3. Problem session

Exercise 3.1. Show that any submersion $f\colon M \to N$ is an open map.

Exercise 3.2.

a) Show that if M is closed (and non-empty) and N is connected then any submersion $f\colon M \to N$ is surjective.
b) Conclude that there are no submersions $f\colon M \to \mathbb{R}^n$ when M is closed (and $n > 0$). Describe (intuitively - meaning low on notational details) an indirect argument using a maximum of $\|f(x)\|$ for $x \in M$.

Exercise 3.3.

a) Assume that M is closed. Show that for a local diffeomorphism $f\colon M \to N$ the pre-image of a point $y \in N$ is a finite set.
b) Prove that the number of elements in this set is locally constant depending on $y \in N$.

Exercise 3.4. Let $p\colon \mathbb{R}^n \to \mathbb{R}$ be a homogeneous real polynomial. I.e.

$$p(tx_1, \dots, tx_n) = t^k p(x_1, \dots, x_n)$$

for some $k \in \mathbb{N}_0$. Prove that $p^{-1}(a)$ are diffeomorphic sub-manifolds for $a > 0$ (hint: Prove that the set $S^n \cap \{x \mid p(x) > 0\}$ is diffeomorphic to this for any $a > 0$).

Exercise 3.5. Show that $f\colon S^2 \to \mathbb{R}^4$ given by

$$f(x, y, z) = (yz, xz, xy, ax^2 + by^2 + cz^2)$$

defines a smooth embedding of the quotient $\mathbb{R}P^2 \to \mathbb{R}^4$. (You may use that the quotient map $S^2 \to \mathbb{R}P^2$ is a local diffeomorphism).

Exercise 3.6. Let $\zeta \to \mathbb{R}P^n$ be the quotient of the trivial 1 dimensional vector bundle $S^n \times \mathbb{R}$ where points (x, t) and $(-x, -t)$ are identified (with the obvious projection to $\mathbb{R}P^n$).

a) Argue that ζ is canonically a smooth vector bundle over $\mathbb{R}P^n$, and show that it is not isomorphic to the trivial 1 dimensional vector bundle. (hint: restrict the bundle to $\mathbb{R}P^1 \subset \mathbb{R}P^n$ [corresponding to the quotient of $S^1 \subset S^n$] and identify it with the Moebius bundle, which we saw in class was not trivial).

This is called the tautological line bundle on $\mathbb{R}P^n$. Line bundle is short for 1-dimensional vector bundle.

Let $\nu_{S^n} \to S^n$ denote the 1 dimensional normal bundle of $S^n \subset \mathbb{R}^{n+1}$. This is a sub-bundle of the trivial bundle $S^n \times \mathbb{R}^{n+1}$. Define $\epsilon = \nu_{S^n}/\sim$ where we identify points (x, y) with $(-x, -y)$.

b) Argue that $\epsilon \to \mathbb{R}P^n$ is canonically a smooth vector bundle over $\mathbb{R}P^n$, and show that it is isomorphic to the trivial bundle $\mathbb{R}P^n \times \mathbb{R}$.
c) Let $\epsilon^1 = \mathbb{R}P^n$ denote the trivial 1-dimensional bundle. Prove that $T\mathbb{R}P^n \oplus \epsilon^1 \cong \zeta^{\oplus(n+1)}$ as vector bundles over $\mathbb{R}P^n$ (i.e. there is an isomorphism of vector bundles over the identity on $\mathbb{R}P^n$). Hint: identify both as the same quotient of trivial bundles on S^n.