4. Problem session

Exercise 4.1. Let $\epsilon \to S^n$ be the 1-dimensional trivial bundle. Prove that $TS^n \oplus \epsilon \to S^n$ is trivial.

Exercise 4.2 (3.6)**.** Let $\zeta \to \mathbb{R}P^n$ be the quotient of the trivial 1 dimensional vector bundle $S^n \times \mathbb{R}$ where points (x,t) and $(-x,-t)$ are identified (with the obvious projection to $\mathbb{R}P^n$).

We already argued that ζ is canonically a smooth vector bundle over $\mathbb{R}P^n$, and showed that it is not isomorphic to the trivial 1 dimensional vector bundle. This is called the tautological line bundle on $\mathbb{R}P^n$. Line bundle is short for 1-dimensional vector bundle.

Let $\nu_{S^n} \to S^n$ denote the 1 dimensional normal bundle of $S^n \subset \mathbb{R}^{n+1}$. This is a sub-bundle of the trivial bundle $S^n \times \mathbb{R}^{n+1}$. Define $\epsilon = \nu_{S^n}/\sim$ where we identify points (x,y) with $(-x,-y)$.

b) Argue that $\epsilon \to \mathbb{R}P^n$ is canonically a smooth vector bundle over $\mathbb{R}P^n$, and show that it is isomorphic to the trivial bundle $\mathbb{R}P^n \times \mathbb{R}$.
c) Prove that $T\mathbb{R}P^n \oplus \epsilon \cong \zeta^{\oplus(n+1)}$ as vector bundles over $\mathbb{R}P^n$ (i.e. there is an isomorphism of vector bundles over the identity on $\mathbb{R}P^n$). Hint: identify both as the same quotient of a bundle on S^n.

Exercise 4.3 (2.8)**.** For any curve $\gamma\colon \mathbb{R} \to M$ (here M is a smooth manifold) define $\gamma'(0) \in T_{\gamma(0)}M$ to be the vector $(D_0\gamma)(1)$. Prove that any vector in the tangent space T_xM can be realized as $\gamma'(0)$ where $\gamma(0) = x$.

Exercise 4.4. Assume that $N \subset \mathbb{R}^k$ is a sub-manifold. Prove that any continuous map $f\colon M \to N$ has a smooth ϵ-approximation - in the sense that there is a map $g\colon M \to N$ such that $\|g(x) - f(x)\| \leq \epsilon$. (hint use tubular neighborhood - and look at early proof that any map to $\mathbb{R}^k$ can be approximated.)

A continuous homotopy from f to g is a continuous map $F\colon M \times I \to N$ such that F restricted to $M \times \{0\}$ and $M \times \{1\}$ is f and g respectively. Prove that the approximation can be done such that the maps f and g are homotopic. (Hint: if the construction of g is done properly it should follow easily).

Exercise 4.5.

a) Prove that for a smooth n-manifold M with boundary: the boundary ∂M is a smooth manifold of dimension $n-1$. (remember to show that the definition of boundary points depends only on a single chart).
b) Prove that the tangent bundle TM of M is defined as an n dimensional bundle over all of M (precisely the same definition as before - with our extended notion of what a smooth chart is) and that canonically $T_x\partial M$ is a sub-bundle of $TM_{|\partial M}$.

Exercise 4.6 (Colar neighborhood theorem)**.** Prove that the boundary $\partial M \subset M$ of a compact manifold w.b. has a neighborhood diffeomorphic to $\partial M \times [0,1)$. (Hint: try and construct the map from a neighborhood to $\partial M \times [0,1)$) What about non-compact (hand waving allowed)?