2. Problem session

Exercise 2.1. $S^k \subset S^n$ is a sub-manifold.

Exercise 2.2. The sub-manifold of a sub-manifold in M is a sub-manifold in M.

Exercise 2.3.

a) An injective immersion is a smooth embedding precisely when it is a homeomorphism onto its image.
b) A proper map $M \to N$ is a homeomorphism onto its image.

Exercise 2.4. A smooth embedding $M \to N$ is proper precisely when it has closed image.

Exercise 2.5. Define the diagonal map $\Delta\colon M \to M \times M$ by

$$\Delta(x) = (x, x).$$

Show that this is an embedding by using exercise 1.3.

Exercise 2.6. Prove that the minimal number of charts one needs in an atlas of S^n is precisely two (hint: try making the charts we already made slightly bigger in S^n, so that they also cover the equator - this can be done by composing with a diffeomorphism of the sphere to itself that moves the equator down a bit. If this fails look up stereo-graphic projection on Wikipedia).

Exercise 2.7. Show that $\mathbb{R}P^n = S^n / \sim$ where $x \sim y$ if $x = y$ or $x = -y$ is canonically a smooth manifold.

Exercise 2.8. For any curve $\gamma\colon \mathbb{R} \to M$ (here M is a smooth manifold) define $\gamma'(0) \in T_{\gamma(0)}M$ to be the vector $(D_0\gamma)(1)$. Prove that any vector in the tangent space T_xM can be realized as $\gamma'(0)$ where $\gamma(0) = x$.

Exercise 2.9. Prove that the union of the (x, y)-plane and the (y, z)-plane in $\mathbb{R}^3$ is not a smooth sub-manifold (hint:look at tangent spaces). Can you prove that it is not a topological manifold? (hint: probably not [unless you know what local homology groups are] - but maybe I am overlooking some clever argument)

Exercise 2.10. Prove that the union of $\mathbb{R}^n \times \{0\}$ and $\{0\} \times \mathbb{R}^n$ in $\mathbb{R}^{2n}$ is not a topological manifold.