## 5. Problem session

**Exercise 5.1.**

a) Prove that the bundle map $\pi\colon\ M\times\mathbb{R}^k \to TM$ defined by orthogonal projections
$$\pi_x\colon\ \mathbb{R}^k \to TM$$
is a submersion.

b) Prove that for any section $v\colon\ M \to TM$ the map
$$V\colon\ M\times\mathbb{R}^k \to TM$$
given by $V(x,s) = v(x) + \pi_x(s)$ is also a submersion.

**Exercise 5.2.** Problem 5 in Milnor.

**Exercise 5.3.**

a) Prove that for any smooth map $f\colon\ \mathbb{R}^n \to \mathbb{R}^n$ with no zero's the winding number
$$W(f_{|S^{n-1}}, 0)$$
is zero.

b) Let
$$f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$
be a complex polynomial with $n > 0$. Prove that $f$ has a root.

**Exercise 5.4.** Prove that the function $f$ defined in the proof of Lemma 33.4 does, indeed, have 1 as a regular value.

**Exercise 5.5.** Problem 1 in Milnor.

**Exercise 5.6.** Problem 6 i Milnor.

**Exercise 5.7.** Prove that for closed $M$ and $N$ we have $\chi(M\times N) = \chi(M)\cdot\chi(N)$.

Hint: Prove that vector fields $v_M\colon\ M \to TM$ and $v_N\colon\ N \to TN$ has a product vector field $v_M \times v_N\colon\ M\times N \to T(M\times N)$ which is transversal to the zero-section if $v_M$ and $v_N$ are. Then prove that the sign of an intersection point is the product of the signs for the two.

**Exercise 5.8.** Problem 2 in Milnor.