). Hence the normalization

$$f_t = \frac{g'_t}{\|g'_t\|}$$

is defined. This defines a homotopy from $f^{\phi_\varepsilon} = f_0$ (which is compactly supported homotopic to f by $f^{t+(1-t)\phi_\varepsilon}$) to the map

$$f_1(x) = \frac{(D_0 f(x) - s_0)\phi_\varepsilon(x) - (1 - \phi_\varepsilon(x))s_0}{\|(D_0 f(x) - s_0)\phi_\varepsilon(x) - (1 - \phi_\varepsilon(x))s_0\|}.$$

Since we may make ε as small as we want we see that the only real remnants left of f in this map is its differential at 0. Now, since all orientation preserving linear maps as $D_0 f\colon \mathbb{R}^n \to T_{-s_0}S^n \cong \mathbb{R}^n$ are homotopic through orientation preserving linear maps - the lemma follows. □

We may similarly define a *minus map* $f\colon \mathbb{R}^n \to S^n$ as the exact same thing except that $D_x f$ at the point x mapped to $-s_0$ is orientation reversing, and we similarly get the lemma:

Lemma 39.4. All minus maps are homotopic through compactly supported maps.

One of the central ideas in Hopf's degree theorem is the following lemma.

Lemma 39.5. A compactly supported map $f\colon \mathbb{R}^n \to S^n$ which has precisely two points $x, y \in \mathbb{R}^n$ mapping to $-s_0$ and such that

- $D_x f$ is orientation preserving and
- $D_y f$ is orientation reversing

is homotopic through compactly supported maps to the constant map at s_0.

Proof. We may assume that $x = (1, 0, \dots, 0)$ and $y = (-1, 0, \dots, 0)$. Indeed, we may pre-compose f with the following isotopies:

- First we translate to make $(x + y)/2$ equal to 0.
- Then we scale to make $\|x\| = 1$, which automatically makes $\|y\| = 1$.
- Then we rotate to make $x = (1, 0 \dots, 0)$, which automatically makes $y = (-1, 0, \dots, 0)$.

The latter is only generally possible when $n > 1$. However, this case can be treated in the following in a similar way (exchanging the role of y and x and some signs).

Now, consider the map $g_0\colon \mathbb{R}^n \to \mathbb{R}^n$ given by

$$g_0(x_1, \dots, x_n) = (x_1^2 - 1, x_2, \dots, x_n).$$

This precisely maps x and y to 0 as the only points. It is also a local diffeomorphism at x and y. Indeed, its differential at these points is:

$$D_x g_0 = \begin{pmatrix} 2 & 0 \\ 0 & I_{n-1} \end{pmatrix} \qquad D_y g_0 = \begin{pmatrix} -2 & 0 \\ 0 & I_{n-1} \end{pmatrix}.$$

The first preserves orientation the second reverses it. We also conclude that the function $\phi_\varepsilon \circ g_1\colon \mathbb{R}^n \to \mathbb{R}$ is 1 at x and y but zero outside a small neighborhood of x and y. It follows that

$$f^{(1-t)+t(\phi_\varepsilon \circ g_1)}$$

defines a compactly supported homotopy from f to a map f' which sends everything outside small neighborhoods of x and y to s_0. It follows that in some chart

$$\mathbb{R}^n \to \mathbb{R}^n$$

around x the map f' is a plus map, and that in some chart

$$\mathbb{R}^n \to \mathbb{R}^n$$

around y the map f' is a minus map. We will pictorially represent this as in Figure 9. It follows from Lemma 39.3 and Lemma 39.4 that all such maps

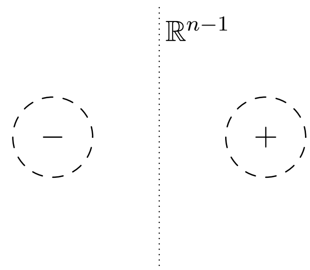


FIGURE 9. The map f' is a plus map in a chart around x and minus map in a chart around y. Outside of the small balls everything is sent to s_0

are homotopic, which is why we are content with simply filling out the small balls with a + or a −.

Now, consider the homotopy $g_t\colon \mathbb{R}^n \to \mathbb{R}^n$ given by

$$g_t(x_1, \ldots, x_n) = (x_1^2 + (2t-1), x_2, \ldots, x_n).$$

Use this to define the compactly supported homotopy $f_t\colon \mathbb{R}^n \to S^n$ given by

$$h_t(x) = \frac{(-1, g_t(x))\phi_\varepsilon(g_t(x)) + (1 - \phi_\varepsilon(g_t(x)))s_0}{\|(-1, g_t(x))\phi_\varepsilon(g_t(x)) + (1 - \phi_\varepsilon(g_t(x)))s_0\|}.$$

The latter is well-defined because $(-1, g_t(x)) = -s_0$ precisely means $g_t(x) = 0$ hence $\phi_\varepsilon(g_t(x)) = 1$.

Now g_1 has no points mapping to 0 - hence h_1 has no points mapping to $-s_0$. So, by Lemma 39.2 it is homotopic through compactly supported maps to the constant map at s_0.

The map g_0 is as above and has precisely the two points x and y mapping to 0 and hence h_1 has precisely these two points mapping to $-s_0$. One may check (exercise) that it is orientation preserving at x and orientation reversing at y. This is a map precisely as f' described in Figure 9. □

Proof of Hopf's degree theorem. Let $f\colon M \to S^n$ be any map. Let $y \in S^n$ be a regular value. By composing with a rotation we may assume that $y = -s_0$. Now let $\{x_1, \ldots, x_k\} = f^{-1}(-s_0)$, and choose disjoint charts

$$\psi_i\colon \mathbb{R}^n \to M$$

such that $\psi_i(0) = x_i$. Then define the smooth map $\Phi\colon M \to \mathbb{R}$ by

$$\Phi(z) = \begin{cases} \phi_1(\psi_i^{-1}(z)) & z \in \operatorname{im}(\psi_i) \\ 0 & \text{otherwise} \end{cases}$$

Then the homtopy

$$f^{(1-t)+t\Phi}$$

is a homotopy from f to a map f' with compact support contained in the images of the charts. Furthermore, in each chart f' is either a plus map or a minus map depending on the sign of $D_{x_i}f$. We may pictorially represent this by drawing M with a small $+$ or $-$ ball at each x_i.

Each of these balls can be moved around slightly (inside their respective charts). Indeed, by precomposing with a translation homotopy in the charts we get a homotopy moving the balls (and the point mapped to $-s_0$).

Now, assume we are given any path $\gamma\colon I \to M$ with $\gamma(0) = x_i$ and γ disjoint from all the other x_j. Then we may cover γ with charts, and move the $+$ or $-$ ball at x_i along the path (using translations in the charts successively). When changing from a chart to another to continue the "move" we may make the $+$ ot $-$ ball smaller to fit inside the next chart. We conclude that we can in fact move (using a homotopy) these $+$ or $-$ balls freely around - as long as they don't touch each other.

By moving a $+$ ball and $-$ ball into the image of some chart $\mathbb{R}^n \to M$ we can (using the chart) and Lemma 39.5 annihilate the two balls, and make the map constant in this chart.

It follows that the map f is homotopic to a map with either

- $\deg(f)$ plus balls or
- $-\deg(f)$ minus balls.

Since we can move these freely around using a homotopy we can put them in our favorit position, and it follows that the homotopy class of f only depends on the degree. Notice in particular that the ordering of the balls does not matter for the map f. □

References

[1] Victor Guillemin and Alan Pollack, *Differential topology*, AMS Chelsea Publishing, Providence, RI, 2010. Reprint of the 1974 original. MR2680546 (2011e:58001)

[2] John Milnor, *On manifolds homeomorphic to the* 7*-sphere*, Ann. of Math. (2) **64** (1956), 399–405. MR0082103 (18,498d)

[3] John W. Milnor, *Topology from the differentiable viewpoint*, Based on notes by David W. Weaver, The University Press of Virginia, Charlottesville, Va., 1965. MR0226651 (37 #2239)