IMMERSIONS IN THE METASTABLE RANGE AND SPIN STRUCTURES ON SURFACES

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1. Introduction

Recall that an immersion of a smooth manifold into another one is a map, which locally is a smooth embedding. Hence, smooth embeddings are immersions, but not vice versa. The basic classifications of immersions and embeddings are inspired by the spaces of immersions and embeddings: Two immersions (embeddings) are regularly homotopic (diffeotopic) if they can be connected by a path in the corresponding space. Smale and Hirsch [4] reduced the classification of immersions up to regular homotopy to homotopy theory. Haefliger [3] did the same for embeddings up to diffeotopy, in the metastable range.

One may consider embeddings up to regular homotopy, but the diffeotopy classification is more refined.

In this paper we consider immersions up to an equivalence, which is a generalization of diffeotopy. It is called *diffeotopy equivalence*. Two immersions are diffeotopy equivalent if they differ by diffeomorphisms, diffeotopic to identity, of the source and target.

We manage to classify generic immersions of a sufficiently highly connected manifold into Euclidean space, under the assumption that the dimensions are such that the dimension of the self intersection is 0, 1 or 2. The diffeotopy equivalence classes are described in terms of the topological type of the self intersection and numerical invariants of additional structures (e.g. spin structure), which appear on the self intersection. This description depends periodically on the dimension of the source manifold. When the dimension of the self intersection is 0 or 1, the period is 2. When it is 2, the period is 4.

The self intersection of a generic immersion in the metastable range is particularly simple. There, a generic immersion has no triple points and at a double point the intersection is transverse. This implies that the self intersection is a submanifold of the target. When the target of a generic immersion in the metastable range is Euclidean space and the source is sufficiently connected, the diffeotopy equivalence class of this map is determined by the local (even infinitesimal) properties of it in a neighborhood of the preimage of the self intersection. This is the content of Theorem 1.2 below, which is the first step towards our classification.

As mentioned, there are structures induced on a self intersection. For example, if we let F denote a component of a self intersection surface of a generic immersion then the preimage of F is either disconnected or connected. In the former case the immersion induces a spin structure on F. There is a quadratic function on $H_1(F;\mathsf{Z}_2)$ associated to this spin structure. The Arf invariant of this function is preserved under diffeotopy equivalence. In the latter case the immersion induces a functional on immersed curves in F. The properties of it depend on the residue class of the dimension of the source manifold modulo 4. We find invariants of this functional that are preserved under diffeotopy equivalence.

There is another application of the functional mentioned above: We use it to show that there are restrictions on the self intersection of a generic immersion, $f\colon V^v \to \mathsf{R}^{2v-2}$, if $v \equiv 1 \mod 4$. If F is a nonorientable component of the self intersection, the genus of F must be even (see Theorem 7.30). It is interesting that the same restriction holds for an immersion, without triple points and with transverse self intersection, of any orientable 3-manifold into the 4-sphere (see Theorem 7.31).

1.1. *Isotopy equivalence.* Let $f, h\colon X \to Y$ be maps of topological spaces. We say that f is *isotopy equivalent* to h if there exist isotopies $G_t : X \to X$ and $K_t : Y \to Y$ such that $G_0 = 1_X$, $K_0 = 1_Y$ and $K_1 f G_1 = h$. The analogous relations in the differential- and PL category will be called *diffeotopy equivalence* and PL *isotopy equivalence* respectively. Since our main concern will be the smooth case, all maps and manifolds are assumed to be smooth, unless something else is explicitly stated.

1.2. *Immersions.* A map $f : V^v \to W^w$ between two manifolds is called an *immersion* if df, the differential of f, is injective at every point of V.

Definition 1.1. A generic immersion $f : V \to W$ is an immersion such that for any $z \in W$ the set $f^{-1}(z)$ contains at most two points and if $x, y \in V$, $x \neq y$ and $f(x) = f(y) = z$ then $dfT_xV + dfT_yV = T_zW$.

If $2w > 3v$ (the metastable range) then any immersion $h\colon V^v \to W^w$ is homotopic, by an arbitrarily small homotopy, to a generic immersion. We write $\widetilde{M}_f = \{x \in V : f^{-1}(f(x)) \neq x\}$ and $M_f = f(\widetilde{M}_f)$. We call M_f the *self intersection* and $\widetilde{M}_f$ the *double locus.*

1.3. *Statements of the main results.* Let $f, h\colon V^v \to W^w$ be generic immersions. As already mentioned, if f and h are diffeotopy equivalent then $M_f \approx M_h$ and $\widetilde{M}_f \approx \widetilde{M}_h$ (where "$\approx$" denotes "is diffeomorphic to").

THEOREM 1.2. *Let $f, h\colon V^v \to \mathsf{R}^w$ be generic immersions. Let $w = v + k$, where $k \geq 3$ and $v + 3 < 2k \leq 2v$. Assume that V is closed and $(v - k + 1)$-connected and that there are diffeotopies $K_t\colon \mathsf{R}^w \to \mathsf{R}^w$ and $G_t : V \to V$ such that $K_1 f G_1|T = h|T$, where T is a tubular neighborhood of $\widetilde{M}_h$. Then f is diffeotopy equivalent to h.*

A proof is presented in Section 3.

If $f : V^v \to \mathsf{R}^{2v}$ is a generic immersion then M_f is a collection of points. If V and R^{2v} are oriented and v is even then there is an induced orientation of M_f. Let $c \subset M_f$ be a component (i.e. c is a point in M_f). When v is even, we let $s(c) \in \mathsf{Z}_2$ be the signature of c. When v is odd, there are no local obstructions to diffeotopy equivalence and we let $s(c) = 0$.

THEOREM 1.3. *Let $f, h\colon V^v \to \mathsf{R}^{2v}$, $v \geq 4$, be generic immersions. Assume that V is closed and simply connected. Then f is diffeotopy equivalent to h if and only if there is a 1-1 correspondence between the components C_f of M_f and C_h of M_h such that if C_f corresponds to C_h then $s(C_f) = s(C_h)$, where s is as described above.*

Theorem 1.3 is proved in Section 4.2. There is an analogue of Theorem 1.3 in the PL category that actually is somewhat stronger. We can replace $v \geq 4$ above by $v \geq 3$. For the necessary definitions in the PL case, see Section 4.3.

If $f : V^v \to \mathsf{R}^{2v-1}$ is a generic immersion then M_f is a collection of circles. Assume that V is a spin manifold and let c be a component of M_f. We shall define an invariant $s(c) \in \mathsf{Z}_2$, which describes the local obstructions to diffeotopy equivalence. The preimage $\widetilde{c} = f^{-1}(c)$ has either one or two components. If $\widetilde{c}$ is disconnected then there is an induced spin structure on c (cf. Proposition 2.10) and, in this case, we let $s(c) = 1$ if the induced spin structure is trivial (spin cobordant to zero) and $s(c) = 0$ otherwise. If $\widetilde{c}$ is connected then, given a framing of the normal bundle of $\widetilde{c}$, the map f induces a structure similar to a framing of the normal bundle of c. Such a structure will be called a twist framing (see Definition 5.3). There are two twist framings up to isotopy. Taking into account also the framing from which the twist framing is induced, it is possible to define $s(c)$ (see Definition 6.7) also in this case. The latter case, $\widetilde{c}$ is connected for some component c of M_f, appears only if $v = \dim V$ is odd.

Theorem 1.4. *Let $f, h : V^v \to \mathsf{R}^{2v-1}$, $v \geq 6$, be generic immersions. Assume that V is closed and 2-connected. Then f is diffeotopy equivalent to h if and only if there is a 1-1 correspondence between the components C_f of M_f and C_h of M_h such that if C_f corresponds to C_h then $f^{-1}(C_f)$ and $h^{-1}(C_h)$ have the same number of components and $s(C_f) = s(C_h)$, where s is as described above.*

Theorem 1.4 is proved in Section 6.2.

If $f : V^v \to \mathsf{R}^{2v-2}$ is a generic immersion then M_f is a collection of closed surfaces. Assume that V is a spin manifold and let F be a component of M_f. Let $g(F) = \text{genus}(F)$. We shall define an invariant $s(F) \in G(n, [v])$, where the group $G(n, [v])$ depends on the number n of components of $\widetilde{F} = f^{-1}(F)$ and the residue class $[v]$ of v modulo 4 in the following way, $G(2, [v]) = \mathsf{Z}_2$ for all $[v]$ and $G(1, [v]) = 0, 0, \mathsf{Z}_2 \oplus \mathsf{Z}_2, \mathsf{Z}_8$ for $[v] = 0, 1, 2, 3$ respectively. The invariant $s(F)$ describes the local obstructions to diffeotopy equivalence and is defined in Table 1 below. The first row in this table is straightforward to explain: If $\widetilde{F}$ has two components then f induces a spin structure on F and hence a Z_2-quadratic form $q_{f,F}$ on $H_1(F; \mathsf{Z}_2)$. Let $\text{Arf}(q_{f,F})$ denote the Arf invariant of the quadratic function $q_{f,F}$ and define $s(F) = \text{Arf}(q_{f,F}) \in \mathsf{Z}_2$, in this case. The other rows should be read in a similar way. The second and third row are explained in Section 7.6, the fourth row in Section 7.5 and the fifth row in Section 7.4.

Theorem 1.5. *Let $f, h : V^v \to \mathsf{R}^{2v-2}$, $v \geq 8$, be generic immersions. Assume that V is closed and 3-connected. Then f and h are diffeotopy equivalent if and only if there is a 1-1 correspondence between the components C_f of M_f and C_h of M_h such that if C_f corresponds to C_h then $f^{-1}(C_f)$ and $h^{-1}(C_h)$ have the same number of components, $g(C_f) = g(C_h)$ and $s(C_f) = s(C_h)$, where s is as described in Table 1.*

Theorem 1.5 is proved in Section 7.7.

2. **Self intersection and double locus of a generic immersion**

Recall our definition of generic immersion (Definition 1.1). If we are not in the metastable range then the term generic is somewhat misleading. For simplicity, we keep this notation also outside the metastable range.

Proposition 2.1. *Let $f : V^v \to W^w$ be a generic immersion of a closed manifold into a manifold. Then M_f is a closed submanifold of W and $\widetilde{M}_f$ is a closed submanifold of V. Moreover, $f|\widetilde{M}_f \colon \widetilde{M}_f \to M_f$ is a (possibly disconnected) double cover of M_f.*

n	$[v]$	F	Local obstruction	$G(n,[v])$	$s(F)$
2	0, 1,2, 3	Orientable	A spin structure on F and hence a Z_2-quadratic function $q_{f,F}$ on $H_1(F;\mathsf{Z}_2)$	Z_2	$\mathrm{Arf}(q_{f,F})$
1	0	Orientable	None	0	0
1	1	Non-orientable, $g(f)$ even	The quadratic function of a spin structure evaluated at $b \in H_1(\widetilde{F};\mathsf{Z}_2)$, with $f_*(b)$ dual to $w_1 \in H^1(F;\mathsf{Z}_2)$	Z_2	$q_{f,\widetilde{F}}(b)$
1	2	Orientable	A $\mathsf{Z}_2 \oplus \mathsf{Z}_2$-quadratic function $q_{f,F}$ on $H_1(F;\mathsf{Z}_2)$	$\mathsf{Z}_2 \oplus \mathsf{Z}_2$	$\mathrm{Arf}(q_{f,F})$
1	3	Non-orientable	A pin structure on F and hence a Z_4-quadratic function $q_{f,F}$ on $H_1(F;\mathsf{Z}_2)$	Z_8	Brown's invariant of $q_{f,F}$

TABLE 1. Local obstructions to diffeotopy equivalence associated to a component F of a self intersection surface.

DEFINITION 2.2. Let $f : V^v \to W^w$ be a generic immersion. We denote by M_f' the union of those components C_f of M_f such that $f^{-1}(C_f)$ is connected. Let $M_f'' = M_f - M_f'$.

2.1. *Orientation.*

PROPOSITION 2.3. *Let $f : V^v \to W^w$ be a generic immersion of an orientable closed manifold into an orientable manifold. If $w-v$ is* even *then M_f and $\widetilde{M}_f$ are orientable.*

PROOF. Since $\widetilde{M}_f$ is a double cover of M_f it suffices to prove that M_f is orientable. Let $x \in M_f$. Near x, the submanifold M_f is the intersection of two oriented submanifolds U_1 and U_2 of dimension v. If we order these, there is a standard way to assign an orientation to a neighborhood of x in M_f. Since $w-v$ is even, this orientation does not depend on the ordering.

REMARK 2.4. If V and W in Proposition 2.3 are oriented then there is an induced orientation of M_f.

PROPOSITION 2.5. *Let $f : V^v \to W^w$ be a generic immersion of an orientable closed manifold into an orientable manifold. If $w-v$ is* odd *then the covering $\widetilde{M}_f \to M_f$ is the orientation double covering.*

Proof. Let $p \in M_f$ and $f^{-1}(p) = \{p_1, p_2\}$. There are neighborhoods $N \subset M_f$, $N_1 \subset V$ and $N_2 \subset V$ of p, p_1 and p_2, respectively such that $N = f(N_1) \cap f(N_2)$. Ordering N_1 and N_2 we get an induced orientation of N. We can pull back this orientation to $f^{-1}(N) \cap N_1 = \widetilde{N}_1$. Changing the ordering we pull back the opposite orientation to $\widetilde{N}_2$. It follows that $\widetilde{M}_f$ is the orientation double cover.

Remark 2.6. If $f: V^v \to W^w$ is a generic immersion of orientable manifolds then $\widetilde{M}_f$ is orientable.

2.2. *Spin.* Let η be an orientable vector bundle over a base space B. Let $E(SO(\eta))$ denote the total space of its principal SO-bundle. A spin structure on η can be viewed as an element in the cohomology group $H^1(E(SO(\eta)); \mathsf{Z}_2)$, which takes the value 1 on the fiber class, i. e. the class represented by inclusion of the nontrivial loop in a fiber. In the special case when the vector bundle is 1-dimensional we consider the spin structure as an element in $H^1(B; \mathsf{Z}_2)$. A spin structure on an orientable manifold is a spin structure on its tangent bundle. The following lemma is proved in [11].

Lemma 2.7. *Let η^n, η^m, η^k be orientable vector bundles over B. Let $n = m + k$ and $\eta^n = \eta^m \oplus \eta^k$. Spin structures on any two of these bundles determine a spin structure on the third.*

As a consequence of Lemma 2.7 we get the spin analogue of Remark 2.6:

Proposition 2.8. *Let $f: V^v \to W^w$ be a generic immersion of spin manifolds. Then $\widetilde{M}_f$ is spin.*

Proof. Let $i: \widetilde{M}_f \to \widetilde{M}_f$ denote the involution (i.e. $f(i(x)) = f(x)$, $x \in \widetilde{M}_f$). Since f is a generic immersion, we have the following splitting

$$(f^*TW)|\widetilde{M}_f = TV|\widetilde{M}_f \oplus i^*N(\widetilde{M}_f \subset V),$$

where $N(\widetilde{M}_f \subset V)$ denotes the normal bundle of $\widetilde{M}_f$ in V. An application of Lemma 2.7 shows that $i^*N(\widetilde{M}_f \subset V)$ is a spin bundle. The decomposition

$$TV|\widetilde{M}_f = N(\widetilde{M}_f \subset V) \oplus T\widetilde{M}_f,$$

then shows that $\widetilde{M}_f$ is spin.

Proposition 2.9. *Let X and Y be transversely intersecting submanifolds of a manifold W. Let $Z = X \cap Y$. If there are spin structures on X, Y and W then there is an induced spin structure on Z.*

PROOF. The restriction of the tangent bundle of W to Z decomposes as follows:

$$TW|Z = TX|Z \oplus N(Z \subset Y),$$
$$TW|Z = TY|Z \oplus N(Z \subset X),$$
$$TW|Z = N(Z \subset X) \oplus N(Z \subset Y) \oplus TZ.$$

Apply Lemma 2.7.

PROPOSITION 2.10. *Let $f\colon V^v \to W^w$ be a generic immersion. Let C_f be a component of M_f such that $f^{-1}(C_f) = \widetilde{C}_f$ is disconnected. If there are spin structures on V and W then there is an induced spin structure on C_f.*

PROOF. There are disjoint tubular neighborhoods N_1 and N_2 of the components of $\widetilde{C}_f$ such that $f|N_i$ is an embedding $i = 1, 2$. Now, C_f is the transverse intersection of the two embedded spin submanifolds $f(N_1)$ and $f(N_2)$ in the spin manifold W.

Spin structures are defined on oriented manifolds. The self intersection in Proposition 2.10 receives an orientation if we order N_1 and N_2 in the proof. If $w - v$ is even the induced orientation is independent of the ordering. If $w - v$ is odd and this ordering is changed then the orientation changes and the spin structure is reversed.

If M is a spin manifold and $\vec{c}$ is a framed loop in M, we can orthonormalize the framing and regard $\vec{c}$ as a loop in the principal bundle. Then $\vec{c}$ represents a homology class $[\vec{c}\,] \in H_1(E(SO(TM)); \mathsf{Z}_2)$. Let X, Y, Z and W be as in Proposition 2.9. Let $\vec{c}_Z$ be a framed loop in Z. If we extend this framing of TZ along c to framings of TX and TY then we obtain two framed curves $\vec{c}_X$ and $\vec{c}_Y$ in X and Y respectively. Taking the sum of the extensions and the framing of TZ we get a framing of TW along c. The framed curve in W, obtained in this way, will be denoted $\vec{c}_W$.

PROPOSITION 2.11. *Let ξ_X, ξ_Y and ξ_W denote the spin structures on X, Y and W respectively. Let ξ_Z denote the spin structure induced on Z. Then*

$$\langle \xi_Z, [\vec{c}_Z] \rangle = \langle \xi_X, [\vec{c}_X] \rangle + \langle \xi_Y, [\vec{c}_Y] \rangle + \langle \xi_W, [\vec{c}_W] \rangle.$$

PROOF. This follows from the proof of Lemma 2.7, given in [11].

REMARK 2.12. Assume that the extension to TX is changed in such a way that the fiber class of $SO(TX)$ is added to $[\vec{c}_X]$, and that $[\vec{c}_Y]$ is fixed. Then $[\vec{c}_W]$ changes by the fiber class in $SO(TW)$. Hence, the formula in Proposition 2.11 is independent of the extensions of the framing of TZ.

REMARK 2.13. If the manifold Z in Proposition 2.11 happens to be S^1 we see that ξ_Z is an element in $H^1(S^1;\mathsf{Z}_2)$. For future reference we note that there are two spin structures on the circle. The trivial spin structure, which is induced from the unique spin structure on the disk and corresponds to the cohomology class ξ which takes the value 1 on the generator of $H_1(S^1;\mathsf{Z}_2)$. The Lie group one, which corresponds to the 0 cohomology class and with which S^1 is not a spin boundary.

2.3. *Straightening.* We can straighten a generic immersion close to its self intersection, using a diffeotopy of the target space. More precisely, consider a generic immersion $f\colon V^v \to W^w$. Let ξ^{w-v} and η^{2v-w} be vector bundles over $\widetilde{M}_f$ and M_f, respectively. Denote their total spaces $E(\xi)$ and $E(\eta)$. Let $\phi\colon E(\xi) \to U \subset V$ and $\psi\colon E(\eta) \to N \subset W$ be embeddings. That is, U and N are tubular neighborhoods of $\widetilde{M}_f$ and M_f, respectively. Assume that $f(U) \subset N$.

LEMMA 2.14. *Under the conditions above there are open disk subbundles $\xi' \subset \xi$, $\eta' \subset \eta$ and a diffeotopy $K_t : W \to W$ such that $\psi^{-1}K_1 f\phi\colon E(\xi') \to E(\eta')$ is a vector bundle map covering $f : \widetilde{M}_f \to M_f$.*

The proof of Lemma 2.14 is a standard application of the techniques used to prove e. g. the Tubular Neighborhood Theorem. Note that if $\xi^{w-v} = N(\widetilde{M}_f \subset V)$ and $\eta^{2v-w} = N(M_f \subset W)$, Lemma 2.14 tells us that any generic immersion $f : V \to W$ is diffeotopy equivalent to a map that equals the normal derivative of f in a neighborhood of the self intersection.

3. Proof of Theorem 1.2.

Using Lemma 2.14 we can find a diffeotopy $R_t\colon \mathsf{R}^w \to \mathsf{R}^w$ that is fixed outside a neighborhood of $h(T)$ and tubular neighborhoods $N \subset T$ of $\widetilde{M}_h$ and U of M_h such that $R_1 h : N \to U$ and $R_1 h : \partial N \to \partial U$. Let $h' = R_1 h$ and $f' = R_1 K_1 f G_1$. Then $f'|N = h'|N$. Assume that there is a diffeotopy $L'_t : \mathsf{R}^w - U \to \mathsf{R}^w - U$, keeping ∂U fixed and such that $L'_1 f'|(V-T) = h'|(V-T)$. Then, clearly, f is diffeotopy equivalent to h. Thus, to complete the proof it is enough to prove that $f', h'\colon V - T \to \mathsf{R}^w - U$ are ambient diffeotopic keeping ∂U fixed. First of all it is necessary that f' and h' are homotopic relative ∂T. They are both solutions to the lifting problem

$$\begin{array}{ccc} \partial T & \rightarrow & \mathsf{R}^w - U \\ \downarrow & \nearrow & \\ V - T & & . \end{array}$$

The obstructions to a homotopy are found in

$$H^r(V-T, \partial T; \pi_r(\mathsf{R}^w - U)) \cong H_{v-r}(V-T; \pi_r(\mathsf{R}^w - U)).$$

Now, $\mathsf{R}^w - U$ is a deformation retract of $\mathsf{R}^w - M_h$. Since M_h is a $(v-k)$-dimensional submanifold, we see by transversality that $\mathsf{R}^w - U$ is at least $(v+k-1) - (v-k) - 1 = (2k-2)$ connected. By assumption $2k - 2 > v$ and thus all obstructions vanish. Let $F' \colon (V-T) \times I \to \mathsf{R}^w - U$ denote the homotopy between f' and h', fixing ∂T. Define $F'' \colon (V-T) \times I \to (\mathsf{R}^w - U) \times I$ by

$$F''(v,t) = (F'(v,t), t).$$

Using smooth approximation we can assume that F'' is smooth and equals f', h' at the top and bottom. Choose a smooth triangulation of $\mathsf{R}^w - U$. We may assume that f' and h' are PL embeddings relative this triangulation (at least they are isotopic to such maps and this is what is really needed). By Theorem 3.1 and Remark 3.2 below we can conclude that F'' is homotopic, keeping the boundary fixed, to a PL embedding, provided some connectedness conditions are fulfilled. More precisely, we shall verify that

(a) $V - T$ is $2(v+1) - (v+k+1) = v - k + 1$ connected,
(b) $\mathsf{R}^w - U$ is $v - k + 2$ connected,
(c) $v + k - v = k \geq 3$.

Condition (c) holds by assumption. Condition (b) holds by the above since $v - k + 2 < 2k - 2$. To see that condition (a) follows from our assumptions, note that $V - T$ is a deformation retract of $V - \widetilde{M}_h$. Hence, $V - T$ is $(v-k+1)$ connected by general position and the fact that V is $(v-k+1)$ connected. Thus, there is a PL embedding

$$F \colon (V-T) \times I \to (\mathsf{R}^w - U) \times I,$$

such that $F_0 = f'$ and $F_1 = h'$. Adjoining copies of $(V-T) \times [0,\epsilon]$ at the ends of $(V-T)\times I$ we can make $F_s = f'$ for s near 0 and similarly at the other end. Hence $F \colon (V-T)\times I \to (\mathsf{R}^w - U)\times I$ is an allowable concordance connecting f' and h'. Proposition 3.3 then implies that f' and h' are PL isotopic (even ambient isotopic). We conclude from Theorem 3.5 that f' is smoothly isotopic to h', keeping ∂T fixed, since $2(v+k) > 3(v+1)$. Proposition 3.4 shows that this differentiable isotopy extends to an ambient diffeotopy that keeps ∂U fixed. Thus we have the required diffeotopy L_t discussed in the beginning of the proof.

We state below the theorems used in the above proof.

THEOREM 3.1. (HUDSON) *Let M and Q be connected PL manifolds of dimensions m and q respectively, M being compact. Let $f\colon M \to Q$ be a map such that $f^{-1}(\partial Q) = \partial M$, and the restriction $f|_{\partial M}$ is a PL embedding. If $q - m \geq 3$,*

$$\pi_r(f) = 0 \qquad \text{for } r \leq 2m - q + 1,$$

and

$$\pi_r(M) = 0 \qquad \text{for } r \leq 3m - 2q + 2,$$

then f is homotopic, keeping ∂M fixed, to a PL embedding. Here, $\pi_r(f)$ denotes the r^{th} relative homotopy group of (I_f, M) where I_f is the mapping cylinder of f.

Theorem 3.1 is proved in [6].

REMARK 3.2. The exact homotopy sequence of the pair (I_f, M) shows that $\pi_r(f) = 0$ if $\pi_{r-1}(M) = 0$ and $\pi_r(Q) = 0$.

PROPOSITION 3.3. (HUDSON) *Let $f, h\colon M \to Q$ be allowably concordant PL embeddings keeping $Y \subset M$ fixed. Suppose that M is compact, $f^{-1}(\partial Q) \subset N \subset Y$ and $\dim(Q) - \dim(M) \geq 3$. Then f and h are ambient isotopic keeping $fY \cup \partial Q$ fixed.*

Proposition 3.3 is proved in [5].

PROPOSITION 3.4. *Let $f, h\colon M \times I \to Q \times I$ be allowably concordant smooth embeddings keeping N fixed. Suppose that M is compact, $f^{-1}\partial Q = N$ and $\dim(Q) - \dim(M) \geq 3$. Then f is ambient diffeotopic to h keeping ∂Q fixed.*

REMARK . Proposition 3.4 follows from Theorem 2.1 in [5] in the same way as Proposition 3.3 (named Corollary 1.3 in [5]) follows from Theorem 1.1 in this paper.

THEOREM 3.5. (HAEFLIGER) *Let $f_0, f_1\colon M^m \to Q^q$ be two differentiable embeddings. Any isotopy connecting f_0 and f_1 can be approximated by a differentiable isotopy connecting f_0 and f_1 if $2q > 3(m+1)$.*

See [1] for reference.

4. **0-dimensional self intersection**

A generic immersion $f\colon V^v \to W^{2v}$ has 0-dimensional self intersection. If V is closed, Proposition 2.1 shows that M_f is a finite collection of points. If v is even and V and W are oriented, Proposition 2.3 shows that there is an induced orientation of M_f.

4.1. *The invariants.*

Definition 4.1. Let $f\colon V^v \to \mathsf{R}^{2v}$ be a generic immersion. Let c be a point in M_f. Assume that V and R^{2v} are oriented.

(a) If v is even, define $s(c) = 0$ if the induced local orientation of R^{2v} at c agrees with that of R^{2v} and $s(c) = 1$ if it does not.
(b) If v is odd, define $s(c) = 0$.

Lemma 4.2. *Let $f : V^v \to W^{2v}$ be a generic immersion. Assume that V and W are oriented. Let $K_t\colon W^{2v} \to W^{2v}$ and $G_t\colon V^v \to V^v$ be diffeotopies. If $c \in M_f$ then $K_1 c \in M_{K_1 f G_1}$ and $s(c) = s(K_1 c)$.*

Proof. Diffeotopies preserve local orientation.

4.2. *Proof of Theorem 1.3.* The only if part follows from Lemma 4.2 and the fact that the diffeomorphism type of the self intersection is preserved under diffeotopy equivalence. Orient V and R^{2v}. Let $z \in M_f$ and $\zeta \in M_h$. Assume that $s(z) = s(\zeta)$. Suppose that $f^{-1}(z) = \{x, y\}$ and $h^{-1}(\zeta) = \{\xi, \eta\}$. When v is odd we get an induced orientation of z by ordering $\{x, y\}$. Changing the ordering changes the induced orientation. Assume that the notation is chosen so that (x, y) induces the same orientation on z as does (ξ, η) on ζ. Choose diffeotopies $G_t\colon V \to V$ and $K_t'\colon \mathsf{R}^{2v} \to \mathsf{R}^{2v}$ such that $G_1(\xi) = x$, $G_1(\eta) = y$ and $K_1'(z) = \zeta$. Then $K_1' f G_1(\xi) = K_1' f G_1(\eta) = \zeta$. By Lemma 2.14 we can, locally, replace the maps h and $K_1' f G_1$ by their normal derivatives at $\{\xi, \eta\}$. Pick framings of $N(\xi \subset V)$ and $N(\eta \subset V)$. Then h and $K_1' f G_1$ induce two framings of $N(\zeta \subset \mathsf{R}^{2v})$. By our assumptions the orientations of these framings agree. It is then easy to find a diffeotopy $K_t'' : \mathsf{R}^{2v} \to \mathsf{R}^{2v}$, supported in a neighborhood N of ζ, such that $K_1'' K_1' f G_1 | U = h | U$ where U is some neighborhood of $\{\xi, \eta\}$. Repeating this procedure for all points in M_h we find diffeotopies $L_t\colon V \to V$ and $K_t : \mathsf{R}^{2v} \to \mathsf{R}^{2v}$ such that $K_1 f L_1 | W = h | W$, where W is a neighborhood of $\widetilde{M}_h$. An application of Theorem 1.2 finishes the proof.

4.3. PL *results.* A PL immersion is a PL map $f : V^v \to W^w$ that locally, is a locally flat PL embedding. (If $w - v \geq 3$, local flatness is automatic.) If $w = 2v$ then we say that the PL immersion f is generic if any self intersection point is an intersection of two v-dimensional faces, of $f(V)$, in general position. Theorem 1.2 can be proved in the PL case. The proof is exactly the same as that given for the smooth case. However, we need not apply Theorem 3.5 and therefore we can weaken the assumption $v + 3 < 2k$ to $v + 3 \leq 2k$. (Haefliger's construction of knotted differential 3-spheres in 6-space (cf. [2]) shows that the condition $v+3 \leq 2k$ is not sufficient in the smooth category.) We can then prove Theorem 1.3 in the PL category with the assumption $v \leq 4$ replaced by $v \leq 3$. It is only a matter of finding the PL counterparts of the smooth concepts used in the above proof. We give a brief description: Let $h\colon V^v \to \mathsf{R}^{2v}$ be a generic PL immersion. If $z \in M_h$ then z is the intersection of two v-disks D_1 and D_2 in general position. We can find a $2v$-disk D such that $D_1 \cup D_2 \subset D$ and $\partial D_1 \cup \partial D_2 \subset \partial D$. Now, $s(z) = \mathrm{lk}(\partial D_1, \partial D_2)$, if v is even, where the linking number is computed in ∂D. Since $v \geq 3$ this linking number determines the isotopy class of the link.

Probably there are PL versions of Theorems 1.4, 1.5 as well. To translate the proofs to the PL case one must study the PL analogue of the smooth normal bundle of the self intersection.

5. **Framed and twist framed circles in R^n**

If c is a circle in the self intersection of a generic immersion there is some induced structure in the normal bundle of c. We study the structures that arise in this way when the dimension of the self intersection is 1 or 2.

5.1. *Framed circles.* Let $c\colon S^1 \to \mathsf{R}^n$, $n \geq 4$. Let $x_i\colon S^1 \to N(c \subset \mathsf{R}^n)$, $i = 1, \dots, n-1$ be such that $X = (x_1, \dots, x_{n-1})$ is a framing of the normal bundle of c. Adding the tangent vector $\dot{c}$ to this framing we obtain a framing $(\dot{c}, X)$ of $T\mathsf{R}^n|c$, if the orientation of $(\dot{c}, X)$ does not agree with that of R^n we consider instead $(-\dot{c}, X)$. We denote this framed curve $\vec{c}$. As in Section 2.2, $\vec{c}$ defines an element (after orthonormalization) $[\vec{c}] \in H_1(SO(T\mathsf{R}^n); \mathsf{Z}_2)$. Since $H^1(\mathsf{R}^n, \mathsf{Z}_2) = 0$, there is a unique spin structure ξ on R^n. Two framed framed circles $\vec{b}$ and $\vec{c}$ are framed isotopic if and only if $\langle \xi, [\vec{b}\,] \rangle = \langle \xi, [\vec{c}] \rangle$. We may also view the vectors in the frame $(\dot{c}, X)$ as column vectors of a matrix. In this way, we get a loop $\sigma(\vec{c})$ in $SO(n)$. Two framed circles $\vec{b}$ and $\vec{c}$ are framed isotopic if and only if $\sigma(\vec{b}\,) \simeq \sigma(\vec{c})$.

5.2. *Paths and loops in $SO(n)$.* Let $\mathbf{I}$ denote the identity matrix.

DEFINITION 5.1. If $n = 4k+1$ define $\mathbf{A}(n) \in SO(n)$ and $\mathbf{B}(n-1) \in SO(n-1)$ by

$$\mathbf{B}(n-1) = \begin{bmatrix} 0 & \mathbf{I}_{2k} \\ \mathbf{I}_{2k} & 0 \end{bmatrix}, \quad \mathbf{A}(n) = \begin{bmatrix} 1 & \\ & \mathbf{B}(n-1) \end{bmatrix}.$$

If $n = 4k + 2$ define $\mathbf{A}(n) \in SO(n)$ and $\mathbf{B}(n-1) \in SO(n-1)$ by

$$\mathbf{B}(n-1) = \begin{bmatrix} 1 & & \\ & 0 & \mathbf{I}_{2k} \\ & \mathbf{I}_{2k} & 0 \end{bmatrix}, \quad \mathbf{A}(n) = \begin{bmatrix} 1 & \\ & \mathbf{B}(n-1) \end{bmatrix}.$$

If $n = 4k$ define $\mathbf{A}(n) \in SO(n)$ and $\mathbf{B}(n-1) \in SO(n-1)$ by

$$\mathbf{B}(n-1) = \begin{bmatrix} -1 & & \\ & 0 & \mathbf{I}_{2k-1} \\ & \mathbf{I}_{2k-1} & 0 \end{bmatrix}, \quad \mathbf{A}(n) = \begin{bmatrix} 1 & \\ & \mathbf{B}(n-1) \end{bmatrix}.$$

The group $\{\mathbf{I}, \mathbf{A}(n)\}$ acts on $SO(n)$ from the right. Denote the corresponding quotient space by $SO(n)/\mathbf{A}(n)$. That is $SO(n)/\mathbf{A}(n)$ is obtained from $SO(n)$ by the identification $R \sim R\mathbf{A}(n)$, $R \in SO(n)$.

Proposition 5.2. *Let $n = 4k + r$, $k \geq 1$ and $r = 0, 1, 2$. Then*

$$\pi_1(SO(n)/\mathbf{A}(n)) = \begin{cases} \mathsf{Z}_4 & \text{if } k \text{ is odd,} \\ \mathsf{Z}_2 \oplus \mathsf{Z}_2 & \text{if } k \text{ is even.} \end{cases}$$

Proof. The exact homotopy sequence of the fibration

$$\mathsf{Z}_2 \to SO(n) \xrightarrow{p} SO(n)/\mathbf{A}(n)$$

gives the exact sequence

$$0 \to \pi_1(SO(n)) \xrightarrow{p_*} \pi_1(SO(n)/\mathbf{A}(n)) \to \pi_0(\mathsf{Z}_2) \to 0.$$

Consider a loop $\sigma\colon [0,1] \to SO(n)/\mathbf{A}(n)$ such that $[\sigma] \notin p_*(\pi_1(SO(n)))$. Assume that $\sigma(0) = p(\mathbf{I})$. When lifting σ to $SO(n)$ we obtain a path $\lambda\colon [0,1] \to SO(n)$ such that $\lambda(0) = \mathbf{I}$ and $\lambda(1) = \mathbf{A}(n)$. There are two homotopy classes relative ∂I of such paths. Hence, $\pi_1(SO(n)/\mathbf{A}(n))$ has 4 elements. The loop $\sigma * \sigma$ lifts to $\lambda * \lambda\mathbf{A}(n)$, where $(\lambda\mathbf{A}(n))(t) = \lambda(t)\mathbf{A}(n)$. (Here $*$ denotes path concatenation. Concatenation of two paths is defined if the first one ends where the second one begins.) Now, if

$$[\lambda * \lambda\mathbf{A}(n)] = \begin{cases} 1 \in \pi_1(SO(n)) & \text{then } \pi_1(SO(n)/\mathbf{A}(n)) \cong \mathsf{Z}_4, \\ 0 \in \pi_1(SO(n)) & \text{then } \pi_1(SO(n)/\mathbf{A}(n)) \cong \mathsf{Z}_2 \oplus \mathsf{Z}_2. \end{cases}$$

We calculate the homotopy class of $\lambda * \lambda\mathbf{A}(n)$ by lifting it to the universal covering $Spin(n)$ of $SO(n)$. To this end, note that $\lambda\colon [0,1] \to SO(n)$ lifts to a path

$$\widetilde{\lambda}\colon [0,1] \to Spin(n) \quad \text{with } \widetilde{\lambda}(0) = +1, \widetilde{\lambda}(1) = \widetilde{\mathbf{A}}(n).$$

Hence, the loop $\lambda * \lambda\mathbf{A}(n)$ lifts to the path $\widetilde{\lambda} * \widetilde{\lambda}\widetilde{\mathbf{A}}(n)$, which ends at $\widetilde{\mathbf{A}}(n)\widetilde{\mathbf{A}}(n)$. Now, if

$$\widetilde{\mathbf{A}}(n)\widetilde{\mathbf{A}}(n) = \begin{cases} +1 & \text{then } [\lambda * \lambda\mathbf{A}(n)] = 0 \in \pi_1(SO(n)), \\ -1 & \text{then } [\lambda * \lambda\mathbf{A}(n)] = 1 \in \pi_1(SO(n)), \end{cases}$$

so it remains to compute this product.

Let $(e_1, \dots, e_n)$ denote the standard basis in R^n. If $n = 4k+1$ then $\mathbf{A}(n)$ is a composition of reflections in the hyperplanes with unit normals $f_i = \frac{1}{\sqrt{2}}(e_{i+1} - e_{2k+i+1})$ and thus

$$\widetilde{\mathbf{A}}(n) = \pm f_1 \dots f_{2k}, \quad \text{when } n = 4k+1.$$

Similarly, we see that

$$\widetilde{\mathbf{A}}(n) = \pm f_1 \dots f_{2k}, \quad \text{when } n = 4k+2,$$
$$\widetilde{\mathbf{A}}(n) = \pm e_2 f_1 \dots f_{2k-1}, \quad \text{when } n = 4k,$$

where $f_i = \frac{1}{\sqrt{2}}(e_{i+2} - e_{2k+i+2})$. Hence, $\widetilde{\mathbf{A}}(n)\widetilde{\mathbf{A}}(n)$ is, in all the cases, a product of the form $u_1 \dots u_{2k} u_1 \dots u_{2k}$, where the u_i are mutually orthonormal unit vectors. It is straightforward to check that

$$u_1 \dots u_{2k} u_1 \dots u_{2k} = (-1)^k.$$

5.3. *Twist framed circles.* Let $c \subset \mathsf{R}^n$, $n = 4k + r$, $r = 0, 1, 2$, be an embedded circle. Let $p \colon [0,1] \to c$ be a parameterization of c. Assume that R^n is oriented.

DEFINITION 5.3. If there is a smooth field of frames $X(p(t))$ in $N(c \subset \mathsf{R}^n)$ such that $X(p(1)) = X(p(0))\mathbf{B}(n-1)$ we say that c is a *twist framed circle* and denote it by (c, q, X), where $q \colon [0,1] \to c$ is given by either $q(t) = p(t)$ or $q(t) = p(1-t)$. The choice being made so that the orientation given by $(\dot{q}, X)$ agrees with that of R^n.

Let (c, q, X) be a twist framed circle. Then $(\dot{q}, X)$ gives a framing of $T\mathsf{R}^n$ at each point of c. Letting the vectors in this frame be column vectors in a matrix the twist framing induces, after orthonormalization, a loop in $SO(n)/\mathbf{A}(n)$. The homotopy class of this loop will be denoted $[c, q, X]$.

Let $V_{n-1}(T\mathsf{R}^n)$ denote the space of $(n-1)$-frames in $T\mathsf{R}^n$.

DEFINITION 5.4. Let (c, q, X) and (b, p, Y) be twist framed circles in R^n. We say that these are *twist framed isotopic* if there is an isotopy

$$h_s \colon [0,1] \to V_{n-1}(T\mathsf{R}^n),$$

such that $h_0 = (q, X)$, $h_1 = (p, Y)$ and for each fixed s,

$$(\pi h_s([0,1]), \pi h_s, h_s)$$

is a twist framed circle. Here $\pi\colon V_{n-1}(T\mathsf{R}^n) \to \mathsf{R}^n$ is the projection.

PROPOSITION 5.5. *Let (c, q, X) and (b, p, Y) be twist framed circles in R^n, $n \geq 4$. Then (c, q, X) and (b, q, Y) are twist framed isotopic if and only if $[c, q, X] = [b, p, Y]$.*

PROOF. Twist framed isotopy induces homotopy in $SO(n)/\mathbf{A}(n)$. This proves the only if part. Assume that $[c, q, X] = [b, p, Y]$. Choose a diffeotopy K_s of R^n such that $K_1 q = p$ and $dK_1(X(q(0))) = Y(p(0))$. The only if part shows that $[b, K_1 q, dK_1 X] = [b, p, Y]$. Hence, the paths $(dK_1\dot{q}, dK_1 X)$ and $(\dot{p}, Y)$ in $SO(n)$ are homotopic with endpoints fixed. It is now easy to complete $(K_s, dK_s X)$ to a twist framed isotopy, moving the frame only.

REMARK 5.6. The proof of Proposition 5.5 shows that if two twist framed circles are twist framed isotopic then there is an ambient diffeotopy of R^n moving one of them to the other.

5.4. *Operations on twist framed circles.* Let (c, q, X) be a twist framed circle in R^n. Letting $\mathbf{B}(n-1)$ act on the framing we get another twist framed circle $(c, q, X\mathbf{B}(n-1))$. Using Proposition 5.5 we get:

PROPOSITION 5.7. *(c, q, X) is twist framed isotopic to $(c, q, X\mathbf{B}(n-1))$.*

If n is even then there is another natural operation on twist framed circles in R^n. It is associated with reversing the parameterization.

DEFINITION 5.8. Let n be even. Define $\mathbf{C}(n-1) \in O(n-1)$ and $\mathbf{C}(n) \in SO(n)$ by

$$\mathbf{C}(n-1) = \begin{bmatrix} -1 & \\ & \mathbf{I}_{n-2} \end{bmatrix}, \quad \mathbf{C}(n) = \begin{bmatrix} -1 & \\ & \mathbf{C}(n-1) \end{bmatrix}.$$

Let (c, q, X) be a twist framed circle in R^n, n even. We can define another twist framed circle (c, q', X') by

$$q'(t) = q(1-t), \quad X'(q'(t)) = X(q(1-t))\mathbf{C}(n-1).$$

LEMMA 5.9. *If $n = 4k+2$ then $[c, q', X'] = -[c, q, X]$.*

PROOF. Let $\lambda\colon [0,1] \to SO(n)$ be the path induced by (c, q, X) and $\sigma = p\lambda$ where $p\colon SO(n) \to SO(n)/\mathbf{A}(n)$. Then the path induced by (c, q', X') is $\lambda^{-1}\mathbf{C}(n)$. Where λ^{-1} denotes λ transversed backwards. A uniform rotation through an angle π in the plane of the first two vectors in the framing $(\dot{q}', X')$ gives a homotopy from $p(\lambda^{-1}\mathbf{C}(n))$ to σ^{-1}.

LEMMA 5.10. *If $n = 4k$ then $[c, q', X'] \neq -[c, q, X]$.*

PROOF. We have exactly as in the proof of Lemma 5.9 with the same notation as there the paths $\lambda, \lambda^{-1}\mathbf{C}(n) \colon [0,1] \to SO(n)$ induced by (c, q, X) and (c, q', X') respectively. Without loss of generality we may assume that $\lambda(0) = \mathbf{I}$ (and thus $\lambda(1) = \mathbf{A}(n)$). Choose a path $\gamma \colon [0,1] \to SO(n)$, with $\gamma(0) = \mathbf{I}$ and $\gamma(1) = \mathbf{C}(n)$. Then, clearly

$$p(\lambda^{-1}\mathbf{C}(n)) \simeq p(\gamma\mathbf{A}(n) * \lambda^{-1}\mathbf{C}(n) * \gamma^{-1}).$$

Note that $\mathbf{C}(n)\mathbf{A}(n) = \mathbf{A}(n)\mathbf{C}(n)$. To prove the lemma it is enough to prove that the loop

$$\delta = \lambda * \gamma\mathbf{A}(n) * \lambda^{-1}\mathbf{C}(n) * \gamma^{-1},$$

is nontrivial in $SO(n)$. Since this implies that

$$[p\lambda] + [p\lambda^{-1}\mathbf{C}(n)] \neq 0 \text{ in } \pi_1(SO(n)/\mathbf{A}(n)).$$

We prove this by lifting δ to $Spin(n)$. We have the lifts $\pm\widetilde{\lambda}$ and $\pm\widetilde{\gamma}$ of λ and γ. We assume that $\widetilde{\lambda}(0) = +1 = \widetilde{\gamma}(0)$. Then $\widetilde{\lambda}(1) = \widetilde{\mathbf{A}}(n)$ and $\widetilde{\gamma}(1) = \pm e_1e_2$. Our lift of the loop is the following. Start at $+1$, follow $\widetilde{\lambda}$ to $\widetilde{\mathbf{A}}(n)$, follow $\widetilde{\gamma}\widetilde{\mathbf{A}}(n)$ to $\widetilde{\gamma}(1)\widetilde{\mathbf{A}}(n)$. Now, $\widetilde{\mathbf{A}}(n) = \pm e_2f_1\dots f_{2k-1}$ by Proposition 5.2. Hence,

$$\begin{aligned}\widetilde{\gamma}(1)\widetilde{\mathbf{A}}(n) &= \pm e_1e_2(e_2f_1\dots f_{2k-1}) = \pm f_1\dots f_{2k-1}e_1\\ &= \pm(-e_2f_1\dots f_{2k-1}e_1e_2) = -\widetilde{\mathbf{A}}(n)\widetilde{\gamma}(1).\end{aligned}$$

So our lifts now follows $-\widetilde{\lambda}^{-1}\widetilde{\gamma}(1)$ until it reaches $-\widetilde{\gamma}(1)$, then it follows $-\widetilde{\gamma}^{-1}$ and ends at -1. This proves that δ is nontrival.

PROPOSITION 5.11. *Let $n = 4k+2$ and (c, q, X) be a twist framed circle in R^n.*

(a) *If k is* even *then (c, q', X') is twist framed isotopic to (c, q, X).*
(b) *If k is* odd *then (c, q', X') is* not *twist framed isotopic to (c, q, X).*

PROPOSITION 5.12. *Let $n = 4k$ and (c, q, X) be a twist framed circle in R^n.*

(a) *If k is* even *then (c, q', X') is* not *twist framed isotopic to (c, q, X).*
(b) *If k is* odd *then (c, q', X') is twist framed isotopic to (c, q, X).*

6. **1-dimensional self intersection**

A generic immersion $f\colon V^v \to W^{2v-1}$ has 1-dimensional self intersection. Since any closed connected 1-manifold is a circle, Proposition 2.1 shows that if V is closed then M_f is a union of circles. If V and W are orientable and v is even then, by Proposition 2.5, $M_f = M_f''$ (cf. Definition 2.2).

6.1. *Definition of the invariants.* Let $f\colon V^v \to W^{2v-1}$ be a generic immersion. Assume that V and W are spin manifolds with spin structures ξ_V and ξ_W, respectively. Consider a component $c \subset M_f''$. Proposition 2.10 shows that there is an induced spin structure on $c \approx S^1$.

Definition 6.1. If $c \subset M_f''$ is a component of M_f, define

$$s(c) = \begin{cases} 1 & \text{if the spin structure induced on } c \text{ is trivial,} \\ 0 & \text{if the spin structure induced on } c \text{ is the Lie group one.} \end{cases}$$

Remark 6.2. We note that

$$s(c) = \langle \xi_c, [\vec{c}\,] \rangle = \langle (f^{-1})^*(\xi_V), [\vec{c}_V^{\,1}] \rangle + \langle (f^{-1})^*(\xi_V), [\vec{c}_V^{\,2}] \rangle + \langle \xi_W, [\vec{c}_W] \rangle,$$

where $\vec{c}_V^{\,i}$, $i = 1, 2$ refer to the extensions of the tangent framing of c to the different sheets of $f(V)$ meeting along c (cf. Remark 2.13).

Lemma 6.3. *Let $f\colon V^v \to W^{2v-1}$ be a generic immersion. Assume that V and W are spin manifolds. Let $c \subset M_f''$ be a component of M_f. If $K_t\colon W \to W$ and $G_t\colon V \to V$ are diffeotopies then $K_1 c$ is a component of $M''_{K_1 f G_1}$ and $s(c) = s(K_1 c)$.*

Proof. Let $b = K_1 c$. We need to compute

$$\begin{aligned} s(b) = \langle \xi_b, [\vec{b}\,] \rangle &= \langle ((K_1 f G_1)^{-1})^*(\xi_V), [\vec{b}_V^{\,1}] \rangle \\ &+ \langle ((K_1 f G_1)^{-1})^*(\xi_V), [\vec{b}_V^{\,2}] \rangle + \langle \xi_W, [\vec{b}_W] \rangle, \end{aligned}$$

Remark 2.12 allows us to take the extensions of the framing of b as those induced by dK_1 acting on the extensions of the framing of c. We have, for $i = 1, 2$,

$$\begin{aligned} \langle ((K_1 f G_1)^{-1})^*(\xi_V), [\vec{b}_V^{\,i}] \rangle &= \langle (K_1^{-1})^*(f^{-1})^*(G_1^{-1})^*(\xi_V), [\vec{b}_V^{\,i}] \rangle \\ &= \langle (K_1^{-1})^*(f^{-1})^*(\xi_V), [\vec{b}_V^{\,i}] \rangle \\ &= \langle (f^{-1})^*(\xi_V), (K_1^{-1})_*(K_1)_*[\vec{c}_V^{\,i}] \rangle \\ &= \langle (f^{-1})^*(\xi_V), [\vec{c}_V^{\,i}] \rangle, \end{aligned}$$

where the second equality follows from the fact that dG_1 is homotopic to identity. Also,

$$\langle \xi_W, (K_1)_*[\vec{c}_W]\rangle = \langle \xi_W, [\vec{c}_W]\rangle,$$

since dK_1 is homotopic to the identity. It follows that $s(b) = s(c)$.

When describing the invariants associated to nontrivially covered components we restrict ourselves to the case when the target manifold is Euclidean space. Let $f\colon V^v \to \mathsf{R}^{2v-1}$ be a generic immersion. Assume that v is odd, R^{2v-1} is oriented and that V is equipped with a spin structure ξ_V. We shall construct a functional ω_f on components of M'_f which takes values in $\pi_1(SO(2v-1)/\mathbf{A}(2v-1))$. Let $c \subset M'_f$ be a component of M_f. Since $v-1$ is even we know by Proposition 2.3 that there is an induced orientation of c. Let $f^{-1}(c) = c_V$. Choose a parameterization $q\colon [0,1] \to c$ of c, such that the orientation given by this parameterization agrees with that induced on c. Then q lifts to two paths $q_i : [0,1] \to V$ such that $q_1(0) = q_2(1)$, $q_1(1) = q_2(0)$ and $q_1 * q_2$ is a parameterization of c_V. Choose a framing Y of $N(c_V \subset V)$ such that, completing Y with the tangent vector of c_V coming from the parameterization $q_1 * q_2$ we get a framing of $TV|c_V$, oriented coherently with the orientation of V. We denote the framed curve obtained in this way by $\vec{c}_V$. Let

$$X(q(t)) = (df\, Y(q_1(t)), df\, Y(q_2(t)))\,.$$

Then (c, q, X) is a twist framed circle.

Let $u \in \pi_1(SO(n))$, $n \geq 3$, denote the nontrivial element. If $p : SO(n) \to SO(n)/\mathbf{A}(n)$ then $p_*(u) \in \pi_1(SO(n)/\mathbf{A}(n))$ is an element of order 2. For $x \in \mathsf{Z}_2$ let $x \cdot p_*(u)$ denote the value at x of the homomorphism $\mathsf{Z}_2 \to \pi_1(SO(n)/\mathbf{A}(n))$ taking 1 to $p_*(u)$.

Definition 6.4. $\omega_f(c) = \langle \xi_V, [\vec{c}_V]\rangle \cdot p_*(u) + [c, q, X] \in \pi_1(SO(2v-1)/\mathbf{A}(2v-1))$.

Lemma 6.5. *The value of $\omega_f(c)$ does neither depend on the choice of parameterization nor on the choice of normal framing of c_V.*

Proof. Let q' and q be two parameterizations. Since their orientations must agree they are homotopic. Hence also the induced parameterizations of c_V are homotopic so neither of the terms on the right hand side in the definition of $\omega_f(c)$ change. Let Y_1 and Y_2 be framings of $N(c \subset V)$ giving rise to framed curves $\vec{c}_V^{\,1}$ and $\vec{c}_V^{\,2}$ respectively. These are framed isotopic, moving the frame only, if and only if

$$\langle \xi_V, [\vec{c}_V^{\,1}]\rangle = \langle \xi_V, [\vec{c}_V^{\,2}]\rangle.$$

Such a framed isotopy clearly gives rise to a twist framed isotopy between (c, q, X_1) and (c, q, X_2) where X_i is the framing induced by df acting on Y_i. Hence,

$$\langle \xi_V, [\vec{c}_V^{\,1}] \rangle = \langle \xi_V, [\vec{c}_V^{\,2}] \rangle \text{ implies } [c, q, X_1] = [c, q, X_2],$$

by Proposition 5.5. Now, if

$$\langle \xi_V, [\vec{c}_V^{\,1}] \rangle \neq \langle \xi_V, [\vec{c}_V^{\,2}] \rangle,$$

then we can change the framing Y_2 by a locally supported rotation of the frame, such that $\langle \xi_V, [\vec{c}_V^{\,2}] \rangle$ changes by 1 and $[c, q, X_2]$ by addition of $p_*(u)$.

Lemma 6.6. *Let $K_t \colon \mathsf{R}^{2v-1} \to \mathsf{R}^{2v-1}$ and $G_t \colon V \to V$ be diffeotopies. Then $K_1 c \subset M'_{K_1 f G_1}$ is a component of $M_{K_1 f G_1}$ and $\omega_{K_1 f G_1}(K_1 c) = \omega_f(c)$.*

Proof. It follows from Lemma 6.5 that we can assume that the twist framing on $K_1 c$ is induced from that of c by appropriate actions of diffeotopies. That the first term on the right hand side in the definition of $\omega_f(c)$ does not change follows from the fact that G_t is a diffeotopy. That the second does not change follows from the fact that the diffeotopies gives rise to a twist framed isotopy and Proposition 5.5.

For $n = 4k$ fix isomorphisms

$$\phi \colon \pi_1(SO(n)/\mathbf{A}(n)) \to \begin{cases} \mathsf{Z}_4 & \text{if } k \text{ is odd,} \\ \mathsf{Z}_2 \oplus \mathsf{Z}_2 & \text{if } k \text{ is even,} \end{cases}$$

such that $\phi(p_*(u)) = 2$, when k is odd and $\phi(p_*(u)) = (1, 1)$, when k is even.

Definition 6.7.

(a) If $v \equiv 1 \mod 4$ and $c \subset M'_f$ is a component of M_f, define

$$s(c) = \begin{cases} 1 & \text{if } \phi(\omega_f(c)) = (1, 0), \\ 0 & \text{if } \phi(\omega_f(c)) = (0, 1). \end{cases}$$

(b) If $v \equiv 3 \mod 4$ and $c \subset M'_f$ is a component of M_f, define

$$s(c) = \begin{cases} 1 & \text{if } \phi(\omega_f(c)) = 1, \\ 0 & \text{if } \phi(\omega_f(c)) = 3. \end{cases}$$

We now turn to the general case: Let $f : V^v \to \mathsf{R}^{2v-1}$ be a generic immersion. Assume that V^v is equipped with a spin structure and that R^{2v-1} is oriented. Let c be a component of M_f.

Lemma 6.8. *Let* $K_t\colon \mathsf{R}^{2v-1}\to\mathsf{R}^{2v-1}$ *and* $G_t\colon V\to V$ *be diffeotopies. Then* K_1c *is a component of* $M_{K_1fG_1}$, $K_1c\subset M'_{K_1fG_1}$ *if and only if* $c\subset M'_f$ *and* $s(c)=s(K_1c)$.

Proof. This follows from Lemmas 6.3, 6.6.

6.2. *Proof of Theorem 1.4.* The assumption, in Theorem 1.4, that V is 2-connected implies that there is a unique spin structure on V. Hence, s is well defined if we fix orientations on V and R^{2v-1}. We assume in the proof below that this is done.

The only if part follows from the fact that the diffeomorphism type of the self intersection is preserved under diffeotopy equivalence and Lemma 6.8. Assume that c and b are components of M_f and M_h respectively. Let $f^{-1}(c)=\widetilde{c}$, $h^{-1}(b)=\widetilde{b}$ and assume that $\widetilde{c}$ and $\widetilde{b}$ have the same number of components. Choose diffeotopies $K'_t\colon\mathsf{R}^{2v-1}\to\mathsf{R}^{2v-1}$ and $G_t\colon V\to V$ such that $K'_1c=b$, $G_1\widetilde{b}=\widetilde{c}$, $K'_1fG_1|\widetilde{b}=h|\widetilde{b}$ and such that the induced orientations agree. By Lemma 2.14 we can, up to diffeotopy equivalence, locally replace $K'_1fG_1=f'$ and h by their normal derivatives on $N(\widetilde{b}\subset V)$.

First, suppose that $c\subset M''_f$, $b\subset M''_h$ and $s(c)=s(b)$. Write $\widetilde{b}=b_1\cup b_2$, where b_1 and b_2 are the components of $\widetilde{b}$. Pick a framing of $N(b_1\cup b_2\subset V)$. Then f' and h induce framings of $N(b\subset\mathsf{R}^{2v-1})$. Consider the formula in Remark 6.2, associated to f' and h respectively. Since $s(b)=s(c)$ Lemma 6.3 tells us that the terms $\langle\xi_b,[\vec{b}\,]\rangle$, associated to f' and h respectively, agree. Moreover,

$$\langle({f'}^{-1})^*\xi_V,[df'(\vec{b}^{\,i}_V)]\rangle=\langle\xi_V,[\vec{b}^{\,i}_V]\rangle=\langle(h^{-1})^*\xi_V,[dh(\vec{b}^{\,i}_V)]\rangle,\ \text{for } i=1,2.$$

Hence, the terms of the form $\langle\xi_{\mathsf{R}^{2v-1}},[\vec{b}_{\mathsf{R}^{2v-1}}]\rangle$ agree as well. But this means that there is a diffeotopy of R^{2v-1} fixed on b and supported in a neighborhood of b taking the framing induced by f' to that induced by h.

Secondly, suppose that $c\subset M'_f$, $b\subset M'_h$ and $s(c)=s(b)$. Pick a framing of $N(\widetilde{b}\subset V)$. Then f' and h induce twist framings of $N(b\subset\mathsf{R}^{2v-1})$. Consider the formula in Definition 6.4, associated to f' and h respectively. Since $s(c)=s(b)$, Lemma 6.6 tells us that $\omega_{f'}(b)=\omega_h(b)$. Note that the terms $\langle\xi_V,[\vec{b}_V]\rangle$ are independent of the immersions. Hence, the terms of the form $[b,q,X]$, associated to f' and h respectively, must agree. Then, by Remark 5.6, there is a diffeotopy of R^{2v-1} fixed on b taking the twist framing induced by f' to that induced by h. Repeating the above construction for all components of M_h we find diffeotopies $K_t\colon\mathsf{R}^{2v-1}\to\mathsf{R}^{2v-1}$ and $L_t\colon V\to V$ such that $K_1fL_1|W=h|W$, where

W is a neighborhood of $\widetilde{M}_h$. An application of Theorem 1.2 finishes the proof.

7. 2-dimensional self intersection

7.1. *The first two invariants.* Let $f: V^v \to W^{2v-2}$ be a generic immersion. If V is closed, Proposition 2.1 shows that M_f is a closed surface. Assume that V and W are orientable and let F be a component of M_f. The topological type of both F and $f^{-1}(F) = \widetilde{F}$ is determined by the genus $g(F)$ of F and the number of components of $\widetilde{F}$: Denote the surface obtained by connected sum of k tori by Σ_k and that obtained by connected sum of k projective planes by Γ_k. Let $g = g(F)$. Proposition 2.1 gives:

$$F \subset M_f'' \text{ implies } F \approx \Sigma_g \text{ and } \widetilde{F} \approx \Sigma_g \cup \Sigma_g.$$

Propositions 2.3, 2.5 give:

(a) If v is even then $F \subset M_f'$ implies $F \approx \Sigma_g$ and $\widetilde{F} \approx \Sigma_{2g-1}$.

(b) If v is odd then $F \subset M_f'$ implies $F \approx \Gamma_g$ and $\widetilde{F} \approx \Sigma_{g-1}$.

7.2. *Disconnected double locus and spin structure.* A spin structure ξ on a surface F induces a quadratic function $q_\xi\colon H_1(F;\mathsf{Z}_2) \to \mathsf{Z}_2$, that is, if $a, b \in H_1(F;\mathsf{Z}_2)$ then $q_\xi(a+b) = q_\xi(a) + q_\xi(b) + a \cdot b$, where $a \cdot b$ denotes the intersection pairing. Let $z \in H_1(SO(TF);\mathsf{Z}_2)$ denote the fiber class. Let $a \in H_1(F;\mathsf{Z}_2)$. Then we can compute $q_\xi(a)$ as follows. Pick a simple closed curve α in F such that $[\alpha] = a$, where $[\alpha]$ denotes the Z_2-homology class of α. Let $\vec{\alpha}$ denote α framed with its unit tangent vector, an orientation of F picks a normal vector to complete it to a framing of TF along α. We get the element $[\vec{\alpha}\,] \in H_1(SO(TF);\mathsf{Z}_2)$ and

$$q_\xi(a) = \langle \xi, [\vec{\alpha}\,] + z \rangle, \tag{†}$$

see [7] for reference. Let $(L^{2g}, \cdot)$ be a symplectic vector space over Z_2. Let $q\colon L \to \mathsf{Z}_2$ be a quadratic function on L. The Arf invariant of q is defined by

$$\operatorname{Arf}(q) = \sum_i q(a_i) q(b_i),$$

where $\{a_i, b_i\}_{i=1}^g$ is a symplectic basis of L. If q_1 and q_2 are quadratic functions on L then there is a symplectomorphism $h\colon L \to L$ such that $q_2 = q_1 h$ if and only if $\operatorname{Arf}(q_1) = \operatorname{Arf}(q_2)$.

Let $f : V^v \to W^{2v-2}$ be a generic immersion and let $F \subset M_f''$ be a component of M_f. Assume that there are spin structures ξ_V and ξ_W on V and W respectively. By Proposition 2.10, there is an induced spin structure ξ_F on F. The quadratic function on $H_1(F;\mathsf{Z}_2)$ induced by ξ_F will be denoted $q_{f,F}$, where f refers to the map.

DEFINITION 7.1. If $F \subset M''_f$, define $s(F) = \mathrm{Arf}(q_{f,F}) \in \mathsf{Z}_2$.

LEMMA 7.2. *Let c be a simple closed curve in $F \subset M''_f$. If $K_t\colon W \to W$ and $G_t\colon V \to V$ are diffeotopies then K_1c is a simple closed curve in K_1F, which is a component of $M''_{K_1fG_1}$ and*

$$q_{K_1fG_1,K_1F}([K_1c]) = q_{f,F}([c]).$$

PROOF. By formula (†) above it is enough to check that

$$\langle \xi_{K_1F}, [K_1\vec{c}]\rangle = \langle \xi_F, [\vec{c}]\rangle,$$

this follows from the calculation in the proof of Lemma 6.3.

LEMMA 7.3. *Let $f, h : V^v \to W^{2v-2}$ be two diffeotopy equivalent generic immersions. Let F be the component of M_f corresponding, under the diffeotopy equivalence, to the component G of M_h then $g(F) = g(G)$. Assume that $F \subset M''_f$, then $G \subset M''_h$ and $s(F) = s(G)$.*

PROOF. This is a direct consequence of Lemma 7.2 and the fact that if $\{a_i, b_i\}$ is a symplectic basis for $H_1(F;\mathsf{Z}_2)$ then $\{K_1a_i, K_1b_i\}$ is a symplectic basis for $H_1(K_1F;\mathsf{Z}_2)$.

7.3. *Connected double locus and a functional on curves.* We restrict ourselves to the case when the target manifold is Euclidean space. Let $f\colon V^v \to \mathsf{R}^{2v-2}$ be a generic immersion. Let F be a component of M'_f. Assume that V is equipped with a spin structure ξ_V and that R^{2v-2} is oriented. We shall construct a functional ω_f on (oriented) immersed, closed curves in F, which takes values in $\pi_1(SO(2v-2)/\mathbf{A}(2v-2))$.

Let $u \in \pi_1(SO(n))$ denote the nontrivial element. Let $p\colon SO(n) \to SO(n)/\mathbf{A}(n)$ be the quotient map. Then $p_*(u) \in \pi_1(SO(n)/\mathbf{A}(n))$ is an element of order 2.

Let $c \subset F$ be an immersed curve whose only self intersections are transverse double points. The inverse image $f^{-1}(c) = \widetilde{c}$ may be either a connected or a disconnected subset of $f^{-1}(F) = \widetilde{F}$. Choose a parameterization $r\colon [0,1] \to c$.

(a) When $\widetilde{c}$ is disconnected r lifts to two parameterizations $r_i\colon [0,1] \to c_i$, $i = 1,2$, where c_1 and c_2 are disjoint immersed closed curves in $\widetilde{F}$. Choose framings Y_i of $N(\widetilde{F} \subset V)|c_i$. Adding the tangent $\dot{r}_i$ and the normal n_i to Y_i, the normal chosen so that $(\dot{r}_i, n_i, Y_i)$ is positively oriented, we get two framed curves $\vec{c}_i$, $i = 1,2$. Let $[\vec{c}_V]$ denote the homology class in $H_1(SO(TV);\mathsf{Z}_2)$ given by

$$[\vec{c}_V] = [\vec{c}_1] + [\vec{c}_2],$$

where $[\vec{c}_i] \in H_1(SO(TV); \mathsf{Z}_2)$ is obtained as usual, by viewing a framed curve as a curve in the principal bundle. Now, f induces the framing

$$X(p(t)) = (\dot{r}, n(r(t)), df\, Y_1(r_1(t)), df\, Y_2(r_2(t)))\,,$$

of $T\mathsf{R}^{2v-2}|c$, where n is chosen so that the framing gives the orientation of R^{2v-2}. This framing induces a loop in $SO(2v-2)$, composing with the quotient map $p\colon SO(2v-2) \to SO(2v-2)/\mathbf{A}(2v-2)$ we get an element $[c, r, X] \in \pi_1(SO(2v-2)/\mathbf{A}(2v-2))$.

(b) When $\widetilde{c}$ is connected, r lifts to $r_i\colon [0,1] \to \widetilde{F}$, $i = 1, 2$ such that $r_1(0) = r_2(1)$, $r_1(1) = r_2(0)$ and $r_1 * r_2$ is a parameterization of $\widetilde{c}$. Choose a framing Y of $N(\widetilde{F} \subset V)|\widetilde{c}$. Adding the tangent vector given by the parameterization and the appropriate normal vector we get a framing of $TV|\widetilde{c}$. This framed curve gives an element $[\vec{c}_V] \in H_1(SO(TV); \mathsf{Z}_2)$. Now, f induces the twist framing

$$X(p(t)) = (\dot{r}, n(r(t)), df\, Y_1(r_1(t)), df\, Y_2(r_2(t)))\,,$$

of c, where n is an appropriately chosen normal vector. As in Section 5.3 we get $[c, r, X] \in \pi_1(SO(2v-2)/\mathbf{A}(2v-2))$.

Remark 7.4. If the framed curve constructed in (a) above is denoted $\vec{c}$ and ξ denotes the unique spin structure on R^{2v-2} then $[c, r, X] = \langle \xi, [\vec{c}] \rangle \cdot p_*(u)$.

Definition 7.5.

$$\omega_f(c, r) = \langle \xi_V, [\vec{c}_V] \rangle \cdot p_*(u) + [c, r, X] + p_*(u) \in \pi_1(SO(2v-2)/\mathbf{A}(2v-2)).$$

Lemma 7.6. *The value of $\omega_f(c, r)$ does neither depend on the choice of framing of $N(\widetilde{F} \subset V)|\widetilde{c}$ nor on the numbering of the liftings of r.*

Proof. The first statement follows in the case of induced twist framing exactly as the corresponding statement in Lemma 6.5. In the case of framing, see Remark 2.12. The second statement follows immediately when there is an induced framing. When there is an induced twist framing it follows from Proposition 5.7.

Lemma 7.7. *Let $f : V^v \to \mathsf{R}^{2v-2}$ be a generic immersion. Let $F \subset M'_f$ be a component and let $c \subset F$ be a closed curve parameterized by r. If $K_t : \mathsf{R}^{2v-2} \to \mathsf{R}^{2v-2}$ and $G_t : V \to V$ are diffeotopies then $K_1 F$ is a component of $M'_{K_1 f G_1}$, $K_1 c \subset K_1 F$ is a closed curve with parameterization $K_1 r$ and*

$$\omega_{K_1 f G_1}(K_1 c, K_1 r) = \omega_f(c, r)$$

Proof. Let $K_1 c = b$. By Lemma 7.6, we can take the framing of $\widetilde{b}$ to be induced by G_1 acting on the framing of $\widetilde{c}$ and similarly for the framing or twist framing induced on b. Then $[\vec{b}_V] = [\vec{c}_V]$, since the curve or curves are framed isotopic. If Y denotes the framing or twist framing of b then $[c, r, X] = [b, K_1 r, Y]$ for the same reason.

Lemma 7.8. *If r and r' are homotopic parameterizations of c then*

$$\omega_f(c, r) = \omega_f(c, r').$$

Proof. An homotopy between r and r' induces homotopies everywhere in the construction preceding Definition 7.5.

There are two nonhomotopic parameterizations of a curve as above. If $r\colon [0,1] \to c$ is a parameterization and $r'(t) = r(1-t)$ then r and r' represents different homotopy classes.

Lemma 7.9. *Let c, r and r' be as described above. Let v denote the dimension of the source space of the generic immersion f.*

(a) *If $v \equiv 0 \mod 4$ or $v \equiv 1 \mod 4$ and $\widetilde{c}$ is connected then*

$$\omega_f(c, r') = \omega_f(c, r) + p_*(u).$$

(b) *If $v \equiv 2 \mod 4$ or $v \equiv 3 \mod 4$ or $\widetilde{c}$ is disconnected then*

$$\omega_f(c, r') = \omega_f(c, r).$$

Proof. This is follows from Lemmas 5.9, 5.10 and Remark 7.4.

Because of Lemmas 7.8, 7.9, we write $\omega_f(c)$, dropping the parameterization, and regard c as an immersed curve, if $v \equiv 2 \mod 4$ or $v \equiv 3 \mod 4$, or as an oriented immersed curve, if $v \equiv 0 \mod 4$ or $v \equiv 1 \mod 4$.

Lemma 7.10. *Let $c \subset F$ be a curve that bounds a disk in F then $\omega_f(c) = 0$.*

Proof. Since c bounds a disk D, $\widetilde{c}$ is disconnected and moreover, D is the intersection of two disjoint sheets of $f(V)$. Hence, there is an induced spin structure on D, and the spin structure on c must be the trivial one. By Remark 2.13 this means that

$$\langle \xi_D, [\vec{c}_D] \rangle = 1.$$

Clearly, we can choose the extensions of the framing so that

$$\langle \xi_V, [\vec{c}_V] \rangle = 0,$$

where $\vec{c}_V$ is obtained by pulling back the extended framings via f. Hence, we must have

$$\langle \xi_{\mathsf{R}^{2v-2}}, [\vec{c}_{\mathsf{R}^{2v-2}}] \rangle = 1,$$

but this means that $[c, r, X] = p_*(u)$ for any parameterization r. We get

$$\omega_f(c) = 0 + p_*(u) + p_*(u) = 0.$$

LEMMA 7.11. *If b and c are isotopic embedded curves in F then $\omega_f(b) = \omega_f(c)$.*

PROOF. The isotopy between b and c extends to an isotopy of F and then to an isotopy of R^{2v-2}. Also, the isotopy, of F lifts and the lifted isotopy extends to an isotopy of V. The lemma now reduces to Lemma 7.7.

Let c be an immersed closed (oriented) curve in F with at least one double point. Smoothing c at this double point, we obtain two immersed, closed (oriented) curves a and b in F.

LEMMA 7.12. $\omega_f(c) = \omega_f(a) + \omega_f(b) + p_*(u)$.

PROOF. Let $x \in F$ be the crossing point. The smoothing takes place in a small disk D about x. Now, $f^{-1}(D) = D_1 \cup D_2$ and we can assume that the framings Y of $N(\widetilde{F} \subset V)|\widetilde{c}$ is chosen to be constant over D_i. The smoothing lifts to two smoothings in D_1 and D_2. If we let the framing of $\widetilde{a}$ and $\widetilde{b}$ be that induced by the framing of $\widetilde{c}$ just changing the part in $T\widetilde{F}$ over D_i, $i = 1, 2$. Then (cf. [7])

$$[\vec{c}_V] = [\vec{a}_V] + [\vec{b}_V].$$

We next consider the situation in $T\mathsf{R}^{2v-2}|F$. We choose the parameterization r of c so that it starts and ends at x. Let r_1 and r_2 denote the induced parameterizations and X_1 and X_2 the induced (twist) framings of a and b respectively. Let $\lambda(a)$, $\lambda(b)$ and $\lambda(c)$ be the paths in $SO(n)$ induced by the framings or twist framings of a, b and c respectively. We treat the different cases:

1. If a and b are both twist framed then c is framed and

$$\lambda(c) \simeq \lambda(a) * \lambda(b)\mathbf{A}(2v - 2).$$

2. If a is twist framed and b is framed then c is twist framed and

$$\lambda(c) \simeq \lambda(a) * \lambda(b)\mathbf{A}(2v - 2).$$

3. If a and b are both framed then c is framed and

$$\lambda(c) \simeq \lambda(a) * \lambda(b).$$

Since $[c, r, X] = [p(\lambda(c))]$ and similarly for a and b we see that, in all the cases above, we have

$$[c, r, X] = [a, r_1, X_1] + [b, r_2, X_2].$$

Finally,

$$\begin{aligned}\omega_f(c) &= \langle \xi_V, [\vec{c}_V] \rangle \cdot p_*(u) + [c, r, X] + p_*(u) \\ &= \langle \xi_V, [\vec{a}_V] + [\vec{b}_V] \rangle \cdot p_*(u) + [a, r_1, X_1] + [b, r_2, X_2] + p_*(u) \\ &= \langle \xi_V, [\vec{a}_V] \rangle \cdot p_*(u) + [a, r_1, X_1] \\ &+ \langle \xi_V, [\vec{b}_V] \rangle \cdot p_*(u) + [b, r_1, X_1] + p_*(u) = \omega_f(a) + \omega_f(b) + p_*(u).\end{aligned}$$

Let a and b be closed immersed (oriented) curves in F that intersect transversely. By smoothing at one of the intersection points we get a closed immersed (oriented) curve c.

Lemma 7.13. $\omega_f(c) = \omega_f(a) + \omega_f(b) + p_*(u)$.

Proof. The proof is similar to the proof of Lemma 7.12.

We next extend the function ω_f.

Definition 7.14. Let $L = c_1 \cup \cdots \cup c_k$ be a disjoint union of embedded (oriented) circles in F, define

$$\omega_f(L) = \sum_i \omega_f(c_i).$$

Lemma 7.15. *Let L_1 and L_2 be two unions of disjoint embedded (oriented) circles in F, intersecting transversely in k points. Then we get a new collection of (oriented) embedded circles L_3 by smoothing at these points and*

$$\omega_f(L_3) = \omega_f(L_1) + \omega_f(L_2) + k \cdot p_*(u).$$

Where $k \cdot p_*(u)$ denotes the value at k of the homomorphism $\mathsf{Z} \to \pi_1(SO(n)/\mathbf{A}(n))$ taking 1 to $p_*(u)$.

Proof. Induction using Lemmas 7.12, 7.13.

For $n = 4k + r$, $r = 0, 2$ choose isomorphisms

$$\phi\colon \pi_1(SO(n)/\mathbf{A}(n)) \to \begin{cases} \mathsf{Z}_4 & \text{if } k \text{ is odd,} \\ \mathsf{Z}_2 \oplus \mathsf{Z}_2 & \text{if } k \text{ is even,} \end{cases}$$

such that $\phi(p_*(u)) = 2$, when k is odd, and $\phi(p_*(u)) = (1,1)$, when k is even.

Definition 7.16. Let $f\colon V^v \to \mathsf{R}^{2v-2}$ be a generic immersion. Assume that V is equipped with a spin structure and that R^{2v-2} is oriented. Let $F \subset M'_f$ be a component of M_f. Let L be a disjoint union of circles in F. (If $v \equiv r \mod 4$, $r = 0, 1$ we assume that the circles are also oriented.) Define

$$\hat{q}_f(L) = \phi(\omega_f(L)).$$

We list the properties of $\hat{q}_f$ that follow from the results above in the following proposition.

Proposition 7.17.

(a) *If L_1 and L_2 are disjoint unions of (oriented) circles in F that intersect in k points then we can get a disjoint union L_3 of (oriented) circles by smoothing all intersections and*

$$\hat{q}_f(L_3) = \hat{q}_f(L_1) + \hat{q}_f(L_2) + k \cdot \begin{Bmatrix} 2 \\ (1,1) \end{Bmatrix},$$

which one of 2 and $(1,1)$ we use depends on the dimension.

(b) *If L is a single circle that bounds a disk then $\hat{q}_f(L) = 0$.*

(c) *If L_1 and L_2 are isotopic collections of circles then $\hat{q}_f(L_1) = \hat{q}_f(L_2)$.*

7.4. *Pin structure.* Throughout this section, let $f : V^v \to \mathsf{R}^{2v-2}$ be as in Definition 7.16, let $F \subset M'_f$ be a component of M_f and let $v \equiv 3 \mod 4$.

Proposition 7.18. *The immersion f induces a pin structure on F and hence a Z_4-quadratic function $q_{f,F}$ on $H_1(F;\mathsf{Z}_2)$.*

Proof. Note that $\pi_1(SO(2v-2)/\mathbf{A}(2v-2)) \cong \mathsf{Z}_4$ and that $\hat{q}_f(c)$ is independent of orientation in this case. Hence $\hat{q}_f$ is a function from disjoint unions of circles in F to Z_4. Since $\hat{q}_f$ satisfies (a) and (b) of Proposition 7.17, we know by Lemma 3.4 in [8] that $\hat{q}_f$ induces a Z_4-quadratic function $q_{f,F}\colon H_1(F;\mathsf{Z}_2) \to \mathsf{Z}_4$. By Theorem 3.2 in the same paper there is a 1-1 correspondence between Z_4-quadratic functions of the intersection form of F and pin structures.

Let L^g be a vector space over Z_2 with a nonsingular bilinear form $(x,y) \mapsto x \cdot y \in \mathsf{Z}_2$. A Z_4-quadratic function is a function $q\colon L \to \mathsf{Z}_4$ such that $q(x+y) = q(x) + q(y) + (x \cdot y) \cdot 2$. Let q be such a function and define the complex number

$$\lambda(q) = \sum_{x \in L} \left(\sqrt{-1}\right)^{q(x)},$$

then $\lambda(q) = \sqrt{2}^g \left(\frac{1+\sqrt{-1}}{\sqrt{2}}\right)^m$, for some $m \in \mathsf{Z}$. Since $\frac{1+\sqrt{-1}}{\sqrt{2}}$ is an 8^{th} root of unity, the residue class of m modulo 8 is a well defined element in Z_8. It is denoted $\beta(q)$ and called *Brown's invariant* of q. For any quadratic function $q : L^g \to \mathsf{Z}_4$, $\beta(q) \equiv g \mod 2$. If q_1 and q_2 are two Z_4-quadratic functions on L then there is an isomorphism h of L, preserving the bilinear form, such that $q_2 = q_1 h$ if and only if $\beta(q_1) = \beta(q_2)$. For a reference on this subject see [10].

Definition 7.19. If $v \equiv 3 \mod 4$ and $F \subset M'_f$, define $s(F) = \beta(q_{f,F}) \in \mathsf{Z}_8$.

Let $h\colon V^v \to \mathsf{R}^{2v-2}$ be another immersion satisfying the hypothesis in Definition 7.16.

Lemma 7.20. *Assume that f is diffeotopy equivalent to h. Let F be a component of M_f and G be the component of M_h corresponding to F under the diffeotopy equivalence then $g(F) = g(G)$. If $F \subset M'_f$ then $G \subset M'_h$ and $s(F) = s(G)$.*

Proof. This follows from Lemma 7.7 and the fact that $K_{1*}|F : H_1(F;\mathsf{Z}_2) \to H_1(K_1F;\mathsf{Z}_2)$ is an isomorphism.

7.5. $\mathsf{Z}_2 \oplus \mathsf{Z}_2$*-quadratic functions.* Throughout this section, let $f : V^v \to \mathsf{R}^{2v-2}$ be as in Definition 7.16, let $F \subset M'_f$ be a component of M_f and let $v \equiv 2 \mod 4$. To describe the invariants in this case it is necessary to introduce some algebra.

Definition 7.21. Let $(L^{2g}, \cdot)$ be a symplectic vector space over Z_2. A function $q\colon L \to \mathsf{Z}_2 \oplus \mathsf{Z}_2$ is called a $\mathsf{Z}_2 \oplus \mathsf{Z}_2$*-quadratic function* if

$$q(x+y) = q(x) + q(y) + (x \cdot y) \cdot (1,1), \quad \text{for all } x, y \in L,$$

where $u \cdot (1,1)$ is the the value at $u \in \mathsf{Z}_2$ of the homomorphism $\mathsf{Z}_2 \to \mathsf{Z}_2 \oplus \mathsf{Z}_2$ taking 1 to $(1,1)$.

Remark 7.22. Let q be a $\mathsf{Z}_2 \oplus \mathsf{Z}_2$-quadratic function on L. If $p_i : \mathsf{Z}_2 \oplus \mathsf{Z}_2 \to \mathsf{Z}_2$, $i = 1,2$ is the projection to the i^{th} summand then $p_i q$, $i = 1,2$ are Z_2-quadratic functions on L.

DEFINITION 7.23. Let q be a $\mathsf{Z}_2 \oplus \mathsf{Z}_2$-quadratic function on L^{2g} and $\{a_i, b_i\}_{i=1}^{g}$ be a symplectic basis. Define $\mathrm{Arf}(q) \in \mathsf{Z}_2 \oplus \mathsf{Z}_2$ by

$$\mathrm{Arf}(q) = \sum_i q(a_i)q(b_i),$$

where $(x, y)(x', y') = (xx', yy')$ for $(x, y), (x', y') \in \mathsf{Z}_2 \oplus \mathsf{Z}_2$.

By Remark 7.22 and the properties of the usual Arf invariant, we see that $\mathrm{Arf}(q)$ is invariant under symplectomorphisms of L.

PROPOSITION 7.24. *The immersion f induces a $\mathsf{Z}_2 \oplus \mathsf{Z}_2$-quadratic function $q_{f,F}$ on $H_1(F; \mathsf{Z}_2)$.*

PROOF. The construction of $q_{f,F}$ is based on $\hat{q}_f$. We have $\pi_1(SO(2v-2)/\mathbf{A}(2v-2)) \cong \mathsf{Z}_2 \oplus \mathsf{Z}_2$, F orientable and $\hat{q}_f(c)$ independent of the orientation of c. We view $\hat{q}_f$ as a function on disjoint unions of embedded circles. The first step is to show that given a collection L of embedded circles we can find a single circle K, homologous to L, such that $\hat{q}_f(L) = \hat{q}_f(K)$. If L has more than one component it is possible to draw an arc connecting two of its components. A regular neighborhood of this arc is a disk D with boundary L_1. By Proposition 7.17 (b) we have $\hat{q}_f(L_1) = 0$. Now, L and L_1 have two pairs of intersection points and if we orient the two components of L so that they transverse D in opposite directions and smooth the intersection points we get a new collection K_1 with one component less than L and two boundaries of disks. Now, K_1 is clearly homologous to L and (a) and (b) of Proposition 7.17 shows that $\hat{q}_f(L) = \hat{q}_f(K_1)$. Continue this process until there is only one component left.

Assume that L_1 and L_2 are homologous. We must show that $\hat{q}_f(L_1) = \hat{q}_f(L_2)$. By the above we can assume that L_1 and L_2 are embedded circles and by Proposition 7.17 (c) and the fact that F is orientable, that they intersect transversely in an even number of points. Smoothing the intersections and applying the above construction we get one zero homologous circle L and $\hat{q}_f(L) = \hat{q}_f(L_1) + \hat{q}_f(L_2)$ so we are done if we can show that $\hat{q}_f(L) = 0$. Since L is zero homologous there is a surface W, with ∂W, a single circle and an embedding $W \to F$ taking ∂W to L. If W is a disk then $\hat{q}_f(L) = 0$. We proceed by induction on the Euler characteristic of W. If $\chi(W) < 1$ there is a handle in W. Let K_1 and K_2 be two disjoint meridians of this handle then $\hat{q}_f(K_1) = \hat{q}_f(K_2)$. Connect both of these to L by disjoint arcs and apply the first construction of the proof twice. We then get a curve L' that bounds a surface W' with $\chi(W') > \chi(W)$ and $\hat{q}_f(L') = 0$ by our inductive assumption. Finally,

$$\hat{q}_f(L') = \hat{q}_f(K_1) + \hat{q}_f(K_2) + \hat{q}_f(L),$$

hence $\hat{q}_f(L) = 0$. Define $q_{f,F}\colon H_1(F;\mathsf{Z}_2) \to \mathsf{Z}_2 \oplus \mathsf{Z}_2$ by $q_{f,F}(a) = \hat{q}_f(\alpha)$ where α is an embedded circle with $[\alpha] = a$. Then $q_{f,F}$ is a $\mathsf{Z}_2 \oplus \mathsf{Z}_2$-quadratic function.

Definition 7.25. If $v \equiv 2 \mod 4$ and $F \subset M'_f$, define $s(F) = \mathrm{Arf}(q_{f,F}) \in \mathsf{Z}_2 \oplus \mathsf{Z}_2$.

Let $h\colon V^v \to \mathsf{R}^{2v-2}$ be another immersion satisfying the hypothesis in Definition 7.16.

Lemma 7.26. *Assume that f is diffeotopy equivalent to h. Let F be a component of M_f and G be the component of M_h corresponding to F under the diffeotopy equivalence then $g(F) = g(G)$. If $F \subset M'_f$ then $G \subset M'_h$ and $s(F) = s(G)$.*

Proof. See Lemma 7.20.

Let L^{2g} be a symplectic vector space over Z_2. Let q_1, q_2 be $\mathsf{Z}_2 \oplus \mathsf{Z}_2$-quadratic functions on L such that there are subspaces $L_i^{2g-1} \subset L$ with $q_i(L_i) \subset \{(0,0),(1,1)\}$ and elements $a^i \in V$ such that $q_i(a^i) \in \{(1,0),(0,1)\}$, $i = 1,2$.

Lemma 7.27. *Under the conditions described above there is a symplectomorphism $h : L \to L$ with $q_2 = q_1 h$ if and only if $\mathrm{Arf}(q_1) = \mathrm{Arf}(q_2)$.*

Proof. Checking all possible cases proves the lemma if $g \leq 2$. Assume that $g > 2$. Pick bases $\{a_1^i, b_1^i, \dots, a_g^i, b_g^i\}$ such that $q_i(a_1^i) = (1,0)$ or $q_i(a_1^i) = (0,1)$. Without loss of generality we can assume that $q_i(a_j^i) = q_i(b_j^i) = (0,0)$ for $3 \leq j \leq g$, $i = 1,2$. The situation then reduces to the case $g = 2$.

Remark 7.28. When $q_{f,F} : H_1(F;\mathsf{Z}_2) \to \mathsf{Z}_2 \oplus \mathsf{Z}_2$ is a quadratic function induced by a generic immersion, we have exactly the situation described above Lemma 7.27, since $\hat{q}_f(c) \notin \{(0,0),(1,1)\}$ if and only if $\widetilde{c}$ is connected and $\widetilde{c}$ is connected if and only if $[c] \notin f_*(H_1(\widetilde{F};\mathsf{Z}_2))$.

7.6. *Absence of local obstructions.* Throughout this section let $f : V^v \to \mathsf{R}^{2v-2}$ be as in Definition 7.16, let $F \subset M'_f$ be a component of M_f and let $v \equiv 0 \mod 4$ or $v \equiv 1 \mod 4$. Theorem 7.30 shows that if $v \equiv 0 \mod 4$ then $g(F)$ is even.

Lemma 7.29.

(a) *If $v \equiv 1 \mod 4$ then there is a disjoint collection of oriented embedded curves in F, $\{\alpha_1, \dots, \alpha_g\}$, such that $\{[\alpha_1], \dots, [\alpha_g]\}$ is a basis of $H_1(F;\mathsf{Z}_2)$ and $\hat{q}_f(\alpha_i) = (1,0)$ for all i.*

(b) *If* $v \equiv 0 \mod 4$ *then there is a collection of oriented embedded curves in* F, $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$, *such that* $\alpha_i \cap \beta_i$ *is exactly one point for each* i *and these are the only intersections in the collection. Also,* $\{[\alpha_1], [\beta_1], \ldots, [\alpha_g], [\beta_g]\}$ *is a basis of* $H_1(F; \mathsf{Z})$, $[\alpha_i] \cdot [\beta_i] = 1$ *and* $\hat{q}_f(\alpha_i) = \hat{q}_f(\beta_i) = 1$, *for all* i.

Proof. Consider case (a). Let α_i be the core of the i^{th} Moebius strip in F then the basis statement follows. Since $\widetilde{F} \to F$ is the orientation double cover

$$\hat{q}_f(\alpha_i) = (1, 0), \quad \text{or} \quad \hat{q}_f(\alpha_i) = (0, 1).$$

If $-\alpha_i$ denotes α_i with the opposite orientation then

$$\hat{q}_f(-\alpha_i) = \hat{q}_f(\alpha_i) + (1, 1),$$

by Lemma 7.9. Then (a) follows.

Consider case (b). We see that $F \approx \Sigma_g$, some g. We can pick embedded curves $\{\gamma_1, \delta_1, \ldots, \gamma_g, \delta_g\}$ as in the statement of the theorem, such that $\widetilde{\gamma_1}$ is connected, and the inverse image of any other curve in the collection is disconnected. Now, let D_i denote a Dehn twist along δ_i, $i > 1$. Let D denote a Dehn twist along a curve in the class $\sum_{i=1}^{g} [\gamma_i]$ and $h = DD_2 \ldots D_g$. Then

$$\hat{q}_f(h\gamma_i) = \pm 1 \quad \text{and} \quad \hat{q}_f(h\delta_i) = \pm 1.$$

If, for example, $\hat{q}_f(h\gamma_i) = 1$ and $\hat{q}_f(h\delta_i) = -1$ then let $\alpha_i = -h\delta_i$ and $\beta_i = h\gamma_i$. The other cases that arise can be treated similarly. We get the desired collection $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$. (Note that there is a diffeomorphism $k \colon F \to F$ such that $k\gamma_i = \alpha_i$ and $k\delta_i = \beta_i$, for all i.)

As we shall see in Section 7.7, Lemma 7.29 implies that there are no local obstructions to diffeotopy equivalence associated to components $F \subset M'_f$ if $v \equiv 0 \mod 4$. We therefore define $s(F) = 0$ in this case.

If $v \equiv 1 \mod 4$ and $F \subset M'_f$ then there is a local obstruction in Z_2: Proposition 2.8 shows that there is an induced spin structure on $\widetilde{F} = f^{-1}(F)$. Let $q_{f,\widetilde{F}}$ denote the corresponding quadratic function. Since F is nonorientable, there is a unique torsion class $a \in H_1(F; \mathsf{Z})$. Since $g(F)$ is even, there is an element $b' \in H_1(\widetilde{F}; \mathsf{Z})$ such that $f_*(b') = a$. Moreover, reducing b' modulo 2 gives a uniquely determined $b \in H_1(\widetilde{F}; \mathsf{Z}_2)$. Define

$$s(F) = q_{f,\widetilde{F}}(b).$$

Then $s(F)$ is invariant under diffeotopy equivalence. Noting that $f_*(b)$ is dual to w_1, we see that $s(F) = s(F')$ is a sufficient condition for the existence of the desired diffeomorphism $p \colon F' \to F$ in the proof below.

7.7. *Proof of Theorem 1.5.* The only if part follows from Lemmas 7.3, 7.20, 7.26. When $F \subset M''_f$, we let $\hat{q}_f(\alpha) = q_{f,F}([\alpha])$ for embedded curves α in F. Here $q_{f,F}$ is the Z_2-quadratic form associated to the spin structure induced on F and $[\alpha]$ is the Z_2-homology class of α. With this convention we can treat all cases simultaneously.

Assume that F' is a component of M_f and that F is a component of M_h such that $\widetilde{F}$ and $\widetilde{F'}$ have the same number of components, $g(F) = g(F')$ and $s(F') = s(F)$. If G is a surface then any automorphism of $H_1(G;\mathsf{Z}_2)$ that preserves the intersection pairing lifts to a autodiffeomorphism of G. Using this and Lemmas 7.27, 7.29 we see that there is a diffeomorphism $p\colon F' \to F$ such that $\hat{q}_f(p^{-1}(\alpha_i)) = \hat{q}_h(\alpha_i)$, where $\{[\alpha_i]\}$ is a basis of Z_2-homology. By transversality arguments any two embeddings of F in R^{2v-2} are ambient diffeotopic. Hence, there is a diffeotopy $K_t\colon \mathsf{R}^{2v-2} \to \mathsf{R}^{2v-2}$ such that $K_1|F' = p$. Since $\hat{q}(\alpha)$ determines how α lifts and $\hat{q}_f p^{-1}$ agrees with $\hat{q}_h$ on a basis of Z_2-homology, the diffeomorphism $p^{-1}\colon F \to F'$ is covered by a diffeomorphism $k\colon \widetilde{F} \to \widetilde{F'}$. That is, we have the commutative diagram

$$\begin{array}{ccc} \widetilde{F'} & \xleftarrow{k} & \widetilde{F} \\ f\downarrow & & \downarrow h \\ F' & \xrightarrow{p} & F. \end{array}$$

A similar argument as above shows that we can find a diffeotopy $G_t : V \to V$ such that $G_1|\widetilde{F} = k$. It follows that $K_1 f G_1|\widetilde{F} = h|\widetilde{F}$. Let $K_1 f G_1 = f'$. We can find diffeotopies P_t and P'_t, fixed on F, of R^{2v-2} such that the normal derivatives of $P_1 h$ and $P'_1 f'$ are constant along the fibers of tubular neighborhoods of $h^{-1}(\alpha_i)$ in $\widetilde{F}$. We therefore assume that f' and h have this property. Lemma 2.14 shows that we can replace f' and h by their normal derivatives close to $\widetilde{F}$. Choose framings of $N(\widetilde{F} \subset V)$ along the curves $f^{-1}(\alpha_i) = \widetilde{\alpha}_i$. Then f' and h induce framings or twist framings of $N(F \subset \mathsf{R}^{2v-2})|\alpha_i$. By Lemmas 7.7, 7.2, $\hat{q}_{f'}(\alpha_i) = \hat{q}_h(\alpha_i)$. This means that the induced framings or twist framings are isotopic. Clearly, we can assume that F is fixed under such an isotopy. We now separate the cases.

First, assume that F is orientable, $g(F) > 0$ and that the basic curves are $\{a_1, b_1, \dots, a_g, b_g\}$. Choose a diffeotopy L^1_t of R^{2v-2}, supported in a neighborhood of a_1, that brings the framing of a_1 induced by f' to that induced by h and that keeps F fixed. Choose a second diffeotopy J^1_t, supported in a neighborhood of b_1 and fixed on F, keeping the framing at the intersection point of a_1 and b_1 fixed, and such that it brings the framing induced by f' on b_1 to that induced by h. Let c_1 be a path connecting a_1 to a_2. Choose a diffeotopy R^1_t supported in a neighborhood of c_1 with the same features as the diffeotopies already used. Continue in this manner until b_g is reached. Then $R_1 = R_1^g L_1^g J_1^{g-1} \dots J_1^1 L_1^1 R_1^1$ is one end of a diffeotopy of R^{2v-2} such that $R_1 f' | \widetilde{F} = h | \widetilde{F}$ and $R_1 f'$ and h are equal in $N(F \subset \mathsf{R}^{2v-2}) | \widetilde{C}$, where $\widetilde{C}$ is a neighborhood in $\widetilde{F}$ of $h^{-1}(\widetilde{a}_1 \cup \widetilde{b}_1 \cup \widetilde{c}_1 \cup \dots \cup \widetilde{b}_g)$. Now, $F - C$, where $C = h(\widetilde{C}) = R_1 f'(\widetilde{C})$ is a disk D with boundary $\partial D = \partial C$. The maps $R_1 f'$ and h induce framings of this disk, from a chosen framing of $N(\widetilde{D} \subset V)$. These framings agree on ∂D. If we can find a homotopy between the two framings then a diffeotopy L_t of R^{2v-2} such that $L_1 R_1 f' = h$ in a neighborhood of $\widetilde{F}$ is easily found. Interpreting the framings in the usual way we must look for a solution of the homotopy problem

$$\begin{array}{ccc} D \times \partial I \cup \partial D \times I & \rightarrow & SO(2v-4) \\ \downarrow & \nearrow & \\ D \times I & & . \end{array}$$

Since $\pi_2(SO(2v-4)) = 0$, there is a solution.

Secondly, if F is nonorientable we construct the desired diffeotopy in almost the same way, just drop the b's and J's in the above construction.

Finally, if $g(F) = 0$ we start from any simple closed curve a in F and get two pairs of framed disks with coinciding boundary framings. We see, as above, that the desired diffeotopy can be constructed.

Repeating this construction for any component of M_h we can find diffeotopies $K_t \colon \mathsf{R}^{2v-2} \to \mathsf{R}^{2v-2}$ and $G_t \colon V \to V$ such that $K_1 f G_1 | W = h | W$ where W is a neighborhood of $\widetilde{M_h}$. An application of Theorem 1.2 completes the proof.

7.8. *Restrictions on the self intersection.*

THEOREM 7.30. *Let $f \colon V^v \to \mathsf{R}^{2v-2}$ be as in Definition 7.16 and let $v \equiv 1 \mod 4$. If $F \subset M_f'$ is a component of M_f then $g(F)$ is even.*

PROOF. Note that $F \approx \Gamma_g$. Assume, to get a contradiction, that g is odd. Pick an embedded curve $\alpha \subset F$ that is dual to the first Stiefel-Whitney class w_1. Since g is odd, a regular neighborhood of α is a Moebius strip, B. Since α was dual to w_1, $F - B$ is an orientable surface with a single boundary circle $c = \partial B$. The covering $f \colon \widetilde{F} \to F$ is the orientation double covering, therefore $f^{-1}(F - B)$ is disconnected and hence there is an induced spin structure on $F - B$. This implies that the spin structure induced by f on $TF | c$ must be the trivial one, which in turn implies that

$$\langle \xi_{(F-B)}, [\vec{c}_{F-B}] \rangle = 1,$$

by Remark 2.13. Noting that $\widetilde{c}$ is disconnected it is easy to see (cf. Lemma 7.10) that

$$\omega_f(c) = 0 \text{ if and only if } \langle \xi_{(F-B)}, [\vec{c}_{F-B}] \rangle = 1.$$

However, using Lemma 7.13,

$$\begin{aligned} \omega_f(c) &= \omega_f(\alpha) + \omega_f(\alpha) + p_*(u) \\ &= p_*(u) \neq 0, \end{aligned}$$

since $\pi_1(SO(2v-2)/\mathbf{A}(2v-2)) \cong \mathsf{Z}_2 \oplus \mathsf{Z}_2$ when $v \equiv 1 \mod 4$. This is a contradiction.

The next theorem shows that there are similar restrictions on the self intersection in other dimensions.

THEOREM 7.31. *Let M^3 be any closed orientable 3-manifold. If $f : M^3 \to S^4$ is an immersion with transverse self intersection consisting of double points (i.e. there are no triple points), then there are no nonorientable components F of M_f with odd genus.*

PROOF. Assume that $f : M^3 \to S^4$ is an immersion with transverse self intersection along F then $f^{-1}(F) = \widetilde{F}$ is an orientable surface by Proposition 2.5. The normal bundle of $\widetilde{F}$ in M^3 is orientable and therefore trivial. If n is a nonvanishing normal vector field of $\widetilde{F}$ in M^3 then define the normal vector field v of F in S^4 by

$$v(x) = df(n(x_1)) + df(n(x_2)),$$

where $f(x_1) = f(x_2) = x$. By the assumption about transversality, this vector field never vanishes. If $g(F)$ is odd then this contradicts the fact, proved in [9], that the Euler number of the normal bundle of an embedded nonorientable surface G in S^4 is congruent to $2\chi(G)$ modulo 4.

8. **Examples of generic immersions**

8.1. *Self intersections along spheres.*

EXAMPLE 8.1. There is a generic immersion $f : S^v \to \mathsf{R}^w$ such that $M_f \approx S^{2v-w}$ and $\widetilde{M}_f \approx S^{2v-w} \cup S^{2v-w}$.

Let $S^v \subset \mathsf{R}^{v+1} \times 0 \subset \mathsf{R}^w$ be a sphere of radius 1. Let $D^v \subset 0 \times \mathsf{R}^v \subset \mathsf{R}^w$ be a disk of radius 2. Then $S^v \cap D^v = S^{2v-w}$ is a sphere of radius 1. Cut out a small v-disk, disjoint from the intersection, from S^v. Join the boundary of this disk and ∂D^v by a v-tube. What is obtained can be viewed as the image of a generic immersion with the required properties.

8.2. *Self intersections along circles.*

EXAMPLE 8.2. There is an immersion $f: S^5 \to \mathsf{R}^9$ with transverse self intersection $M_f \approx S^1$ and $\widetilde{M}_f \approx S^1$.

To construct the map, consider the following decomposition

$$S^5 = S^1_a \times D^4_a \cup D^2_b \times S^3_b.$$

Choose a path $\lambda\colon [0,1] \to SO(8)$ that is constant near the ends, starting at the identity matrix $\mathbf{I}$ and ending at the matrix $\mathbf{B}(8)$. We denote the column vectors in the matrix $\lambda(t)$ by $F^i(t)$, $i = 1, \dots, 8$. Let θ be a coordinate on S^1_a ranging from 0 to 1 and $(x_1, x_2, x_3, x_4) \in \mathbf{D}^4$, the unit disk in R^4, be coordinates on D^4_a. Choose coordinates on $S^1 \times D^8$ as (ω, y) where $0 \le \omega \le 1$ and $y \in \mathbf{D}^8$, the unit disk in R^8. Define the map $f'\colon S^1_a \times D^2_a \to S^1 \times D^8$ by

$$f'(\theta, x) = \begin{cases} \big(2\theta, \sum_{i=1}^4 x_i F^i(2\theta)\big) & \text{if } 0 \le \theta \le \frac{1}{2}, \\ \big(2\theta - 1, \sum_{i=1}^4 x_i F^{i+4}(2\theta - 1)\big) & \text{if } \frac{1}{2} \le \theta \le 1. \end{cases}$$

Then f' has transverse self intersection along $S^1 \times 0$ and $f'^{-1}(S^1_c \times 0) = S^1_a \times 0$. To complete the construction, we consider $S^1 \times D^8$ as a submanifold of R^9 and extend f' to $D^2_b \times S^3_b$ as an embedding of this space into $\mathsf{R}^9 - S^1 \times D^8$. Note that f' is already defined on $\partial D^2_b \times S^3_b$ and embedds this boundary into $\partial(\mathsf{R}^9 - S^1 \times D^8)$. Standard obstruction theory shows that this extends to a map g from $D^2_b \times S^3_b$ to $\mathsf{R}^9 - S^1 \times D^8$. Then Theorem 3.1 shows that g is actually homotopic relative to the boundary to an embedding f''. By the analogue of Theorem 3.5 for embeddings, we can assume that f'' is smooth and then use it to extend f'. This gives the desired f.

8.3. *Self intersections along orientable surfaces.* Let T^2 denote the standard torus.

EXAMPLE 8.3. There is a generic immersion $f\colon S^4 \to \mathsf{R}^6$ with $M_f \approx T^2$ and $\widetilde{M}_f \approx T^2 \cup T^2$.

Consider $\mathbf{D}^4$, the unit disk in C^2, with coordinates (z, w) and the embeddings $f_i\colon \mathbf{D}^4 \to \mathsf{C}^2 \times \mathsf{R}^2$, $i = 1, 2$, defined by

$$\begin{aligned} &f_1(z, w) = (z, w, 0, 0)\,, \\ &f_2(z, w) = \\ &\left(z, w, \frac{1}{4} - |z|^2 - \alpha(|z|^2 + |w|^2)|w|^2, \frac{1}{4} - |w|^2 - \alpha(|z|^2 + |w|^2)|z|^2\right), \end{aligned}$$

where $\alpha\colon [0,1] \to [0,1]$ satisfies $0 \leq \alpha \leq 1$, $\alpha(t) = 0$ if $0 \leq t \leq \frac{3}{4}$ and $\alpha(t) = 1$ if $\frac{7}{8} \leq t \leq 1$. We see that $f_1(\mathbf{D}^4) \cap f_2(\mathbf{D}^4) \approx T^2$ and that the self intersection is transverse. Now, for $x \in \partial\mathbf{D}^4$ we have

$$f_2(x) = f_1(x) - (0, 0, -\frac{1}{2}, -\frac{1}{2}).$$

Hence, we can connect the two disks by an embedding of $S^3 \times I$. This gives us the desired f.

Example 8.4. There is a generic immersion $f : S^6 \to \mathsf{R}^{10}$ with $M_f \approx T^2$ and $\widetilde{M}_f \approx T^2$.

The construction of this map involves the path λ described in Example 8.2. We keep the notation. Consider closed tubular neighborhoods of embedded tori in S^6 and R^{10}, denote them by U and V respectively. They are of the form $U = S^1 \times S^1 \times D^4$ and $V = S^1 \times S^1 \times D^8$. Let (θ, ω) be coordinates on $S^1 \times S^1$, $(x_1, \dots, x_4) \in \mathbf{D}^4$ coordinates on D^4 and $v \in \mathbf{D}^8$ be coordinates on D^8. Define the map $\phi\colon U \to V$ by

$$f'(\theta,\omega,x) = \begin{cases} (2\theta, \omega, \sum_{i=1}^4 x_i F_i(2\theta)) & \text{for } 0 \leq \theta \leq \frac{1}{2}, \\ (2\theta - 1, \omega, \sum_{i=1}^4 x_i F_{i+4}(2\theta - 1)) & \text{for } \frac{1}{2} \leq \theta \leq 1. \end{cases}$$

It is straightforward to check that this map has transverse self intersection $M_f = S^1 \times S^1 \times 0 \subset V$ and that $f^{-1}(M_f) = S^1 \times S^1 \times 0 \subset U$. To complete the construction we shall extend the map to $S^6 - U$ as an embedding into $\mathsf{R}^{10} - V$. First we apply obstruction theory to see that it extends as a continuous map. The key fact is that $S^{10} - T^2$ is 6-connected, so that all obstructions vanish. When this is done we have a continuous map $f''\colon S^6 - U \to S^{10} - V$ that is a smooth embedding close to the boundary. Then we apply Theorem 3.1 to see that it is homotopic to an embedding. The key facts here are that $S^6 - T^2$ is 2-connected and $\dim(S^{10} - V) - \dim(S^6 - U) = 4 > 3$. Then proceed as in Example 8.2 to get the desired map.

8.4. *Self intersections along nonorientable surfaces.* Theorem 7.30 shows that a nonorientable surface, which is a self intersection of a generic immersion, must have even genus if the dimension v of the source manifold satisfies $v \equiv 1 \mod 4$. The following example shows that the condition of genericity is necessary.

Example 8.5. Denote the coordinates in R^8 by $y = (y_1, \dots, y_8)$ and consider S^5 as the unit sphere in R^6 with coordinates $(x_1, \dots, x_6)$. Define the family of maps $f_\epsilon\colon S^5 \to \mathsf{R}^8$, $-1 < \epsilon < 1$, by the equations

$$\begin{aligned} y_1 &= x_1^2 - x_2^2, & y_5 &= x_3(x_4^2 + x_5^2 + x_6^2 + \epsilon^2), \\ y_2 &= x_1 x_2, & y_6 &= x_4, \\ y_3 &= x_1 x_3 - x_2(x_4^2 + x_5^2 + x_6^2 + \epsilon^2), & y_7 &= x_5, \\ y_4 &= x_2 x_3 + x_1(x_4^2 + x_5^2 + x_6^2 + \epsilon^2), & y_8 &= x_6. \end{aligned}$$

If $\epsilon \neq 0$ then f_ϵ is an embedding: The last three equations determine x_4, x_5, x_6, the fifth then determines x_3. Finally, considering x_3, x_4, x_5, x_6 as fixed, the third and fourth equation form a linear system in x_1, x_2 with nonzero determinant. Similarly, f_0 is injective on the subset $(x_4^2 + x_5^2 + x_6^2 \neq 0)$ of S^5. The restriction $f_0|(x_4 = x_5 = x_6 = 0) : S^2 \subset S^5 \to \mathsf{R}^4 \subset \mathsf{R}^8$ is an immersion with image $\mathsf{R}P^2$. Thus, $f_0 : S^5 \to \mathsf{R}^8$ is an immersion with self intersection $\mathsf{R}P^2$. However, an arbitrarily small perturbation of f_0 gives a map f_ϵ, which is an embedding. Hence, the self intersection is not transverse.

Example 8.6. There is a generic immersion $f : S^7 \to \mathsf{R}^{12}$ with $M_f \approx \mathsf{R}P^2$ and $\widetilde{M}_f \approx S^2$.

To construct this map we write

$$S^7 = S^1_a \times D^6_a \cup D^2_b \times S^5_b$$

Then we choose a path $\lambda : [0,1] \to SO(11)$, starting at the identity $\mathbf{I}$ and ending at the matrix $\mathbf{B}(11)$. We denote the column vectors of the matrix $\lambda(t)$ by $F^i(t)$, $i = 1, \ldots, 11$. Let $(\theta, x) \in [0,1] \times \mathbf{D}^6$ be coordinates on $S^1_a \times D^6_a$. Let $(\omega, v) \in [0,1] \times \mathbf{D}^{11}$ be coordinates on $S^1 \times D^{11} \subset \mathsf{R}^{12}$. Let K denote the manifold $\mathsf{R}^{12} - S^1 \times D^{11}$. Define the map $f' \colon S^1_a \times D^6_a \to S^1 \times D^{11}$ by

$$f'(\theta, x) = \begin{cases} \left(2\theta, x_1 F_1(2\theta) + \sum_{i=2}^6 x_i F_i(2\theta)\right), \\ \left(2\theta - 1, -x_1 F_1(2\theta - 1) + \sum_{i=2}^6 x_i F_{i+5}(2\theta - 1)\right), \end{cases}$$

where the first equation holds for $0 \leq \theta \leq \frac{1}{2}$ and the second for $\frac{1}{2} \leq \theta \leq 1$. The self intersection manifold of f' is a Moebius strip B with boundary circle ∂B in $S^1 \times \partial D^{11}$. The inverse image of this set is the band $S^1_a \times I \times 0$ where I is the unit interval in the x_1-direction in $\mathbf{D}^6$. Moreover, the intersection is transverse. The next step is to extend f' to $D^2_b \times S^5_b$. To this end we write

$$D^2_b \times S^5_b = D^2_b \times (D^5_n \cup S^4 \times I \cup D^5_s).$$

Here S^5_b is considered as the boundary of D^6_a and D^5_n is a small disk centered on the point $(1,0,0,0,0,0)$ in the coordinates above, D^5_s is the antipodal disk. The circle $\partial B \subset \partial K$ is homotopically trivial in K and hence bounds a 2-disk. By transversality arguments there is actually an embedded disk $E^2 \subset K$ with $\partial E^2 = \partial B$. Moreover, E^2 can be chosen so that $df'(\theta,\pm 1,0,0,0,0,0)\frac{\partial}{\partial x_1}$ is an outward normal vector field to ∂E^2. The framing induced on ∂E by f' induces a loop of the form $\lambda * \lambda\mathbf{A}(12)$ in $SO(12)$, where $\lambda\colon [0,1]\to SO(12)$ is a path from $\mathbf{I}$ to $\mathbf{A}(12)$. Since $\pi_1(SO(12)/\mathbf{A}(12))\cong \mathsf{Z}_4$ this is the nontrivial loop. This means that the normal framing of ∂E extends to a normal framing of E. At this point we can find an extension f'' of f' to $D^2_b\times(D^5_n\cup D^5_s)$. Namely, map $D^2_b\times n$ and $D^2_b\times s$ onto E^2. Take the five vectors of D^5_n to the first five vectors in the extended frame and the five of D^5_s to the last five. Then f' is a map with transverse self intersection manifold $\mathsf{R}P^2$ and we need only extend it as an embedding to $D^2_b\times S^4\times I$ into $W=K-N$, where N is a tubular neighborhood of E in K. Note that f' is already defined on $\partial D^2_b\times S^4\times I\cup D^2_b\times S^4\times\partial I$ so we meet first the extension problem

$$\begin{array}{ccc} \partial D^2_b\times S^4\times I\cup D^2_b\times S^4\times\partial I & \xrightarrow{f'} & W \\ \downarrow & \nearrow & \\ D^2_b\times S^4\times I & & \end{array}.$$

The space W is a deformation retract of $K-E$ and K is 10-connected by general position. Hence, again by general position we see that W is 9-connected. Obstructions to extending this map are found in $H^{q+1}(S^4\times D^3, S^4\times\partial D^3;\pi_q(W))$ and thus vanish. Pick an extension g. Then g is an embedding close to the boundary. The argument from Example 8.2 can now be repeated to find the desired f.

EXAMPLE 8.7. There is a generic immersion $f\colon S^9\to\mathsf{R}^{16}$ such that $M_f\approx K^2$ and $\widetilde{M_f}\approx T^2$, where K^2 denotes the Klein bottle.

Choose a path $\lambda\colon[0,1]\to SO(15)$ such that $\lambda(0)=\mathbf{I}$ and $\lambda(1)=\mathbf{B}(15)$. As in Example 8.6, we can construct a map $g\colon S^1\times D^8\to S^1\times D^{15}$ such that $M_g=B$, where B is a Moebius strip with $\partial B\subset S^1\times\partial D^{15}$. Furthermore, $g^{-1}(B)=\widetilde{B}=S^1\times I\times 0\subset S^1\times D^8$. Let $\mathsf{R}^n_{i+}=\{x\in\mathsf{R}^n\colon x_i\geq 0\}$ and $\mathsf{R}^n_{i-}=\{x\in\mathsf{R}^n\colon x_i\leq 0\}$. There is a natural inclusion $\mathsf{R}^9\subset S^9$. Choose an embedding $\phi\colon S^1\times D^8\to\mathsf{R}^9_{9+}$ such that $\phi(S^1\times 0)\subset\mathsf{R}^9_{9+}\cap\mathsf{R}^9_{8+}$. Choose coordinates $(\theta,x_1,x)\in S^1\times D^8$ and assume that ϕ is such that $\phi(\theta,0,x)\in\mathsf{R}^9_{9+}\cap\mathsf{R}^9_{8+}$, $p_8(\phi(\theta,x_1,x))=x_1$, where p_j is the projection on the j^{th} coordinate and that $p_k\phi(\theta,-x_1,x)=p_k\phi(\theta,x_1,x)$, $k\neq 8$. There is then an embedding $\psi\colon S^1\times[0,1]\to\mathsf{R}^9_{9+}$ such that $\psi(S^1\times 0)\subset\mathsf{R}^9_{9+}\cap\mathsf{R}^9_{9-}$ and $\psi(S^1\times 1)=\phi(S^1,1,0)$. Choose an embedding $\alpha: S^1\times D^{15}\to\mathsf{R}^{16}_{16+}$. Then there is an embedding $\beta\colon S^1\times I\to\mathsf{R}^{16}_{16+}$ that agrees with $\alpha(\partial B)$ in one end and maps the other to $\mathsf{R}^{16}_{16+}\cap\mathsf{R}^{16}_{16-}$. Let $f'=\alpha g\phi$. Then we can extend f' to $\psi(S^1\times[0,1])$ using β. Furthermore, f' induces a framing of $\alpha(\partial B)$ and this framing extends to a framing of $\beta(S^1 \times I)$, which may be taken constant near the ends. Now, extend f' by $f'(x, -x_8, x_9) = f'(x, x_8, x_9)$ for $x_9 \geq 0$. Using the framing of $\beta(S^1 \times I)$ we may then extend f' to a tubular neighborhood of the "tube" where it is defined. Taking the double of this map we get a map $f'' \colon N \to U$ where N is a tubular neighborhood of $T^2 \subset \mathsf{R}^9 \subset S^9$ and U is a tubular neighborhood of $K^2 \subset \mathsf{R}^{16}$ such that $M_{f''} = K^2$, $\widetilde{M}_{f''} = T^2$, $f'' \colon \partial N \hookrightarrow \partial U$. Using Theorems 3.1, 3.5 as in the above examples we get the desired f.

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