Exercise 3.5. Show that $f\colon S^2 \to \mathbb{R}^4$ given by

$$f(x, y, z) = (yz, xz, xy, ax^2 + by^2 + cz^2)$$

for distinct $a, b, c \in \mathbb{R}$ defines a smooth embedding of the quotient $\mathbb{R}P^2 \to \mathbb{R}^4$. (You may use that the quotient map $S^2 \to \mathbb{R}P^2$ is a local diffeomorphism).

Proof. Let $\pi\colon S^2 \to \mathbb{R}P^2$ be the quotient. There are Several things we need to check for f.

- Well-defined: We check that f is defined on the quotient

$$\begin{aligned} f(x, y, z) &= (yz, xz, xy, ax^2 + by^2 + cz^2) = \\ &= ((-y)(-z), (-x)(-z), (-x)(-y), a(-x)^2 + b(-y)^2 + c(-z)^2) = \\ &= f(-x, -y, -z). \end{aligned}$$

  Call $\tilde{f}$ the map defined on the quotient.
- Smooth: The restriction π_i^+ of the projection π to $U_i^+ = \{(x_1, x_2, x_3) \in S^2 \mid x_i > 0\}$ is a diffiomorphism onto its image, and these cover $\mathbb{R}P^2$. So the compositions $f \circ (\pi_i^+)^{-1}$ which equals the restritions of $\tilde{f}$ are smooth.
- Injective: Recall that

$$\begin{aligned} \mathbb{R}P^2 &= S^2/(x \sim -x) = \\ &= (\mathbb{R}^3 - \{0\})/((x, y, z) \sim (tx, ty, tz)) \qquad \text{(for all } t \in \mathbb{R}) \end{aligned}$$

  So reconstructing a point in $\mathbb{R}P^2$ means reconstructing $(x, y, z) \in S^2$ up to scaling by a real number $t \in \mathbb{R}$. Denote the coordinates of f by $f(x, y, z) = (f_1, f_2, f_3, f_4)$.

  If all f_1, f_2 and f_3 are non-zero we reconstruct $[x, y, z]$ as

$$[x, y, z] = [1, f_1/f_2, f_1/f_3] = [1, y/x, z/x]$$

  If precisely one of f_1, f_2 and f_3 is zero we get a contradiction because one of x, y and z has to be zero which implies that at least two of f_1, f_2 and f_3 is zero.

  If precisely two of f_1, f_2 and f_3 are zero, say assume by symmetry f_2 and f_3, then we see that $x = 0$, $y \neq 0$, $z \neq 0$ and that we can reconstruct both y^2 and z^2 uniquely by the linear equations:

$$\begin{aligned} y^2 + z^2 &= 1 \\ by^2 + cz^2 &= f_4 \end{aligned}$$

  Indeed, the determinant of the coefficient matrix is non-zero because $b \neq c$. Now

$$[0, y, z] = [0, 1, z^2/yz] = [0, 1, z^2/f_1].$$

If all $f_1 = f_2 = f_3 = 0$ we get that precisely two of x, y, z is non-zero. So (x, y, z) is either $(0,0,1), (0,1,0)$ or $(1,0,0)$, and which is determined by whether f_4 is a, b or c.

- f (and hence $\tilde{f}$) is an immersion: This is equivalent to the differential being injective. Let $(x, y, z) \in S^2$ be given. If we think of f as defined on all of $\mathbb{R}^3$ we see that
$$D_{(x,y,z)}f = \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \\ 2ax & 2by & 2cz \end{pmatrix}$$
This has rank 3 when $xyz \neq 0$ (due to the determinant of the top 3 rows). So in this case it is injective, and thus also injective when restricted to $T_{(x,y,z)}S^2$. By symmetry we thus assume that $x = 0$, then the kernel of the above matrix is given by vectors (v_1, v_2, v_3) such that $v_1 = 0$ and
$$zv_2 + yv_3 = 0 \qquad \text{and} \qquad 2byv_2 + 2czv_3 = 0. \tag{3.1}$$
In addition the tangent space is given by those vectors which are orthogonal to the point in question $(0, y, z)$ - that is $yv_2 + zv_3 = 0$. So we se that the kernel intersected with the tangent space is the kernel of the matrix
$$\begin{pmatrix} z & y \\ 2by & 2cz \\ y & z \end{pmatrix}$$
The first and last row are linearly independent unless $z = \pm y$, but then the last two rows are linearly independent (unless $z = y = 0$, which is not the case since $y^2 + z^2 = 1$). So the kernel restricted to the tangent space is 0.
- $\tilde{f}$ is an embedding. It is an injective immersion from a closed manifold. We have seen before that this means it is an embedding.

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