THE GROUP OF IMMERSIONS OF HOMOTOPY $(4k-1)$-SPHERES

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Abstract. Regular homotopy classes of immersions of homotopy n-spheres into $(n+q)$-space form an Abelian group under connected summation. The subgroup of immersions of those homotopy spheres which bound parallelizable manifolds is explicitly calculated for codimension two immersions of $(4k-1)$-dimensional homotopy spheres. This group is proved to determine the corresponding groups in all codimensions. The results show in particular that in codimensions smaller than $2k+1$, the group of immersed homotopy $(4k-1)$-spheres is *not* the direct sum of the group of immersed standard spheres and the group of homotopy spheres.

1. Introduction

Let f and g be immersions of two oriented n-manifolds M and N, respectively, into $(n+q)$-space. Kervaire [12] defines the *connected sum* $f \sharp g$ which is an immersion of $M \sharp N$ into $(n+q)$-space. The immersion $f \sharp g$ is well-defined up to regular homotopy (a regular homotopy is a continuous 1-parameter family of immersions) and its regular homotopy class depends only on the regular homotopy classes of f and g.

Since homotopy n-spheres form a group θ_n under connected sum, the set of regular homotopy classes of immersions of all homotopy n-spheres into $\mathbb{R}^{n+q}$ form a group $\mathbf{I}\theta_n^q$ in a natural way. The identity element is the class of the standard embedding of the standard sphere, addition is connected sum of immersions, and the existence of an inverse operation follows from the Smale-Hirsch h-principle, see [5]. In this paper we consider the cases $n = 4k-1$.

Homotopy $(4k-1)$-spheres which bound parallelizable $4k$-manifolds form a subgroup $bP_{4k} \subset \theta_{4k-1}$. Let $\mathbf{I}b^q_{4k-1} \subset \mathbf{I}\theta^q_{4k-1}$ denote the corresponding group of immersions. Also, let $\mathbf{I}^q_{4k-1} \subset \mathbf{I}b^q_{4k-1}$ denote the group of immersions of the standard $(4k-1)$-sphere. The connected sum of two embeddings is regularly homotopic to an embedding in codimensions at least 2. For $q \geq 2$, let $\mathbf{E}\theta^q_{4k-1} \subset \mathbf{I}\theta^q_{4k-1}$, $\mathbf{E}b^q_{4k-1} \subset \mathbf{I}b^q_{4k-1}$, and $\mathbf{E}^q_{4k-1} \subset \mathbf{I}^q_{4k-1}$ denote the subgroups of regular homotopy classes representable by embeddings.

Smale's work [20] and the additivity of the Smale invariant (which classifies immersions of standard spheres up to regular homotopy) under connected sum [12] imply that $\mathbf{I}^q_{4k-1} \approx \pi_{4k-1}(V_{4k+q-1,4k-1})$, where $V_{4k+q-1,4k-1}$ is the Stiefel manifold of $(4k-1)$-frames in $(4k+q-1)$-space. Kervaire [11] proved that $\mathbf{E}^q_{4k-1} = 0$ for $q \geq 2k+1$ and Haefliger [6] showed that $\mathbf{E}^q_{4k-1} \approx \mathbb{Z}$, for $2 \leq q \leq 2k$.

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Hughes and Melvin [9] determined the index of $\mathbf{E}^2_{4k-1}$ in $\mathbf{I}^2_{4k-1}$ by expressing the Smale invariant of a codimension two embedding of the standard $(4k-1)$-sphere in terms of the signature of a Seifert-surface of the embedding. In [4], the authors generalized this showing how the singularities of a generic map of an oriented $4k$-manifold with spherical boundary into $\mathbb{R}^{6k-1}$ or $\mathbb{R}^{6k}$, which is an immersion in a neighborhood of its boundary and maps the boundary $(4k-1)$-sphere into $\mathbb{R}^{4k+1}$, determines the Smale invariant of the immersed boundary. The same method applies more generally to immersions of any homotopy $(4k-1)$-sphere which bounds a parallelizable $4k$-manifold (see Lemma 2.4 and Remark 2.6). Combining it with classical theorems by Kervaire and Milnor [13] and [14], we are able to determine some of the immersion groups mentioned above. As we shall see, most of their properties are determined by the group of immersions in codimension two where the situation is the following.

Theorem 1.1. *For $k \geq 2$,*

(a) $\mathbf{I}b^2_{4k-1} \approx \mathbb{Z} \oplus \mathbb{Z}_r$, *where $r = 2^{s(k)}$, and $2^{s(k)-3}$ is the highest power of 2 which divides k if $k \geq 3$ and $s(k) = 2$ if $k = 2$.*

(b) $\mathbf{E}\theta^2_{4k-1} = \mathbf{E}b^2_{4k-1} = \mathbb{Z}$ *and* $\mathbf{E}^2_{4k-1} \subset \mathbf{E}b^2_{4k-1}$ *is a subgroup of index $n(k)$, where* $n(k) = 2^{2k-2}(2^{2k-1}-1)\,\mathrm{Numerator}\left(\frac{4B_k}{k}\right)$ *and B_k is the k^{th} Bernoulli number.*

Theorem 1.1 is proved in Subsection 4.A. Kervaire and Milnor [14] prove that bP_{4k} is cyclic of order $n(k)$. This together with Theorem 1.1 implies that $\mathbf{I}b^2_{4k-1}$ is *not* isomorphic to the direct sum $\mathbf{I}^2_{4k-1} \oplus bP_{4k}$. (Recall $\mathbf{I}^2_{4k-1} \approx \mathbb{Z}$.)

In codimension one the group of immersions $S^{4k-1} \to \mathbb{R}^{4k}$ satisfies $\mathbf{I}^1_{4k-1} = \mathbf{I}^2_{4k-1} \oplus \mathbb{Z}$, and the regular homotopy classes representable by embeddings no longer form a group. Let $\mathbf{I}\theta^1_{4k-1} \to \mathbf{I}\theta^2_{4k-1}$ be the homomorphism induced by the inclusion $\mathbb{R}^{4k} \to \mathbb{R}^{4k+1}$.

Theorem 1.2. *For $k \geq 2$, only the standard $(4k-1)$-sphere embeds in $\mathbb{R}^{4k}$ and there are exactly two different regular homotopy classes containing embeddings, the standard embedding s and the orientation reversed standard embedding $\hat{s}$. If $\langle[\hat{s}]\rangle \subset \mathbf{I}\theta^1_{4k-1}$ denotes the infinite cyclic subgroup generated by the class of $\hat{s}$ then*

$$\begin{aligned}
&\langle[\hat{s}]\rangle = \mathrm{Ker}(\mathbf{I}\theta^1_{4k-1} \to \mathbf{I}\theta^2_{4k-1}),\\
&\mathbf{I}\theta^1_{4k-1} \approx \mathbf{I}\theta^2_{4k-1} \oplus \langle[\hat{s}]\rangle,\\
&\mathbf{I}b^1_{4k-1} \approx \mathbf{I}b^2_{4k-1} \oplus \langle[\hat{s}]\rangle.
\end{aligned}$$

Theorem 1.2 is proved in Subsection 4.B.

To formulate the results in "medium-high" codimensions, let $G(k,q)$, for $q \geq 2$, denote the group which is the direct complement of $\mathbf{I}^2_{4k-1} \approx \pi_{4k-1}(V_{4k+1,4k-1}) \approx \mathbb{Z}$ in $\mathbf{I}^q_{4k-1} \approx \pi_{4k-1}(V_{4k+q-1,4k-1}) \approx \mathbb{Z} \oplus G(k,q)$ (see Subsection 2.C).

Theorem 1.3. *For $3 \leq q \leq 2k$,*

(a) $\mathbf{I}\theta^q_{4k-1} \approx \mathbf{I}\theta^2_{4k-1} \oplus G(k,q)$ *and* $\mathbf{I}b^q_{4k-1} \approx \mathbf{I}b^2_{4k-1} \oplus G(k,q)$,

(b) $\mathbf{E}b^q_{4k-1}$ *is infinite cyclic, and* $\mathbf{E}^q_{4k-1} \subset \mathbf{E}b^q_{4k-1}$ *is a subgroup of index $n(k)$.*

Theorem 1.3 is proved in Subsection 4.C.

In high codimensions the immersion groups of homotopy spheres are direct sums of the corresponding immersion groups of the standard sphere and the group of homotopy spheres itself.

Theorem 1.4. *For $q \geq 2k+1$, $\mathbf{I}\theta^q_{4k-1} \approx \theta_{4k-1} \oplus \mathbf{I}^q_{4k-1}$, $\mathbf{I}b^q_{4k-1} \approx bP_{4k} \oplus \mathbf{I}^q_{4k-1}$, and $(\mathbf{E}\theta^q_{4k-1}, \mathbf{E}b^q_{4k-1}) \approx (\theta_{4k-1}, bP_{4k})$.*

Theorem 1.4 is proved in Subsection 4.D. The inclusion $\mathbb{R}^{4k+1} \to \mathbb{R}^{4k-1+q}$, $q > 2$ induces a homomorphism $\mathbf{I}\theta^2_{4k-1} \to \mathbf{I}\theta^q_{4k-1}$. As a consequence of Hirsch lemma, [8] Theorem 6.4, the image of this homomorphism can be identified with the subgroup of codimension q framed immersions. For $3 \leq q \leq 2k+1$ this subgroup is determined in Theorem 1.3.

Theorem 1.5. *For $q \geq 2k+1$, $\mathrm{Im}(\mathbf{I}\theta^2_{4k-1} \to \mathbf{I}\theta^q_{4k-1}) \approx \theta_{4k-1} \oplus \mathrm{Im}(\mathbf{I}^2_{4k-1} \to \mathbf{I}^q_{4k-1})$, and $\mathrm{Im}(\mathbf{I}b^2_{4k-1} \to \mathbf{I}b^q_{4k-1}) \approx bP_{4k} \oplus \mathrm{Im}(\mathbf{I}^2_{4k-1} \to \mathbf{I}^q_{4k-1})$.*

Theorem 1.5 is proved in Subsection 4.D. We note that for $q \geq 4k$, Theorem 1.5 is rather trivial since $\mathbf{I}^q_{4k-1} = 0$ and $\mathbf{I}\theta^q_{4k-1} \approx \theta_{4k-1}$. The first case of interest is thus $q = 4k-1$, where $\mathrm{Im}(\mathbf{I}^2_{4k-1} \to \mathbf{I}^{4k-1}_{4k-1}) = 0$ if $k \geq 3$, but $\mathrm{Im}(\mathbf{I}^2_7 \to \mathbf{I}^7_7) = \mathbb{Z}_2$ (since there exists an element of Hopf-invariant one in $\pi^s(7)$, which is representable by an immersion into $S^7 \to \mathbb{R}^{14}$ with a single double point).

2. Geometric regular homotopy invariants

In this section it is shown that any codimension two immersion of a homotopy sphere which bounds a parallelizable manifold is regularly homotopic to a connected sum of an immersed standard sphere and an embedding. Also, a geometrically defined regular homotopy invariant is discussed.

2.A. Generating $\mathbf{I}b^2_{4k-1}$.

Lemma 2.1. *If $f\colon S^{4k-1} \to \mathbb{R}^{4k+1}$ represents $[f] \in \mathbf{I}^2_{4k-1}$ and $r\colon S^{4k-1} \to S^{4k-1}$ is an orientation reversing diffeomorphism then $f \circ r$ represents $-[f]$.*

Proof. This follows in a straightforward manner from the definition of the Smale invariant, see Lemma 3.4.2 in [2]. □

Remark 2.2. The corresponding statement for $S^n \to \mathbb{R}^{n+2}$ is true except in the first non-trivial case $S^2 \to \mathbb{R}^4$, where it is false and the inverse is given by composing on the right with a hyperplane reflection.

Lemma 2.3. *Let $f\colon \Sigma^{4k-1} \to \mathbb{R}^{4k+1}$ be an immersion of a homotopy sphere which bounds a parallelizable manifold. Then f is regularly homotopic to a connected sum of an embedding $e\colon \Sigma^{4k-1} \to \mathbb{R}^{4k+1}$ and an immersion $g\colon S^{4k-1} \to \mathbb{R}^{4k+1}$ of the standard sphere.*

Proof. Let M_+ be a parallelizable $4k$-manifold with $\partial M_+ = \Sigma^{4k-1}$. By handle theory, we may assume that M_+ is $(2k-1)$-connected, and that it is a regular neighborhood of a $2k$-dimensional cell complex. Thus, there exists an embedding $E_+\colon M_+ \to \mathbb{R}^{4k+1}$. Let M_- be M_+ with the opposite orientation and let Σ^{4k-1}_- denote the boundary of M_-. Then E_+ gives an embedding $E_-\colon M_- \to \mathbb{R}^{4k+1}$. Let $e_+ = E_+|\Sigma^{4k-1}$ and $e_- = E_-|\Sigma^{4k-1}_-$. Let $M_+ \sharp M_-$ denote the boundary connect sum. Then $M_+ \sharp M_-$ is a manifold with boundary $\Sigma^{4k-1} \sharp \Sigma^{4k-1}_- \approx S^{4k-1}$. By Novikov additivity of signature, $\sigma(M_+ \sharp M_-) = 0$. Therefore $e_+ \sharp e_-$ is an immersion of the standard sphere which is regularly homotopic to the standard embedding (the latter statement follows from the Hughes and Melvin formula for the Smale invariant mentioned in Section 1).

Let e and $\hat{e}$ be the embeddings which are the connected sums of $n(k)-1$ copies of the embeddings e_+ and e_-, respectively. Since the order of the group bP_{4k} is $n(k)$, the connected sum of the immersions f and e, $g = f \sharp e$, is an immersion of the standard sphere and if $\simeq$ denotes regular homotopy we have

$$f \simeq f \sharp e \sharp \hat{e} \simeq g \sharp \hat{e},$$

where $\hat{e}$ is an embedding of the connected sum of $n(k)-1$ copies of the inverse of Σ_-^{4k-1} in bP_{4k}, which is diffeomorphic to Σ^{4k-1}. $\square$

2.B. **Geometric formulas.** The Smale invariant is an isomorphism $\Omega\colon \mathbf{I}^2_{4k-1} \to \pi_{4k-1}(V_{4k+1,4k-1}) \approx \mathbb{Z}$. A geometric formula for Ω was given in [4]. This formula applies, as we shall see, also to immersed homotopy spheres bounding parallelizable manifolds. To state the formula we need to discuss signatures and generic maps.

Let M be a closed oriented $4k$-manifold, let $[M] \in H_{4k}(M;\mathbb{Z})$ denote its orientation class, and let $\sigma(M)$ denote its signature. If M is almost parallelizable then all its Pontryagin classes, except possibly the top one $p_k \in H^{4k}(M;\mathbb{Z})$, vanish. The signature theorem implies that there exists a constant c_{4k} such that $\langle p_k, [M]\rangle = c_{4k}\sigma(M)$ for any almost parallelizable $4k$-manifold M.

A map of a $4k$-manifold into $\mathbb{R}^{6k}$ is *generic* if it is stable. (Note that we are in the nice dimensions where stable maps are dense in the space of all maps, see [15].) Let M be an oriented $4k$-manifold with boundary ∂M which is a homotopy sphere. Let $g\colon (M,\partial M) \to (\mathbb{R}^{6k}_+, \mathbb{R}^{6k-1})$ be a generic map which is an immersion in a neighborhood of the boundary. We associate three numbers with g.

First, g has no multiple points of multiplicity larger than 3 and the triple points of g form an oriented (since the codimension is even) 0-dimensional submanifold of $\mathbb{R}^{6k}$. Let $t(g)$ denote the algebraic number of triple points of g.

Second, the rank of the differential of g satisfies $\mathrm{Rank}(dg_x) \geq 4k-1$ at every $x \in M$. The set of points $y \in M$ where $\mathrm{Rank}(dg_y) = 4k-1$ is a $(2k-1)$-dimensional submanifold $\tilde{S} \subset M$ and $g(\tilde{S})$ is an embedded submanifold of $\mathbb{R}^{6k}_+$, disjoint from $\mathbb{R}^{6k-1}$. If $\tilde{D} \subset M$ denotes the set of points $z \in M$ such that $g^{-1}(g(z))$ contains at least two points then $\tilde{D} \cup \tilde{S}$ is an immersed submanifold (with boundary) of M with double points at preimages of triple points of g. The image $D = g(\tilde{D} \cup \tilde{S})$ is an immersed submanifold with boundary of $\mathbb{R}^{6k}_+$ and S is the part of ∂D which is disjoint from $\mathbb{R}^{6k-1}$. Let n denote the outward normal vector field of S in $TD \subset T_D\mathbb{R}^{6k}$. The orientation of M and the standard orientation in $\mathbb{R}^{6k}$ induces an orientation on D. This orientation, in its turn, induces an orientation on S. Let S' denote a copy of S shifted slightly along n. Note that $S' \cap g(M) = \emptyset$. Define

$$l(g) = \mathrm{lk}(g(M), \Sigma'),$$

where lk denotes linking number in $(\mathbb{R}^{6k}_+, \mathbb{R}^{6k-1})$.

Third, the restriction ∂g of g to ∂M is a generic (self-transverse) immersion. The set of points $y \in \partial M$ where $\partial g^{-1}(\partial g(y))$ contains at least two points is a $(2k-1)$-submanifold $\tilde{\Delta}$ of ∂M and $\Delta = \partial g(\tilde{\Delta}) \subset \mathbb{R}^{6k-1}$ is a submanifold with induced orientation. The normal bundle of ∂g has dimension $2k$ and therefore admits a nonzero section v over $\tilde{\Delta}$. Let E_0 denote the total space of the bundle of nonzero vectors in the normal bundle of ∂g. Since ∂M is a homotopy sphere, $H_{2k-1}(E_0;\mathbb{Z}) = \mathbb{Z}$. Let $[v] \in \mathbb{Z}$ be the homology class of $v(\tilde{\Delta})$ in E_0. For $p \in \Delta$, define $w(p) = v(p_1) + v(p_2)$, where $\partial g(p_1) = \partial g(p_2) = p$. Then w is a normal vector field of Δ in $\mathbb{R}^{6k-1}$. Let Δ'_v be a copy of Δ shifted slightly along w. Then $\Delta'_v \cap \partial g(\partial M) = \emptyset$. Define

$$L(\partial g) = \mathrm{lk}(\Delta'_v, \partial g(\partial M)) - [v],$$

where the linking number lk is computed in $\mathbb{R}^{6k-1}$. Then L is independent of the choice of v, see [3].

Lemma 2.4. *Let $f\colon \Sigma^{4k-1} \to \mathbb{R}^{4k+1}$ be an immersion of a homotopy sphere, let $\iota\colon \mathbb{R}^{4k+1} \to \mathbb{R}^{6k-1}$ denote the inclusion, and let $a_k = 2$ if k is odd and $a_k = 1$ if k is even.*

(a) *Assume that Σ^{4k-1} bounds a compact parallelizable $4k$-manifold M. If $g\colon (M, \partial M) \to (\mathbb{R}^{6k}_+, \mathbb{R}^{6k-1})$ is any generic map which is an immersion on some neighborhood of the boundary and such that the restriction ∂g of g to the boundary is regularly homotopic to $\iota \circ f$ define*

$$\Omega(f) = \frac{1}{a_k(2k-1)!}\Big(c_{4k}\sigma(M) + 3t(g) - 3l(g) + L(\partial g)\Big). \tag{2.1}$$

Then $\Omega(f)$ depends only on the class $[f] \in \mathbf{I}b^2_{4k-1}$ of f and induces a homomorphism $\Omega\colon \mathbf{I}b^2_{4k-1} \to \mathbb{Z}$ which agrees with the Smale invariant when restricted to $\mathbf{I}^2_{4k-1}$.

(b) *Let $t(k)$ denote the order of the group of all homotopy $(4k-1)$-spheres θ_{4k-1}. Then the connected sum $t(k)\cdot f$ of $t(k)$ copies of f is an immersion of the standard sphere. Define*

$$\Omega([f]) = \frac{1}{t(k)}(t(k)\cdot f) \tag{2.2}$$

then $\Omega\colon \mathbf{I}\theta^2_{4k-1} \to \mathbb{Q}$ is a homomorphism agreeing with Ω as defined in (a) *when restricted to $\mathbf{I}b^2_{4k-1}$.*

Proof. Theorem 1.1 in [4] and Remark 6.15 there imply (a), except for the fact that Ω is integer-valued. This follows from Lemma 2.3 which implies that $\mathbf{I}b^2_{4k-1}$ is generated by its subgroups $\mathbf{I}^2_{4k-1}$ and $\mathbf{E}b^2_{4k-1}$ on which Ω is integer-valued by Theorem 1.1 in [4] and Kervaire and Milnor [13], respectively. The statement (b) is immediate. □

Let M be an oriented parallelizable manifold with boundary Σ^{4k-1}, a homotopy sphere. Let M_- be M with the opposite orientation and oriented boundary Σ^{4k-1}_-. Let $i\colon \Sigma^{4k-1}_- \to \Sigma^{4k-1}$ be the map induced by the identity map.

Corollary 2.5. *If $f\colon \Sigma^{4k-1} \to \mathbb{R}^{4k+1}$ represents an element $[f] \in \mathbf{I}b^2_{4k-1}$ then $f \circ i\colon \Sigma^{4k-1}_- \to \mathbb{R}^{4k+1}$ represents $-[f]$.*

Proof. The connected sum $f \sharp (f \circ i)$ is an immersion of the standard sphere. Let $g\colon M \to \mathbb{R}^{6k}_+$ be a map as in Lemma 2.4 (a) which is regularly homotopic to $\iota \circ f \sharp (f \circ i)$ on the boundary. Changing the orientation of M changes the sign of its signature, changes the sign of $t(g)$, preserves the orientations of S and Δ, and changes the orientations of $g(M)$ and $\partial g(\partial M)$. Hence, by Lemma 2.4, $f \sharp (f \circ i)$ is regularly homotopic to the standard embedding of the standard sphere. □

Remark 2.6. The left hand side of (2.1) can also be expressed in terms of singularities of maps into $\mathbb{R}^{6k-1}$ as follows (see [4], Theorem 1.1). If $g\colon M \to \mathbb{R}^{6k-1}$ is a generic map then the rank of the differential of f satisfies $\mathrm{Rank}(df_x) \geq 4k-1$ at every $x \in M$. The set of points $y \in M$ where $\mathrm{Rank}(dg_y) = 4k-1$ is a $2k$-dimensional submanifold $\Sigma^1 \subset M$. The kernel of the differential $\mathrm{Ker}(dg)$ is a 1-dimensional subbundle of the bundle $TM|\Sigma^1$. At isolated points in Σ^1, $\mathrm{Ker}(dg)$ is tangent to $T\Sigma^1$. Such points are called $\Sigma^{1,1}$*-points* or simply *cusps*. The orientations of M and $\mathbb{R}^{6k-1}$ induce an orientation on the cusps and we let $\sharp\Sigma^{1,1}(g)$ denote the algebraic number of cusps of f.

Let $f\colon \Sigma^{4k-1} \to \mathbb{R}^{4k+1}$ be an immersion of a homotopy sphere, let $\iota\colon \mathbb{R}^{4k+1} \to \mathbb{R}^{6k-1}$ denote the inclusion, and let $a_k = 2$ if k is odd and $a_k = 1$ if k is even.

Assume that there exists a compact parallelizable $4k$-manifold M with $\partial M = \Sigma^{4k-1}$. If $g\colon M \to \mathbb{R}^{6k-1}$ is a generic map which is an immersion on some neighborhood of the boundary and such that the restriction ∂g of g to the boundary is regularly homotopic to $\iota \circ f$ then

$$\Omega(f) = \frac{1}{a_k(2k-1)!}\Big(c_{4k}\sigma(M) + \sharp\,\Sigma^{1,1}(g)\Big). \tag{2.3}$$

2.C. **Extension of Ω to codimensions $3 \leq q \leq 2k$.** The Formulas (2.1) and (2.2) extend immediately to immersions $S^{4k-1} \to \mathbb{R}^{4k-1+q}$, $3 \leq q \leq 2k$ and provide non-trivial homomorphisms $\Omega\colon \mathbf{I}b^q_{4k-1} \to \mathbb{Z}$ and $\Omega\colon \mathbf{I}\theta^q_{4k-1} \to \mathbb{Q}$.

Of course, these extended formulas will not, in general, give complete regular homotopy invariants of immersed standard spheres in $\mathbb{R}^{4k-1+q}$ but they do provide splittings $\mathbf{I}^q_{4k} \approx \mathbf{I}^2_{4k-1} \oplus G(k,q)$, where $G(k,q)$ is as in Theorem 1.3 .

3. Embeddings, the J-homomorphism, and commutative diagrams

In this section we prove Theorem 1.1 (b) and present five commutative diagrams which will be used in the proofs of the theorems stated in Section 1.

3.A. **Proof of Theorem 1.1 (b).** Since an embedded orientable $(4k-1)$-submanifold P in $\mathbb{R}^{4k+1}$ has an embedded compact orientable $4k$-dimensional Seifert-surface $M \subset \mathbb{R}^{4k+1}$ (i.e., $\partial M = P$) which is necessarily parallelizable it follows that $\mathbf{E}\theta^2_{4k-1} = \mathbf{E}b^2_{4k-1}$.

We show that the homomorphism $\Omega\colon \mathbf{E}b^2_{4k-1} \to \mathbb{Z}$ (see Subsection 2.B) is injective. Let $f\colon \Sigma^{4k-1} \to \mathbb{R}^{4k+1}$ be an embedding and let M be a Seifert-surface of $f(\Sigma^{4k-1})$. Then $\Omega(f)$ is a multiple of the signature of M. We conclude that if $\Omega(f) = 0$ then $\Sigma^{4k-1} = S^{4k-1}$ and the Smale invariant of f is zero.

To calculate the index of $\mathbf{E}^2_{4k-1}$ in $\mathbf{E}b^2_{4k-1}$, let M be a parallelizable $4k$-manifold with boundary Σ^{4k-1} and minimal signature embedded in $\mathbb{R}^{4k+1}$ (see Lemma 2.3). Using Ω we find that the corresponding embedding f of Σ^{4k-1} represents a generator of $\mathbf{E}b^2_{4k-1}$. Using Ω again and $bP_{4k} \approx \mathbb{Z}_{n(k)}$ (see Section 1), we see that the connected sum of $n(k)$ copies of f is a generator of $\mathbf{E}^2_{4k-1}$. □

3.B. **The J-homomorphism.** The Hopf-Whitehead J-homomorphism is a homomorphism $J\colon \pi_{4k-1}(SO) \to \pi^s(4k-1)$, where SO denotes the infinite special orthogonal group (note that $\pi_{4k-1}(SO(N)) \approx \pi_{4k-1}(SO) \approx \mathbb{Z}$, for $N \geq 4k+1$) and $\pi^s(4k-1)$ is the stable homotopy group of spheres in codimension $4k-1$ (that is, the group $\pi_{N+4k-1}(S^N)$, for N large). It is defined as follows.

For $N > 1$, consider the standard sphere $S^{4k-1} \subset \mathbb{R}^{N+4k-1}$ with its standard normal framing $X = (n, e_{4k+1}, \dots, e_{N+4k-1})$, where n is the outward normal vector field of $S^{4k-1} \subset \mathbb{R}^{4k}$ and e_i is the unit vector in the i^{th} coordinate direction.

Given an element $\xi \in \pi_{4k-1}(SO)$, represent it by a smooth map $g\colon S^{4k-1} \to SO(N)$, for some sufficiently large N. By the Pontryagin construction, the standard embedding $S^{4k-1} \subset \mathbb{R}^{N+4k-1}$ with the normal framing gX represents an element η in $\pi_{N+4k-1}(S^N)$. Let $J(\xi) = \eta$.

As a consequence of the Adams conjecture, formulated in [1] and proved in [19], the order $j(k)$ of $\operatorname{Im}(J)$ in dimension $4k-1$ equals Denominator$(\frac{B_k}{4k})$ where B_k is the k^{th} Bernoulli number.

3.C. **A homomorphism $\alpha\colon \mathbf{I}\theta^2_{4k-1} \to \pi^{\mathrm{s}}(4k-1)$.** Let $\xi \in \mathbf{I}\theta^2_{4k-1}$. Represent ξ by an immersion $f\colon \Sigma^{4k-1} \to \mathbb{R}^{4k+1}$. Since $\pi_n(S^1) = 0$ for $n \geq 2$, the immersion f has a homotopically unique normal framing (n_1, n_2). Let $\iota\colon \mathbb{R}^{4k+1} \to \mathbb{R}^{4k-1+N}$ denote the inclusion. If N is large enough then $\iota \circ f$ with normal framing $(n_1, n_2, e_{4k+2}, \ldots, e_{4k+N-1})$ is framed regularly homotopic to an embedding with some framing. By the Pontryagin construction this framed embedding can be viewed as an element η in $\pi_N(S^{4k+n-1}) \approx \pi^{\mathrm{s}}(4k-1)$. Define $\alpha\colon \mathbf{I}\theta^2_{4k-1} \to \pi^{\mathrm{s}}(4k-1)$ as $\alpha(\xi) = \eta$. Then α is a homomorphism.

Lemma 3.1. *The homomorphism $\alpha\colon \mathbf{I}\theta^2_{4k-1} \to \pi^{\mathrm{s}}(4k-1)$ is surjective.*

Proof. Let η be an element in $\pi^{\mathrm{s}}(4k-1)$ and represent it by a framed $(4k-1)$-manifold $P \subset \mathbb{R}^{4k+N-1}$. Using framed surgery on this odd-dimensional manifold, an embedded homotopy sphere $e\colon \Sigma^{4k-1} \subset \mathbb{R}^{4k+N-1}$ framed cobordant to P is produced. The Hirsch lemma implies that there exists a framed regular homotopy of e to a framed immersion f mapping into $\mathbb{R}^{4k+1} \subset \mathbb{R}^{4k+N-1}$ with framing as in the definition of α. □

3.D. **The restriction of α to $\mathbf{I}_{4k-1}$.**

Lemma 3.2. $\alpha(\mathbf{I}^2_{4k-1}) = \operatorname{Im}(J)$.

Proof. Let $\xi \in \mathbf{I}^2_{4k-1}$. Represent ξ by an immersion $f\colon S^{4k-1} \to \mathbb{R}^{4k+1}$ and normal framing (n_1, n_2). Let $\iota\colon \mathbb{R}^{4k+1} \to \mathbb{R}^N$ denote the inclusion. If N is large enough then $\iota \circ f$ with normal framing $(n_1, n_2, e_{4k+2}, \ldots, e_{4k+N-1})$ is framed regularly homotopic to the standard embedding with some framing. Thus ξ maps into $\operatorname{Im}(J) \subset \pi^{\mathrm{s}}(4k-1)$.

Using Hirsch lemma as in the proof of Lemma 3.1 we see that the opposite inclusion $\operatorname{Im}(J) \subset \alpha(\mathbf{I}^2_{4k-1})$ holds as well. □

3.E. **The quotient $\mathbf{I}\theta^2_{4k-1}/\mathbf{I}b^2_{4k-1}$.**

Lemma 3.3. *The sequence*

$$0 \longrightarrow \mathbf{I}b^2_{4k-1} \longrightarrow \mathbf{I}\theta^2_{4k-1} \xrightarrow{\alpha'} \pi^{\mathrm{s}}(4k-1)/\operatorname{Im}(J) \longrightarrow 0,$$

where α' equals α composed with the quotient homomorphism is exact.

Proof. After Lemma 3.1 only exactness in the middle is an issue. We first show that $\alpha(\mathbf{I}b^2_{4k-1}) \subset \operatorname{Im}(J)$. By Lemma 2.3 any class in $\mathbf{I}b^2_{4k-1}$ can be represented by a connected sum of an immersion of the standard sphere and an embedding. Any embedding has a Seifert-surface which gives a normally framed cobordism to a framed standard sphere. Therefore both summands map to $\operatorname{Im}(J)$ and hence so does their sum.

Second, if $f\colon \Sigma^{4k-1} \to \mathbb{R}^{4k+1}$ is an immersion with $\alpha([f]) \in \mathrm{Im}(J)$ then a normally framed embedding of Σ^{4k-1} into $\mathbb{R}^{4k+N-1}$ for some large N is framed cobordant to a framed embedding of the standard sphere. It follows that $\Sigma^{4k-1} \sharp S^{4k-1}$ bounds a parallelizable manifold. Hence, Σ^{4k-1} does too, and $[f] \in \mathbf{I}b^2_{4k-1}$. $\square$

3.F. **Commutative diagrams.** Recall $n(k) = 2^{2k-2}(2^{2k-1}-1)\,\mathrm{Numerator}\left(\frac{4B_k}{k}\right)$, $j(k) = \mathrm{Denominator}\left(\frac{B_k}{4k}\right)$, and the isomorphism $\left(\mathbf{I}^2_{4k-1}, \mathbf{E}^2_{4k-1}\right) \approx (\mathbb{Z}, j(k)\mathbb{Z})$, see Section 1.

Lemma 3.4. *The following diagram commutes and has exact rows and columns*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathbf{E}^2_{4k-1} \arrow[r, "\times n(k)"] \arrow[d, "\times j(k)"'] & \mathbf{E}\theta^2_{4k-1} \arrow[r] \arrow[d] & bP_{4k} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{I}^2_{4k-1} \arrow[r] \arrow[d, "\alpha"'] & \mathbf{I}\theta^2_{4k-1} \arrow[r] \arrow[d, "\alpha"] & \theta_{4k-1} \arrow[r] & 0 \\
 & \mathrm{Im}(J) \arrow[r] \arrow[d] & \pi^{\mathrm{s}}(4k-1) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \tag{3.1}$$

Proof. This follows from Theorem 1.1 (b) (see Subsection 3.A), Lemma 3.1, and Lemma 3.2. $\square$

Lemma 3.5. *The following diagram commutes and has exact rows and columns*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathbf{E}^2_{4k-1} \arrow[r, "\times n(k)"] \arrow[d, "\times j(k)"'] & \mathbf{E}b^2_{4k-1} \arrow[r] \arrow[d] & bP_{4k} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathbf{I}^2_{4k-1} \arrow[r] \arrow[d, "\alpha"'] & \mathbf{I}b^2_{4k-1} \arrow[r] \arrow[d, "\alpha"] & bP_{4k} \arrow[r] & 0 \\
 & \mathrm{Im}(J) \arrow[r, "\mathrm{id}"] \arrow[d] & \mathrm{Im}(J) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \tag{3.2}$$

Proof. This follows from Theorem 1.1 (b) (see Subsection 3.A), Lemma 3.2, and Lemma 3.3. $\square$

Lemma 3.6. *For $2 \leq q \leq 2k$, the following diagrams commute and have exact rows and columns*

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & \mathbf{I}^2_{4k-1} \arrow[r] \arrow[d] & \mathbf{I}\theta^2_{4k-1} \arrow[r] \arrow[d] & \theta_{4k-1} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \mathbf{I}^q_{4k-1} \arrow[r] & \mathbf{I}\theta^q_{4k-1} \arrow[r] & \theta_{4k-1} \arrow[r] & 0 \end{tikzcd} \tag{3.3}$$

and

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & \mathbf{I}^2_{4k-1} \arrow[r] \arrow[d] & \mathbf{I}b^2_{4k-1} \arrow[r] \arrow[d] & bP_{4k} \arrow[r] \arrow[d, "\mathrm{id}"] & 0. \\ 0 \arrow[r] & \mathbf{I}^q_{4k-1} \arrow[r] & \mathbf{I}b^q_{4k-1} \arrow[r] & bP_{4k} \arrow[r] & 0 \end{tikzcd} \tag{3.4}$$

Proof. The injectivity of the vertical arrows follows from Lemma 2.4. ☐

4. Proofs

In this section the theorems stated in Section 1 are proved.

4.A. **Proof of Theorem 1.1.** Statement (b) was proved in Subsection 3.A. Consider (a). It follows from the second row of Diagram (3.2) that $\mathbf{I}b^2_{4k-1}$ is generated by two elements and has rank 1. Thus it has the form $\mathbb{Z} \oplus \mathbb{Z}_r$ for some integer r. Let ϕ be a free generator of $\mathbf{I}b^2_{4k-1}$ and let τ be a generator of the torsion part $\mathrm{Tor}(\mathbf{I}^2_{4k-1}) \approx \mathbb{Z}_r$.

Fix an isomorphism $\mathbf{I}^2_{4k-1} = \mathbb{Z}$ and let $\mathbf{I}^2_{4k-1} \to \mathbf{I}b^2_{4k-1}$ be given by $1 \mapsto a\phi + b\tau$, where a is an integer and b a $\mathrm{mod}(r)$ residue class. Similarly, choose an isomorphism $\mathbf{E}b^2_{4k-1} = \mathbb{Z}$ and let $\mathbf{E}b^2_{4k-1} \to \mathbf{I}b^2_{4k-1}$ be $1 \mapsto c\phi + d\tau$ for $c \in \mathbb{Z}$ and $d \in \mathbb{Z}_r$.

Claim.

$$r = \gcd(n(k), j(k))$$

Proof. First, r divides $\gcd(n(k), j(k))$. Indeed, $ar = n(k)$, since on the one hand $\mathbf{I}b/\mathbf{I} \approx \mathbb{Z}_{n(k)}$, and on the other this quotient has ar elements. Similarly, $cr = j(k)$.

Second, assume that p is a prime and m is a positive integer such that p^m divides $\gcd(n(k), j(k))$, but p^m does not divide r. Then the two equalities above imply that p divides both a and c. It follows that, if $\mathrm{pr}\colon \mathbf{I}b^2_{4k-1} \to \mathbf{I}b^2_{4k-1}/\,\mathrm{Tor}(\mathbf{I}b^2_{4k-1})$ is the quotient homomorphism then $\mathrm{pr}(\mathbf{I}^2_{4k-1})$ and $\mathrm{pr}(\mathbf{E}b^2_{4k-1})$ cannot generate $\mathbf{I}b^2_{4k-1}/\,\mathrm{Tor}(\mathbf{I}b^2_{4k-1})$, since every element in these images are divisible by p . This however contradicts Lemma 2.3 which implies that $\mathbf{I}^2_{4k-1}$ and $\mathbf{E}b^2_{4k-1}$ generates $\mathbf{I}b^2_{4k-1}$. Hence p^m must divide r. ☐

Claim. $r = 2^{s(k)}$, *where*

$$s(k) = \begin{cases} 2 & \text{if } k = 2, \\ u + 3 & \text{if } k = 2^u \cdot (\mathrm{odd}). \end{cases}$$

Proof. In the first case, $j(2) = 24$ and $n(2) = 28$ imply $r = 4$. In the general case,

$$n(k) = 2^{2k-2}(2^{2k-1} - 1)\,\text{Numerator}\left(\tfrac{4B_k}{k}\right). \tag{4.1}$$

Since $j(k) = \text{Denominator}\left(\frac{B_k}{4k}\right)$, we see that any *odd* common divisor l of $n(k)$ and $j(k)$ must divide $2^{2k-1} - 1$. On the other hand it is known that ([18], Lemma B.1, p. 282) $2^{2k}(2^{2k} - 1)\frac{B_k}{2k}$ is an integer and hence l divides also $(2^{2k} - 1)$. But $2^{2k} - 1$ and $2^{2k-1} - 1$ are relatively prime. Thus $l = 1$ and r is a power of 2.

To determine the exponent of r we use the following result of Adams (see [18] Problem B-2, p. 286): if the integer N is large enough then $j(k)$ is the greatest common divisor of the numbers

$$2^N(2^{2k} - 1),\ 3^N(3^{2k-1}),\ 4^N(4^{2k} - 1),\ 5^N(5^{2k} - 1),\ \dots \tag{4.2}$$

Let $k = 2^u \cdot t$, where t is odd. For any odd number s, let $S = s^t$. By Equation (4.2), $j(k)$ divides $s^N(s^{2k} - 1)$. Hence, the greatest power of 2 dividing $j(k)$ must divide $s^{2k} - 1$, which can be expressed as

$$s^{2k} - 1 = (S - 1)(S + 1)(S^2 + 1)(S^4 + 1)\dots(S^{2^u} + 1).$$

Note that the product

$$(S^2 + 1)(S^4 + 1)\dots(S^{2^u} + 1)$$

is divisible by 2^u but not by 2^{u+1}. Furthermore $(S - 1)(S + 1)$ is always divisible by 8 but not by 16 (take $s = 3$). It follows that 2^{u+3} is the greatest power of 2 which divides $j(k)$.

Finally, (4.1) implies that 2^{2k-2} divides $n(k)$ and if $k = 2^u \cdot (\text{odd}) > 2$ then $2k - 2 > u + 3$. The claim follows. □

The two claims prove (a). □

4.B. **Proof of Theorem 1.2.** If $f\colon P \to \mathbb{R}^{n+1}$ is an immersion of an oriented n-manifold let $\nu_f\colon P \to S^n \subset \mathbb{R}^{n+1}$ be the Gauss map which maps a point $x \in P$ to the positive unit normal vector $\nu_f(x) \in S^n$ of the affine tangent plane of $f(N)$ at $f(x)$. The *normal degree* of f is $\deg(\nu_f)$.

Milnor [16] proved that $d = \deg(\nu_f) - 1$ is additive under connected summation of immersions. It follows that d induces a homomorphisms $d\colon \mathbf{I}\theta^1_{4k-1} \to \mathbb{Z}$. Further, he proved that $\deg(\nu_f)$ is odd for any immersion of a non-parallelizable manifold and that any odd value is attained. Since S^{4k-1} is not parallelizable for $k \geq 3$, $d\colon \mathbf{I}\theta^1_{4k-1} \to 2\mathbb{Z}$ and is an epimorphism. For $k = 2$, S^7 is parallelizable, $SO(8) \approx S^7 \times SO(7)$ and it follows from the h-principle that $d\colon \mathbf{I}\theta^1_7 \to \mathbb{Z}$ is an epimorphism.

Let $f, g\colon \Sigma^{4k-1} \to \mathbb{R}^{4k}$ be two immersions of a homotopy sphere. Fix a point $p \in \Sigma^{4k-1}$, a small disk D around p, and a framing X of $T\Sigma^{4k-1}$ over $\Sigma^{4k-1} - \text{Int}(D)$. By a regular homotopy, arrange so that f and g agree in a neighborhood of D. Let

$$\Phi^f_g\colon (\Sigma - \text{Int}(D)) \cup_{\partial D} (\Sigma - \text{Int}(D)) \to V_{4k,4k-1},$$

be the map which is $df(X)$ and $dg(X)$ on the first and second copy of $(\Sigma - \text{Int}(D))$, respectively. Adding the normal vector field to Φ^f_g we may regard it as a map into $SO(4k)$ ($V_{4k,4k-1} \approx SO(4k)$).

Consider the inclusion $\iota\colon \mathbb{R}^{4k} \to \mathbb{R}^{4k+1}$, let $\Psi^g_f = (\Phi^g_f, e_{4k+1})$, where e_{4k+1} is the unit normal of $\iota(\mathbb{R}^{4k}) \subset \mathbb{R}^{4k+1}$, then Ψ^f_g is a mapping into $SO(4k + 1)$.

Now, $(\Sigma-\mathrm{Int}(D))\cup_{\partial D}(\Sigma-\mathrm{Int}(D))$ is homeomorphic to a sphere and the Smale-Hirsch h-principle implies that f and g ($\iota\circ f$ and $\iota\circ g$) are regularly homotopic if and only if Φ^f_g (Ψ^f_g) are null-homotopic.

Consider the following diagram associated to the fibrations $SO(4k+1)\to S^{4k}$ and $SO(4k)\xrightarrow{\mathrm{pr}} S^{4k-1}$,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_{4k}(S^{4k}) & \xrightarrow{\partial} & \pi_{4k-1}(SO(4k)) & \xrightarrow{\iota_*} & \pi_{4k-1}(SO(4k+1)) & \longrightarrow & 0 \\
 & & & & \Big\downarrow{\scriptstyle \mathrm{pr}} & & & & . \\
 & & & & \pi_{4k-1}(S^{4k-1}) & & & &
\end{array} \tag{4.3}$$

The row is exact: the zero on the right follows from $\pi_{4k-1}(S^{4k})=0$. To see the zero on the left, let $\mathbf{Vect}_d(S^n)$ denote isomorphism classes of oriented d-dimensional vector bundles over S^n. Then

$$\pi_{4k}(SO(4k))\approx \mathbf{Vect}_{4k}(S^{4k+1})\to \mathbf{Vect}_{4k+1}(S^{4k+1})\approx \pi_{4k}(SO(4k+1))$$

is surjective since any orientable $(4k+1)$-dimensional vector bundle over S^{4k+1} has vanishing Euler class and therefore admits a non-zero section.

Exactness of the row implies that $\pi_{4k-1}(SO(4k))\approx\mathbb{Z}\oplus\mathbb{Z}$. Moreover, the composition $\partial\circ\mathrm{pr}\colon \pi_{4k}(S^{4k})\to\pi_{4k-1}(S^{4k-1})$ is simply multiplication by 2, see Husemoller [6], Chapter 8, Theorem 10.1. Also, note that $d(f)=\mathrm{pr}[\Phi^f_s]$.

We next show that $[\hat{s}]$ generates $\mathrm{Ker}(\mathbf{I}^1_{4k-1}\to\mathbf{I}^2_{4k-1})$. It is obvious that $\hat{s}$ is regularly homotopic to s in $\mathbb{R}^{4k+1}$. Thus $\Phi^{\hat{s}}_s\in\mathrm{Ker}(\iota_*)$. Let $\alpha\in\pi_{4k}(S^{4k})$ be the preimage $\partial^{-1}([\Phi^{\hat{s}}_s])$. Using normal degrees we find

$$-2=d(\hat{s})=\mathrm{pr}(\Phi^{\hat{s}}_s)=\mathrm{pr}\circ\partial(\alpha).$$

Since $\mathrm{pr}\circ\partial$ is multiplication by 2, the element α must be a generator of $\pi_{4k}(S^{4k})$ and thus $\partial(\alpha)=[\Phi^{\hat{s}}_s]$ generates $\mathrm{Im}(\partial)=\mathrm{Ker}(\iota_*)$. The Smale-Hirsch h-principle then implies that $[\hat{s}]$ generates $\mathrm{Ker}(\mathbf{I}^1_{4k-1}\to\mathbf{I}^2_{4k-1})$.

It is easy to see that $\mathrm{Ker}(\mathbf{I}^1_{4k-1}\to\mathbf{I}^2_{4k-1})=\mathrm{Ker}(\mathbf{I}\theta^1_{4k-1}\to\mathbf{I}\theta^2_{4k-1})$ and the theorem, except for the statement on embeddings, follows.

Let $f\colon\Sigma^{4k-1}\to\mathbb{R}^{4k}$ be an embedding. The Schoenflies theorem (see [17]) implies that f is isotopic to an embedding with image equal to the unit sphere in $\mathbb{R}^{4k}$. Hence $\Sigma^{4k-1}\approx S^{4k-1}$ and the normal degree of f equals ±1. Moreover, Lemma 2.4 implies that $\iota\circ f$, where $\iota\colon\mathbb{R}^{4k}\to\mathbb{R}^{4k+1}$ is regularly homotopic to the standard embedding of the sphere. Since $\mathrm{Ker}(\mathbf{I}^1_{4k-1}\to\mathbf{I}^2_{4k-1})$ is generated by $[\hat{s}]$ and the normal degree of f is ±1 we conclude f is regularly homotopic to s or to $\hat{s}$. □

4.C. **Proof of Theorem 1.3.** Use Diagram (3.3) (Diagram (3.4)): $\mathbf{I}^q_{4k-1}\approx\mathbf{I}^2_{4k-1}\oplus G(k,q)$. Clearly, $\mathrm{Im}(G(k,q))$ and $\mathrm{Im}(\mathbf{I}\theta^2_{4k-1})$ ($\mathrm{Im}(\mathbf{I}b^2_{4k-1})$) generate $\mathbf{I}\theta^q_{4k-1}$ ($\mathbf{I}b^q_{4k-1}$) and $\mathrm{Im}(G(k,q))\cap\mathrm{Im}(\mathbf{I}\theta^2_{4k-1})=0$ ($\mathrm{Im}(G(k,q))\cap\mathrm{Im}(\mathbf{I}b^2_{4k-1})=0$).

4.D. **Proofs of Theorems 1.4 and 1.5.** As a consequence of Haefliger and Hirsch [7], embeddings of homotopy spheres $\Sigma^{4k-1}\to\mathbb{R}^{4k-1+q}$ exists and are unique up to regular homotopy in codimensions $q>2k$. (For embeddings of the standard sphere, uniqueness is due to Kervaire [12].) This provides splitting maps $\theta_{4k-1}\to\mathbf{I}\theta^q_{4k-1}$ and $\theta_{4k-1}\to\mathrm{Im}(\mathbf{I}\theta^2_{4k-1}\to\mathbf{I}\theta^q_{4k-1})$, the restrictions of which are splitting maps $bP_{4k}\to\mathbf{I}b^q_{4k-1}$ and $bP_{4k}\to\mathrm{Im}(\mathbf{I}b^2_{4k-1}\to\mathbf{I}b^q_{4k-1})$. □

5. Generators, examples, and corollaries

5.A. **Brieskorn spheres.** Let $z = (z_1, \dots, z_{2k+1}) \in \mathbb{C}^{2k+1}$ and let S^{4k+1} denote the unit sphere in $\mathbb{C}^{2k+1}$. The Brieskorn spheres

$$\{z\colon z_1^{6j-1} + z_2^3 + z_3^2 + \cdots + z_{2k+1}^2 = 0\} \cap S^{4k+1}, \quad \text{for } j = 1, 2, \dots$$

give, after removing a suitable point ∞ from S^{4k+1}, embeddings into $\mathbb{R}^{4k+1} = S^{4k+1} - \infty$ corresponding to $j \in \mathbb{Z} \approx \mathbf{E}\theta^2_{4k-1}$. In particular, the simplest embedding $S^{4k-1} \to \mathbb{R}^{4k+1}$ non-regularly homotopic to the standard embedding is obtained for $j = n(k)$.

For the immersion group of (standard) 3-spheres in $\mathbb{R}^5$ an explicit construction (similar to twist-spinning in knot theory) of representatives of all classes were given in [2].

Problem. *Find an effective construction of generators of the groups $\mathbf{I}^2_{4k-1}$ and $\mathbf{I}b^2_{4k-1}$.*

Example. *If $g\colon D^8 \to \mathbb{R}^{11}$ is a generic map of the 8-disk with ∂g a diffeomorphism onto $e(S^7)$ where $e\colon S^7 \to \mathbb{R}^9 = S^9 - \infty$ is the generator:*

$$\{z\colon z_1^{167} + z_2^3 + z_3^2 + z_4^2 + z_5^2 = 0\} \cap (S^9 - \infty)$$

(which is a standard sphere since $j = 28 = n(2)$, in the notation above), then the algebraic number of cusps of g is 1440.

Proof. The simplest embedding has Smale invariant $j(2) = 240$, by Formula (2.3) we find that g has $6 \cdot 240 = 1440$ cusps. □

5.B. **Cusps and obstructions to homotopies through generic maps.**

Corollary 5.1. *If $g\colon M \to \mathbb{R}^{6k-1}$ is a generic map of a parallelizable $4k$-manifold with ∂M a homotopy sphere such that g is an immersion in some neighborhood of the boundary and $f = g|\partial M$ is an immersion with trivial normal bundle then the algebraic number of cusps of g is divisible by $(2k-1)!a_k$.*

Proof. The normal bundle of f is trivial. Hence, by the Hirsch lemma, f is regularly homotopic to an immersion $\hat{f}\colon \partial M \to \mathbb{R}^{4k+1} \subset \mathbb{R}^{6k-1}$. By surgery below middle dimension, M can be changed into a $(2k-1)$ connected manifold N, with $\sigma(N) = \sigma(M)$. There exists an embedding $e\colon N \to \mathbb{R}^{4k+1}$ and if $h = e|\partial N$ then since $M \sharp N_-$ is a Seifert-surface of $f \sharp \hat{e}$ we find from (2.3)

$$\sharp\Sigma^{1,1}(g \sharp -e) = \sharp\Sigma^{1,1}(g) = (2k-1)!a_k\Big(\Omega(h) - \Omega(\hat{f})\Big).$$

□

Corollary 5.2. *Let f_0 and f_1 be non-regularly homotopic immersions of a homotopy sphere into $\mathbb{R}^{4k+1}$ and let $\iota\colon \mathbb{R}^{4k-1} \to \mathbb{R}^{6k-2}$ denote the inclusion. Then $\iota \circ f_0$ and $\iota \circ f_1$ cannot be joined by a path in the space of generic (stable) maps in $C^\infty(\Sigma^{4k-1}, \mathbb{R}^{6k-2})$. In fact, any such path transversal to the set of non-generic maps intersects it at least $a_k(2k-1)!|[\Psi^f_g]|$ times, where $|[\Psi^f_g]|$ denotes the index of the subgroup generated by the homotopy class $[\Psi^f_g] \in \pi_{4k-1}(SO(4k+1)) \approx \mathbb{Z}$. (Here Ψ^f_g is as in Subsection 4.B).*

Proof. Similar to the proof of Corollary 5.1. □

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