1. Problem session

We will probably not go through complete solutions to all. It is, however, very heathly to convince one self of all these facts. So, try and solve all problems before hand. This way we can skip the ones (if any) everybody can solve on their own.

Exercise 1.1.

a) Show that the cartesion product $M_1 \times M_2$ of two topological manifolds M_1 and M_2 is a topological manifold.
b) Show that there is a (canonical) smooth structure on $M_1 \times M_2$ if both M_1 and M_2 has given smooth structures.
c) Show that the product $f_1 \times f_2$ is smooth if f_1 and f_2 are smooth.

Exercise 1.2.

a) Show that the charts defined in the first lecture on S^n is a smooth atlas.
b) Generalize the theorem that any continuous map $f\colon [a,b] \to \mathbb{R}$ has a maximum to: any map $f\colon S^n \to \mathbb{R}$ has a maximum.

Exercise 1.3.

Let $i\colon \mathbb{R} - \{0\} \to \mathbb{R}$ be the inclusion. Define

$$X = \mathbb{R} \sqcup_i \mathbb{R}.$$

This is homeomorphic to the space from Example 1.3.

a) Show that X does not satisfy M1.
b) Show that X satisfies M2 and M3.

Exercise 1.4.

Show that the composition of smooth functions are smooth (the new more general version of smooth).

Exercise 1.5.

Show that any homeomorphism $h\colon \mathbb{R}^n \to \mathbb{R}^n_h$ is a diffeomorphism. Here $\mathbb{R}^n$ denotes $\mathbb{R}^n$ with the standard smooth structure, and $\mathbb{R}^n_h$ denotes $\mathbb{R}^n$ with the maximal smooth atlas given by homeomorphisms $\psi\colon \mathbb{R}^n \overset{\circ}{\supset} U \to V \subset \mathbb{R}^n$ which satisfies that $h \circ \psi$ is a diffeomorphism (in the normal/classical sense).

Exercise 1.6. Let M be a (smooth) manifold. Show that for any two closed disjoint subsets $C_0 \subset M$ and $C_1 \subset M$ there exists a (smooth) map $f\colon M \to \mathbb{R}$ such that

- $f(x) = 1$ when $x \in C_1$ and
- $f(x) = 0$ when $x \in C_0$.