2. Problem session

Exercise 2.4. A smooth embedding $f\colon M \to N$ is proper precisely when it has closed image.

This exercise anoyed me a little when solving - it felt somehow difficult to turn it the right way in my head. Here is a short and concise solution.

Proof. First we replace smooth embedding by a homeomorphism onto its image (to emphasize that this is all we need).
"$\Leftarrow$": Assume $\operatorname{im} f$ is closed and $K \subset N$ is compact.

$$\begin{aligned} \operatorname{im} f \text{ is closed } &\Rightarrow K \cap \operatorname{im} f \text{ is closed - hence compact } \Rightarrow \\ &\Rightarrow f^{-1}(K) = f^{-1}(K \cap \operatorname{im} f) \cong K \cap \operatorname{im} f \text{ is compact.} \end{aligned}$$

"$\Rightarrow$": Now assume that f is proper, and $K \subset N$ is any compact set

$$\begin{aligned} f \text{ is proper } &\Rightarrow f(f^{-1}(K)) \text{ is compact } \Rightarrow \\ &\Rightarrow K \cap \operatorname{im} f \text{ is compact} \Rightarrow \\ &\Rightarrow K \cap \operatorname{im} f \text{ is closed} \end{aligned}$$

We thus conclude that $\operatorname{im} f \subset K$ is closed in any compact K - hence $\operatorname{im} f \subset \mathring{K}$ is closed. Since N is locally compact we see that $\operatorname{im} f$ is locally closed hence closed. □

Originally when I possed this problem I Thought the solution would be nicer, and if I missed an easier solution I am sorry. However, the fact that this was more difficult than I thought made me dislike the problem.

Exercise 2.10. Prove that the union of $\mathbb{R}^n \times \{0\}$ and $\{0\} \times \mathbb{R}^n$ in $\mathbb{R}^{2n}$ is not a topological manifold.

This went very fast in class because we ran out of time. So, here is the detailed proof:

Proof. A pointed neighborhood of x is $U - \{x\}$ where U is a neighborhood of x. Let $X_n = \mathbb{R}^n \times \{0\} \cup \{0\} \times \mathbb{R}^n$.

- Any point in a 1-dimensional manifold has a pointed neighborhood with two components. Indeed, $\mathbb{R}^1 - \{0\}$ has two components.
- Any point in a n-dimensional manifold ($n > 1$) has a connected pointed neighborhood. Indeed, $\mathbb{R}^n - \{0\}$ is connected.
- X_1 is not a manifold because: any pointed neighborhood of $\{0\}$ has at least four components. Indeed, $X_1 - \{0\}$ has four components - so in the subspace toplogy $U - \{0\}$ has at least 4 components.
- $X_n, n \geq 2$ is not an n-manifold for $n > 1$ because any pointed neighborhood around $\{0\}$ has at least two components. Indeed, $X_n - \{0\}$ has two components.
- $X_n, n \geq 2$ is not a 1-manifold because:

- For any point x in a 1-manifold there is a neighborhood $V \ni x$ such that any pointed neighborhood around x *in* V has at least two components. Indeed, let $V \cong \mathbb{R}^1$ then $V - \{x\}$ has two components.
- This fact does not hold for X_n. Indeed, any neighborhood V around $\{0\} \neq x \in X_n$ has a smaller neighborhood homeomorphic to $\mathbb{R}^n$ and $\mathbb{R}^n$ minus a point is connected.

$\square$