MINIMIZING SINGULARITIES OF GENERIC PLANE DISKS WITH IMMERSED BOUNDARIES

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Abstract. A cooriented circle immersion into the plane can be extended to a stable map of the disk which is an immersion in a neighborhood of the boundary and with outward normal vector field along the boundary equal to the given coorienting normal vector field. We express the minimal number of fold components of such a stable map as a function of its number of cusps and of the normal degree of its boundary. We also show that this minimum is attained for any cooriented circle immersion of normal degree not equal to one.

1. Introduction

We say that a map $F\colon M \to \mathbb{R}^2$, where M is a compact 2-manifold with boundary, is *admissible* if it is an immersion in some neighborhood of ∂M and if it has only stable singularities. The singularities of such a map F form a closed codimension one submanifold $\Sigma(F)$ of M and the kernel field of its differential dF is tangent to $\Sigma(F)$ at isolated points called *cusps* of F, see Subsection 2.1. We call the components of $\Sigma(F)$ the *fold components* of F. Let $N(F)$ denote the number of fold components of F and let $C(F)$ denote its number of cusps.

A *cooriented circle immersion* is an immersion $f\colon S^1 \to \mathbb{R}^2$ equipped with a normal vector field ν. The *normal degree* $W(f)$ of a cooriented immersion f is the degree of the map $\nu\colon S^1 \to S^1$ where the source is oriented by the tangent vector field τ such that the frame $(\nu, df(\tau))$ represents the positive orientation of $\mathbb{R}^2$. Note that this normal degree equals the tangential degree of f or, in other words, its Whitney index [5].

Let (f,ν) be a cooriented circle immersion. Let D be the unit disk with $\partial D = S^1$ and let n be the outward normal vector field of ∂D in D. Then there exists a map $F\colon D \to \mathbb{R}^2$ such that $F|\partial D = f$ and such that $dF(n) = \nu$. Note that such a map is an immersion in some neighborhood of ∂D and that, after an arbitrarily small deformation vanishing in some neighborhood of ∂D, F is admissible. We call such a map F an *admissible map with boundary values* (f,ν).

Theorem 1. *Let (f,ν) be a cooriented circle immersion and let F be any admissible map with boundary values (f,ν). Then $C(F) \neq W(f) \bmod 2$ and the number of folds of F is bounded in the following way.*

(a) *If $W(f) \leq 0$ then*

$$N(F) \geq \max\left\{\frac{1}{2}\Big(|W(f)| + 1 - C(F)\Big),\, 1\right\}.$$

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(b) *If* $W(f) > 1$, *or if* $W(f) = 1$ *and* F *is not an immersion, then*

$$N(F) \geq \max\left\{\frac{1}{2}\Big(W(f) + 3 - C(F)\Big), 1\right\}.$$

Moreover, for any fixed $m \geq 0$, $m \neq W(f) \bmod 2$ *there exists an admissible* F *with boundary values* (f, ν) *such that* $C(F) = m$ *and such that equality holds in the inequality in* (a) [(b)] *if* $W(f) \leq 0$ [$W(f) > 0$].

Theorem 1 is proved in Section 3. Similar results for stable maps of closed surfaces were found by Eliashberg [2]. Effective conditions for a circle immersion f with $W(f) = 1$ to bound immersed disks were found by Blank, see [4].

2. Notation, orientation conventions, and elementary bordisms

In this section the notions and basic techniques used in the proof of Theorem 1 are described.

2.1. **Singularities of stable maps.** Let $F\colon M \to \mathbb{R}^2$ be a locally stable map of a 2-manifold to the plane. If $\partial M \neq \emptyset$ then let F be an immersion in some neighborhood of ∂M. A theorem of Whitney, see [1], shows that for any point $p \in M$ there are coordinates $x = (x_1, x_2)$ around $p \in M$ and (y_1, y_2) around $F(p) \in \mathbb{R}^2$ such that $F = (F_1(x), F_2(x))$ locally has one of the following forms.

$$(F_1(x), F_2(x)) = (x_1, x_2), \qquad (p \text{ is a regular point}),$$
$$(F_1(x), F_2(x)) = (x_1^2, x_2), \qquad (p \text{ is a fold point}),$$
$$(F_1(x), F_2(x)) = (x_1^3 + x_1 x_2, x_2), \qquad (p \text{ is a cusp point}).$$

The structure of the singularity set $\Sigma(F)$ of F mentioned in the introduction is a straightforward consequence of this local characterization, see [1]. If $p \in M$ is a cusp point of F then, following [2], we define *the characteristic vector* at $F(p)$ to be the vector w tangent to $F(\Sigma(F))$ at $F(p)$ which points into the region where the map has local multiplicity 1. For convenience, we will also call a vector v in T_pM such that $dF(v) = w$ a characteristic vector at p, see Figure 1.

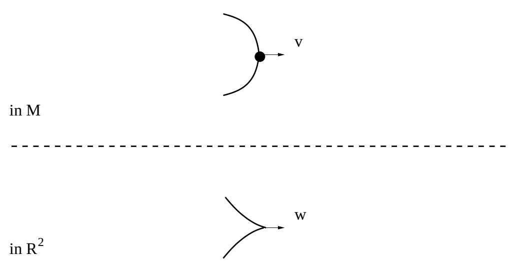

Figure 1. The characteristic vector at a cusp.

2.2. **Punctured disks.** For $m \geq 0$, define an m-punctured disk D_m to be the unit disk D with m open balls, with mutually disjoint closures, removed from its interior: $D_m = D - (\cup_{j=1}^m B_j)$. Then D_m is a 2-manifold with boundary. We consider the boundary component $\partial D \subset \partial D_m$ of D_m as distinguished. We call it the *outer* boundary component of D_m and denote it ∂D_m^0. Other boundary components of D_m are called *inner*, we denote their union ∂D_m^*.

2.3. **The tree of an admissible map.** Let $F\colon D_m \to \mathbb{R}^2$ be an admissible map. The *tree* $\Gamma(F)$ of F is the directed graph defined as follows.

- **V.** The vertices of $\Gamma(F)$ are the distinguished boundary component ∂D_m^0 and the fold components of F.
- **E.** There is an edge connecting two vertices α and β of $\Gamma(F)$ if there exists a component $E \subset (D_m - \Sigma(F))$ such that $\alpha \cup \beta \subset \partial E$ and, if $\alpha \neq \partial D_m^0$ and $\beta \neq \partial D_m^0$ then α separates β from ∂D_m^0. Moreover, we direct an edge with one end on ∂D_m^0 away from ∂D_m^0 and we direct an edge between $\alpha \neq \partial D_m^0$ and $\beta \neq \partial D_m^0$ as above from α to β.

It is easy to verify that $\Gamma(F)$ is a directed tree. We say that a fold γ in $\Gamma(F)$ is a *level k fold* if the shortest path in $\Gamma(F)$ from ∂D_m^0 to γ is k edges long.

2.4. **Orienting immersions and coorienting their boundaries.** Let M be an orientable 2-manifold with boundary ∂M. Let $F\colon M \to \mathbb{R}^2$ be an immersion. We orient M by pulling back the standard orientation of $\mathbb{R}^2$ and we orient ∂M by the tangent vector τ such that (n, τ), where n is the outward normal of ∂M in M, represents the positive orientation on M.

Let $F\colon D_m \to \mathbb{R}^2$ be an admissible map and let $\gamma \subset \Sigma(F)$ be a fold component of F. Note that γ subdivides D_m into two components. We denote these components $M(\gamma)^+$ and $M(\gamma)^-$ with notation chosen so that $\partial D_m^0 \subset \partial M(\gamma)^+$. Consider a tubular neighborhood $N(\gamma) = \gamma \times I$ of γ. We write $\partial N = \gamma^+ \cup \gamma^-$ where $\gamma^\pm \subset M(\gamma)^\pm$.

2.5. **Simple convex double points and standard curves.** Let $f\colon S^1 \to \mathbb{R}^2$ be a self transverse immersion. Then the multiple points of f are transverse double points. A double point q of f is called *innermost* if there exists an arc $A_q \subset S^1$ such that $f|\operatorname{int}(A_q)$ is injective and such that $\partial A_q = f^{-1}(q)$. It is easy to see that any self transverse circle immersion has an innermost double point.

An innermost double point q together with a specified arc A_q as above of a self transverse circle immersion f is called *convex* if the planar disk D_q bounded by $f(A_q)$ satisfies

$$\frac{\operatorname{area}(B \cap D_q)}{\operatorname{area}(B)} < \frac{1}{2},$$

for all sufficiently small disks B centered at q. Note that the orientation of S^1 induces an orientation on $f(A_q)$, which is an embedded circle with one corner at q. Smoothing the corner we get an oriented circle embedding. We say that q is *positive (negative)* if the Whitney index of this circle embedding equals 1 (-1).

A convex double point q of f is called *simple* if $\operatorname{int}(A_q)$ does not contain any double point preimages. A self transverse immersion $f\colon S^1 \to \mathbb{R}^2$ is a *standard curve* if all its double points q are innermost and admit arcs A_q so that they are convex and simple. Note that if $f\colon S^1 \to \mathbb{R}^2$ is a standard curve with double points

$\{q_j\}_{j=1}^n$ and if A_{q_j} is the specified arc corresponding to q_j then f maps $S^1 - \bigcup_{j=1}^n A_{q_j}$ to a simple closed planar curve with corners at $q_1, \dots, q_n$.

2.6. **Elementary bordisms of curves.** We define three elementary bordisms of curves. A *bordism* from an immersed curve $f_0 \colon S^1 \to \mathbb{R}^2$ to a an immersed curve $f_1 : S^1 \to \mathbb{R}^2$ is an admissible map $F \colon S^1 \times [0,1] \to \mathbb{R}^2$ such that $F|S^1 \times \{0\} = f_0$ and $F|S^1 \times \{1\} = f_1$.

($\mathbf{0}$) A bordism $F \colon S^1 \times I \to \mathbb{R}^2$ of type ($\mathbf{0}$) introduces two self intersection points p and q, and q is a simple convex double point of f_1. The map F is an immersion, see Figure 2.

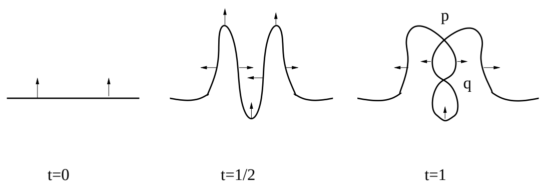

FIGURE 2. An elementary bordism of type ($\mathbf{0}$).

($-$) A bordism $F \colon S^1 \times I \to \mathbb{R}^2$ of type ($-$) removes a convex double point of f_0. The map F has one fold component homotopic to $S^1 \times \{\frac{1}{2}\}$ with one cusp with characteristic vector pointing towards $S^1 \times \{0\}$, see Figure 3.

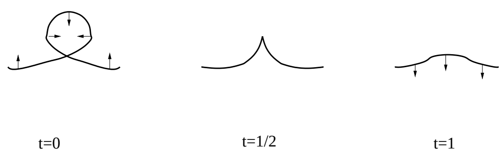

FIGURE 3. An elementary bordism of type ($-$).

(+) A bordism $F\colon S^1 \times I \to \mathbb{R}^2$ of type (+) introduces one convex double point on an embedded arc of f_0. The map F has one fold component homotopic to $S^1 \times \{\frac{1}{2}\}$ with one cusp with characteristic vector pointing towards $S^1 \times \{1\}$, see Figure 4.

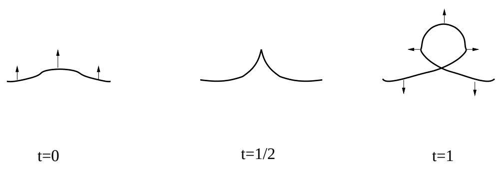

FIGURE 4. An elementary bordism of type (+).

We next describe how the Whitney indices of f_0 and f_1, boundaries of an elementary bordism $F\colon S^1 \times I \to \mathbb{R}^2$, are related. Let n be the outward normal vector field of $\partial(S^1 \times I)$ in $S^1 \times I$. Coorient f_0 by the vector field $dF(n)$ and f_1 by the vector field $-dF(n)$. A straightforward check gives the following. If f_0 and f_1 are related by a bordism of type $(\mathbf{0})$ then

$$W(f_1) = W(f_0). \tag{1}$$

If f_0 and f_1 are related by a bordism with one fold without cusps then

$$W(f_1) = -W(f_0). \tag{2}$$

If f_0 and f_1 are related by a bordism of type $(-)$ then

$$W(f_1) = -W(f_0) + 1. \tag{3}$$

If f_0 and f_1 are related by a bordism of type $(+)$ then

$$W(f_1) = -W_0(f_0) - 1. \tag{4}$$

Note that the elementary bordisms $(\mathbf{0})$, $(-)$, and $(+)$ above can be combined. For example one can construct bordisms which introduces $2j$ double points via $(\mathbf{0})$, creates k simple convex double point via $(+)$, removes l convex double points via $(-)$, and which has one fold component homotopic to $S^1 \times \{\frac{1}{2}\}$ with $k + l$ cusps, k with characteristic vectors pointing towards $S^1 \times \{1\}$ and l with characteristic vectors pointing towards $S^1 \times \{0\}$. Moreover, the Whitney indices of the boundary components of such a cobordism satisfy $W(f_1) = -W(f_0) - k + l$.

2.7. **Cusp elimination.** Following [2], we define a surgery operation which decreases the number of cusps of an admissible map $F\colon M \to \mathbb{R}^2$. Let F be such a map and let $p \neq q$ be points in $\Sigma(F)$ which are cusps. Assume that there exists a smooth arc a in M such that $a \cap \Sigma(F) = \partial a = \{p, q\}$ and assume that the inward normals of ∂a in a are characteristic vectors of p and q. Then we define a new map $F'\colon M \to \mathbb{R}^2$ by redefining F in a neighborhood of a, see Figure 5. Note that F' has two cusps less than F and that $\Sigma(F')$ is obtained from $\Sigma(F)$ by surgery on $\{p, q\} \approx S^0$.

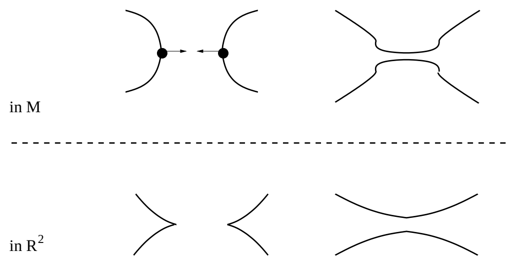

FIGURE 5. Cusp elimination.

2.8. **Euler characteristic and index.** Let $F\colon M \to \mathbb{R}^2$ be an immersion of a manifold with boundary. Let ν be the normalized component of $dF(n)$, where n is the outward normal vector field of ∂M in M, which is orthogonal to ∂M. Orient ∂M by a tangent vector field τ such that (ν, τ) gives the positive orientation of $\mathbb{R}^2$. Then in analogy with the Poincaré-Hopf theorem, see [3],

$$\chi(M) = \mathrm{ind}(\nu), \tag{5}$$

where $\chi(M)$ is the Euler characteristic of M and where $\mathrm{ind}(\nu)$ is the degree of the map $\nu\colon \partial M \to S^1$.

3. PROOF OF THEOREM 1

In this section we present a sequence of lemmas which together constitute a proof of Theorem 1.

Lemma 1. *Let $F\colon D \to \mathbb{R}^2$ be an admissible map. Then the Whitney index $W(F|\partial D)$ of $F|\partial D$ satisfies $W(F|\partial D) \neq C(F) \bmod 2$.*

Proof. To simplify notation, we use the following convention in this proof: if $a \in \mathbb{Z}$ then $\hat{a} = a \bmod 2 \in \mathbb{Z}_2$. Consider the tree $\Gamma(F)$ of F. For each k let $n(k)$ denote the number of fold curves of level k in $\Gamma(F)$ and let $c(k)$ denote the number of cusps on the fold curves of level k. We denote the set of fold curves of level k by

$\{\gamma_j(k)\}_{j=1}^{n(k)}$. Define

$$\hat{W}^{\pm}(k) = \sum_{j=1}^{n(k)} \hat{W}(F(\gamma_j(k)^{\pm})).$$

(Note that if $f\colon S^1 \to \mathbb{R}^2$ is an immersion then $\hat{W}(f)$ is independent of the orientation on S^1.) Let $E(k-1,k)$ denote the submanifold of D which is bounded by

$$\bigcup_{j=1}^{n(k-1)} \gamma_j(k-1)^- \cup \bigcup_{j=1}^{n(k)} \gamma_j(k)^+.$$

Note that $\chi(E(k-1,k)) = n(k-1) - n(k)$ and that $F|E(k-1,k)$ is an immersion.

Let m be the largest integer such that $n(m) \neq 0$. Then, for γ a fold curve of level m, $\hat{W}(F(\gamma^-)) = 1$ since $F(\gamma^-)$ bounds an immersed disk. Thus

$$\hat{W}^-(m) = \hat{n}(m).$$

It then follows from (2)-(4) that

$$\hat{W}^+(m) = \hat{n}(m) + \hat{c}(m).$$

Assume inductively that

$$\hat{W}^+(j) = \hat{n}(j) + \sum_{k=j}^{m} \hat{c}(k).$$

Equation (5) then implies

$$\begin{aligned}
\hat{W}^-(j-1) &= \hat{W}^+(j) + \hat{\chi}(E(j-1,j)) \\
&= \hat{n}(j) + \sum_{k=j}^{m} \hat{c}(k) + \hat{n}(j) + \hat{n}(j-1) \\
&= \hat{n}(j-1) + \sum_{k=j}^{m} \hat{c}(k),
\end{aligned}$$

and another application of (2)-(4) gives

$$\hat{W}^+(j-1) = \hat{n}(j-1) + \sum_{k=j-1}^{m} \hat{c}(k).$$

The final stage of this inductive procedure gives

$$\hat{W}(F|\partial D) = 1 + \sum_{k=1}^{m} \hat{c}(k) = 1 + \hat{C}(F),$$

which is the statement of the lemma. □

Lemma 2. *Let $F\colon D \to \mathbb{R}^2$ be an admissible map with $C(F) = m$. Then there exists an admissible map $G\colon D_m \to \mathbb{R}^2$ such that $F = G$ in some neighborhood of $\partial D = \partial D_m^0$, such that $C(G) = 0$, and such that for all inner boundary components $\delta \subset \partial D_m^*$, $W(G|\delta) = 0$.*

Proof. Let $q \in D$ be a cusp of F. Let $a \subset D$ be a short line segment with one end point at q and directed along a characteristic vector at q. If a is sufficiently short then $a \cap \Sigma(F) = q$. Let B be a ball centered at some point in a such that $F|B$ is an embedding and such that $B \cap \Sigma(F) = \emptyset$. We define the admissible map $F'\colon D_1 \to \mathbb{R}^2$ by letting it agree with F on $D - B$ and by replacing of $F|B$ by the map H of the cylinder $S^1 \times I$ shown in Figure 6. Note that the boundary

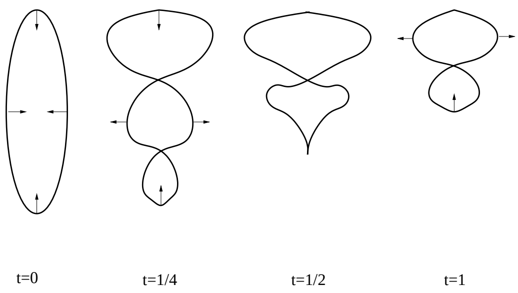

FIGURE 6. The cylinder map glued in.

component of $S^1 \times I$ where H does not agree with $F|\partial B$ is a curve of Whitney index zero. Applying the cusp elimination procedure to the map F' we obtain a map $G'\colon D_1 \to \mathbb{R}^2$ which agrees with F in some neighborhood of ∂D. Repeating this construction at each cusp of F gives the desired map G. $\square$

Lemma 3. *Let $F\colon D_m \to \mathbb{R}^2$ be an admissible map which is not an immersion and let (f,ν) be the cooriented circle immersion $f = F|\partial D_m^0$ and $\nu = dF(n)$ where n is the outward normal vector field of ∂D_m^0. Assume that $W(F|\delta) = 0$ for each inner boundary component $\delta \subset \partial D_m^*$ and that $C(F) = 0$. Then the number of fold components of F is bounded in the following way.*

(a) *If $W(f) \le 0$ then*
$$N(F) \ge \max\left\{\frac{1}{2}\Big(|W(f)| + 1 - m\Big), 1\right\}.$$

(b) *If $W(f) > 0$ then*
$$N(F) \ge \max\left\{\frac{1}{2}\Big(W(f) + 3 - m\Big), 1\right\}.$$

Proof. In the proof below we use the following notational convention: if $F\colon D_m \to \mathbb{R}^2$ is a map and if γ is an oriented curve in D_m such that $F|\gamma$ is an immersion then we write $W(\gamma)$ for $W(F|\gamma)$, suppressing the map from the notation.

The assumption that F is not an immersion implies $N(F) \ge 1$. The lemma thus follows once we establish

$$\begin{aligned} &|W(f)| + 1 - m \le 2N(F) \quad \text{if } W(f) \le 0,\\ (6)\qquad &|W(f)| + 3 - m \le 2N(F) \quad \text{if } W(f) > 0. \end{aligned}$$

We prove (6) by induction on $N(F)$. Assume that $N(F)=1$ and denote its unique fold component γ. Note that γ subdivides D_m into two components $M(\gamma)^+$ and $M(\gamma)^-$, where we use the notation introduced in Subsection 2.4. Furthermore, let $m^\pm$ be the number of inner boundary components of D_m which are subsets of $\partial M(\gamma)^\pm$. Consider slight shrinkings of $M(\gamma)^\pm$ (still denoted by the same symbols) such that $\gamma^\pm \subset \partial M(\gamma)^\pm$. Noting that $F|M(\gamma)^\pm$ are immersions and using the boundary coorientation conventions in Subsection 2.4, we compute, using (5) and the assumption on vanishing Whitney indices for inner boundary components,

$$W(\partial D_m^0) + W(\gamma^+) = \chi(M(\gamma)^+) = -m^+$$

and

$$W(\gamma^-) = \chi(M(\gamma)^-) = 1 - m^-.$$

Since F has no cusps, (2) implies

$$W(\gamma^+) = W(\gamma^-).$$

(Note that the orientation of γ^+ in the present set up differs from the corresponding orientation in Subsection 2.6.) Thus,

$$W(\partial D_m^0) = m^- - (m^+ + 1).$$

Now, if $W(\partial D_m^0) \le 0$ then

$$|W(\partial D_m^0)| + 1 - m = 2 - 2m^- \le 2,$$

and if $W(\partial D_m^0) > 0$ then

$$|W(\partial D_m^0)| + 3 - m = 2 - 2m^+ \le 2.$$

This establishes (6) for $N(F)=1$.

Let $F\colon D_m \to \mathbb{R}^2$ be a map with $N(F)=N$ and assume that (6) holds for all admissible maps $G\colon D_r \to \mathbb{R}^2$ with $N(G) < N$ (for all r). Consider the component E_0 of $D_m - \Sigma(F)$ such that $\partial E_0 - (\partial D_m^*)$ consists of the outer boundary component ∂D_m^0 and level one folds. Let $\{\gamma_j\}_{j=1}^{n(1)}$ be the set of level one folds of F and let m_0 be the number of inner boundary components of D_m which are subsets of ∂E_0. By (5) we have

$$W(\partial D_m^0) + \sum_{j=1}^{n(1)} W(\gamma_j^+) = \chi(E_0) = 1 - n(1) - m_0.$$

Since F has no cusps (2) implies that

$$W(\gamma_j^+) = W(\gamma_j^-).$$

Thus,

$$W(\partial D_m^0) = -\sum_{j=1}^{n(1)} (W(\gamma_j^-) + 1) - (m_0 - 1), \tag{7}$$

which implies

$$|W(\partial D_m^0)| \le \sum_{j=1}^{n(1)} (|W(\gamma_j^-)| + 1) + |m_0 - 1|, \tag{8}$$

Note that $M(\gamma_j)^-$ is a punctured disk with outer boundary γ_j^-. Let m_j be the number of inner boundary components of $M(\gamma_j)^-$. Moreover, $F_j = F|M(\gamma_j^-)$ is an admissible map. Let $N_j = N(F_j)$ and note that $N_j < N$. Let $X \subset \{1, \dots, n(1)\}$ be the subset of j such that $N_j = 0$ and $m_j = 0$, let $Y \subset \{1, \dots, n(1)\}$ be the subset of j such that $N_j = 0$ and $m_j > 0$ and let $Z = \{1, \dots, n(1)\} - (X \cup Y)$. For $j \in X \cup Y$ F_j is an immersion and (5) gives

$$W(\gamma_j^-) = 1 - m_j. \tag{9}$$

For $r \in Z$, the inductive assumption gives

$$|W(\gamma_r^-)| + 1 \le 2N_r + m_r. \tag{10}$$

Equations (8), (9), and (10) imply (with $|A|$ denoting the number of elements in the set A),

$$\begin{aligned}|W(\partial D_m^0)| &\le \sum_{j\in X}(1+1) + \sum_{k\in Y}((m_k - 1) + 1) + \sum_{r\in Z}(2N_r + m_r) + |m_0 - 1|\\ &\le 2|X| + m + 1 + \sum_{r\in Z} 2N_k = 2(N - |Y| - |Z|) + m + 1.\end{aligned}$$

Thus, if $|Y| + |Z| \ge 2$ then $|W(\partial D_m^0)| + 3 - m \le 2N$, and it remains only to check that (6) holds when $|Y| + |Z| < 2$ (and $N > 1$).

Assume first $|Y| = |Z| = 0$. Then $W(\gamma_j^-) = 1$ for $1 \le j \le n(1)$. Equation (7) then implies

$$W(\partial D_m^0) = -2n(1) - m_0 + 1 = -2N - m + 1 < 0,$$

since $N > 1$, and

$$|W(\partial D_m^0)| + 1 - m = 2N.$$

Hence (6) holds if $|Y| = |Z| = 0$.

Assume secondly that $|Y| = 1$ and $|Z| = 0$. Choose notation so that $1 \in Y$. Then $W(\gamma_1^-) = 1 - m_1$ by (5), and (7) gives

$$W(\partial D_m^0) = -2(n(1) - 1) + (m_1 - 2) + 1 - m_0 = m_1 - m_0 + 1 - 2N.$$

Therefore, if $W(\partial D_m^0) > 0$ then

$$W(\partial D_m^0) + 3 - m = 4 - 2m_0 - 2N \le 2N,$$

since $N > 1$, and if $W(\partial D_m^0) \le 0$ then

$$|W(\partial D_m^0)| + 1 - m = 2N - 1 - 2m_1 \le 2N.$$

Hence (6) holds if $|Y| = 1$ and $|Z| = 0$.

Assume thirdly that $|Y| = 0$ and $|Z| = 1$. Choose notation so that $1 \in Z$. Then (7) gives

$$\begin{aligned}W(\partial D_m^0) &= -2(n(1) - 1) - (W(\gamma_1^-) + 1) - m_0 + 1\\ &= 2 - 2n(1) - W(\gamma_1^-) - m_0.\end{aligned}$$

Suppose first that $W(\gamma_1^-) > 0$. Then $W(\partial D_m^0) < 0$ and

$$\begin{aligned}|W(\partial D_m^0)| + 1 - m &= 2n(1) + W(\gamma_1^-) + m_0 - 2 + 1 - m\\ &\le 2n(1) + (2N_1 - 1 + m_1) + m_0 - 1 - m = 2N - 2 \le 2N,\end{aligned}$$

where the inductive assumption and $N_1 > 0$ (since $1 \in Z$) was used. Hence (6) holds if $W(\gamma_1^-) > 0$. Suppose secondly that $W(\gamma_1^-) \le 0$. Then

$$W(\partial D_m^0) = 2 - 2n(1) + |W(\gamma_1^-)| - m_0.$$

If $W(\partial D_m^0) > 0$ then

$$\begin{aligned} W(\partial D_m^0) + 3 - m &= 5 - 2n(1) + |W(F(c_1^-))| - m_0 - m \\ &\leq 5 - 2n(1) + (2N_1 - 1 + m_1) - m_0 - m \\ &= 4 - 2n(1) + 2N_1 - 2m_0 \leq 2N, \end{aligned}$$

since $n(1) \geq 1$ (and hence $N - N_1 \geq 1$). If, on the other hand, $W(\partial D_m^0) \leq 0$ then

$$\begin{aligned} |W(\partial D_m^0)| + 1 - m &= 2n(1) + m_0 - |W(\gamma_1^-)| - 2 + 1 - m \\ &= 2n(1) - 2m_1 - |W(\gamma_1^-)| - 1 \leq 2N. \end{aligned}$$

We conclude that (6) holds in the case $|Y| = 0$ and $|Z| = 1$.

We have thus shown that (6) holds also when $|Y| + |Z| < 2$ for $N(F) = N$. This completes the proof of the lemma. □

Lemma 4. *Let (f, ν) be a cooriented circle immersion with $W(f) = 0$. Then there exists an admissible map $F\colon D \to \mathbb{R}^2$ with boundary values (f, ν), such that $N(F) = C(F) = 1$. Moreover, if γ denotes the fold component of such a map then the characteristic vector at the cusp points into $M(\gamma)^+$.*

Proof. We first note that there exists such a map $F\colon D \to \mathbb{R}^2$ if $f\colon S^1 \to \mathbb{R}^2$ is the figure eight curve with any coorienting vector field, see Figure 7. Note that if γ is

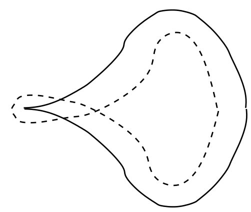

Figure 7. A disk bounded by a figure eight curve. The boundary is dashed and the fold is solid.

the fold component of F then the characteristic vector of the cusp of F points into $M(\gamma)^+$.

Let (f, ν) be any cooriented circle immersion with $W(f) = 0$. We first transform f to a curve of standard from using a combination of elementary bordisms. In the construction below we use the coorientation convention for the boundary of a bordism described in Subsection 2.6.

First note that f has an innermost double point, see Subsection 2.5. Let q be such a double point and let $A_q \subset S^1$ be the corresponding arc where f is injective. Fix some arc $B_q \subset S^1$ such that $B_q \cap A_q = \emptyset$ and such that there are no double point preimages of f in B_q. We construct a bordism $F\colon S^1 \times I \to \mathbb{R}^2$ from $f = f_0$

to f_1 which removes q and introduces simple convex double points with preimages in B_q. Moreover, all fold components γ of F has the following property

(P) γ is homotopic to $S^1 \times \{\frac{1}{2}\}$ and has exactly two cusps with characteristic vectors pointing into different components of $S^1 \times I - \gamma$.

Assume first that q is a convex positive innermost double point. Then we use a bordism which is a combination of elementary bordisms of types $(-)$ and $(+)$: we shrink the loop $f(A_q)$ into a cusp and we create, through a cusp in B_q, a small simple convex double point in B_q. Note that if f_1 is the resulting cooriented circle immersion then $W(f_1) = 0$ by (3) and (4), and that the fold component of F satisfies (**P**).

Assume secondly that q is a convex negative innermost double point. We use a initial bordism of type $(\mathbf{0})$ along B_q which creates double points p', and q' which is a simple convex positive double point in B_q. We compose this initial bordism with a bordism as above removing q' and introducing another simple convex double point in B_q. Note that this bordism satisfies (**P**). Moreover, the coorientation of the resulting f_1 is opposite to that of f and therefore q is a convex positive double point of f_1. Also, p' is a simple convex double point of f_1 in B_q. We now repeat the construction from above removing q and introducing yet another convex double point in B_q.

Assume thirdly that q is a non-convex positive innermost double point. Then we apply an initial bordism shrinking $f(A_q)$, as shown in the first line of Figure 8, creating two small positive convex simple double points. Composing this with

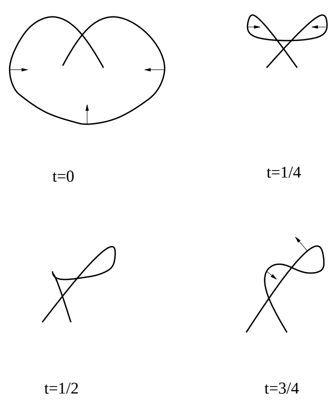

FIGURE 8. The cylinder map glued in.

a bordism as in the first case we remove one of these double points and introduce one convex double point in B_q. We finally continue the bordism as indicated in the last picture in Figure 8 and remove the remaining two double points. The resulting bordism has the desired properties.

Assume fourthly that q is a non-convex negative innermost double then we first reverse the orientation of f as in the second case and then proceed as in the third case. Again the resulting bordism has the desired properties.

Note that for all the bordisms above, the number of double points of f_1 in the complement of the distinguished arc B_q is strictly smaller than the corresponding number of double point for $f = f_0$. To continue the construction we consider the immersion $\tilde{f}$ obtained by deleting the small kinks in B_q from f. Let q_1 be an innermost double point of $\tilde{f}$. If $B_q \cap A_{q_1} = \emptyset$ then q_1 is an innermost double point also of f and the above construction can be continued with $B_{q_1} \subset B_q$. If $B_q \cap A_{q_1} \neq \emptyset$ then $B_q \subset A_{q_1}$. Choose B_{q_1} disjoint from A_q and use initial bordisms as in the first and second cases above to remove all the simple convex double points from B_q at the cost of introducing new convex simple double points in B_{q_1}. The construction then proceeds as above.

This procedure is iterated until f has no double points except for simple and convex double points with preimages in the distinguished arc. This way we have constructed a bordism $F\colon S^1 \times I \to \mathbb{R}^2$ of the required form. Moreover, $W(f_1) = 0$ and thus the sum of the signs of all the double points must equal ± 1.

We remove a pair of a negative and a positive double point using a bordism of the following type. Contract the loop of the positive double point and add a new double point on the negative loop, see Figure 9. (This is a combination of

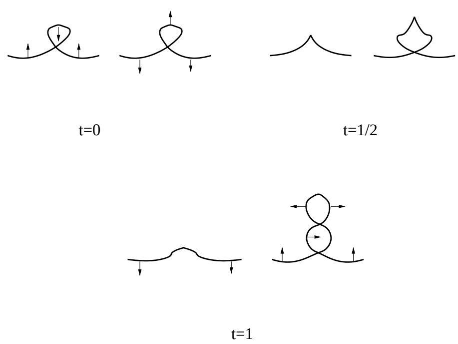

FIGURE 9. Removing two convex double points of opposite signs.

elementary bordisms of types $(-)$ and $(+)$). The resulting curve looks like the result of an $(\mathbf{0})$ elementary bordism with reversed orientation we can thus cancel the two double points. Also this bordism satisfies $(\mathbf{P})$. We continue this removal procedure until we reach a curve (ambient) isotopic to the figure eight curve. Thus, there exists a bordism $F\colon S^1 \times I \to \mathbb{R}^2$ such that $f_0 = f$ and f_1 is a figure eight curve and such that all fold components of F satisfies $(\mathbf{P})$. We complete the map of the cylinder to a map of the disk the disk by filling $F(S^1 \times \{0\})$ by the solution for the figure eight curve discussed above.

We have constructed an admissible map $G\colon D \to \mathbb{R}^2$ such that $\Gamma(G)$ is homeomorphic to an interval with integer endpoints and vertices at the integers between the end points. Moreover there is one characteristic vector pointing in each direction on all fold components except the last (i.e. the innermost) one which have an outwards characteristic vector. Applying cusp cancellation we arrive at the desired $F\colon D \to \mathbb{R}^2$, see Figure 10. $\square$

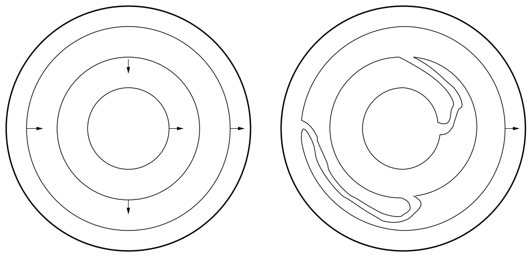

FIGURE 10. Eliminating all cusps but one.

Lemma 5. *Let* (f,ν) *be a cooriented immersion and let* m *be any integer such that* $m \neq W(f) \bmod 2$. *Then there exists an admissible map* $F\colon D \to \mathbb{R}^2$ *with boundary values* (f,ν) *such that* $C(F)=m$ *and such that*

(a) *if* $W(f) \leq 0$ *then*

$$N(F) = \max\left\{\frac{1}{2}\Big(|W(f)|+1-C(F)\Big),\,1\right\}.$$

(b) *if* $W(f)>0$, *then*

$$N(F) = \max\left\{\frac{1}{2}\Big(W(f)+3-C(F)\Big),\,1\right\}.$$

Proof. Assume first that $W(f)>0$. Let $m' = \max\{m - W(f) - 1, 0\}$. Note that m' is even. Create $W(f)+1+\frac{m'}{2}$ simple convex positive double points using a combination of elementary bordisms of type $(\mathbf{0})$. Compose this bordism with a bordism that remove all these $W(f)+1+\frac{m'}{2}$ double points and which introduces $1+\frac{m'}{2}$ simple convex double points. (A combination of elementary bordisms of types $(-)$ and $(+)$.) If $G\colon S^1\times I \to \mathbb{R}^2$ denotes this bordism ($g_0 = f$) then $N(G)=1$, $C(G) = W(f)+2+m'$, and using the orientation convention in Subsection 2.6

$$W(g_1) = -W(f) + \left(W(f)+1+\frac{m'}{2}\right) - \left(1+\frac{m'}{2}\right) = 0.$$

Moreover, $(W(f)+1+\frac{m'}{2})$ of the cusps have characteristic vectors pointing towards $S^1\times\{0\}$ and $(1+\frac{m'}{2})$ of them have characteristic vectors pointing towards $S^1\times\{1\}$. Since $W(g_1)=0$ we can complete the map of the cylinder to a map of a disk gluing in a map as in Lemma 4. We obtain a map $G\colon D\to\mathbb{R}^2$ with $N(F)=2$. Since at least one of the cusps on the level one fold component has its characteristic vector pointing towards the level two fold component and the characteristic vector on the level two fold component is outwards we apply cusp elimination and obtain a map $F\colon D\to\mathbb{R}^2$ with $N(F)=1$ and $C(F)=W(f)+1+m'$. If $m'>0$ then $W(f)+1+m'=m$ and this is the desired map. Assume $m'=0$ then F is a map with $N(F) = 1$ and $C(F) = W(f) + 1$ and all characteristic vectors of the cusps point towards ∂D. Applying cusp elimination to p pairs of such vectors we obtain

$$C(F) = W(f) + 1 - 2p, \quad N(F) = 1 + p.$$

Thus

$$W(f) + 3 - C(F) = 2(p + 1) = 2N(F),$$

as desired.

Assume second that $W(f) \leq 0$. Let $m' = \max\{m - |W(f)| + 1, 0\}$. Note that m' is even. Create $\frac{m'}{2}$ simple convex positive double points using a combination of elementary bordisms of type (**0**). Compose this bordism with a bordism that remove all these $\frac{m'}{2}$ double points and which introduces $|W(f)| + \frac{m'}{2}$ simple convex double points. (A combination of elementary bordisms of types $(-)$ and $(+)$.) If $G\colon S^1 \times I \to \mathbb{R}^2$ denotes this bordism ($g_0 = f$) then $N(G) = 1$, $C(G) = W(f) + m'$, and using the orientation convention in Subsection 2.6

$$W(g_1) = -W(f) - \left(|W(f)| + \frac{m'}{2}\right) + \frac{m'}{2} = 0.$$

Moreover, $\frac{m'}{2}$ of the cusps have characteristic vectors pointing towards $S^1 \times \{0\}$ and $(|W(f)| + \frac{m'}{2})$ of them have characteristic vectors pointing towards $S^1 \times \{1\}$. Since $W(g_1) = 0$ we can complete the map of the cylinder to a map of a disk gluing in a map as in Lemma 4. We obtain a map $G\colon D \to \mathbb{R}^2$ with $N(F) = 2$.

Noting that $|W(f)| = 0$ implies $m' \geq 2$ we conclude $|W(f)| + \frac{m'}{2} > 0$. Thus, at least one of the cusps on the level one fold component has its characteristic vector pointing towards the level two fold component. Since the characteristic vector on the level two fold component is outwards we may apply cusp elimination and obtain a map $F\colon D \to \mathbb{R}^2$ with $N(F) = 1$ and $C(F) = |W(f)| + m' - 1$. If $m' > 0$ then $m = |W(f)| + m' - 1$ and this is the desired map. Assume $m' = 0$ (and hence $W(f) \neq 0$), then F is a map with $N(F) = 1$ and $C(F) = |W(f)| - 1$ and all characteristic vectors of the cusps point inwards. Applying cusp elimination to p pairs of such vectors we obtain

$$C(F) = |W(f)| - 1 - 2p, \quad N(F) = 1 + p.$$

Thus

$$|W(f)| + 1 - C(F) = 2(p + 1) = 2N(F).$$

This completes the proof of the lemma. □

Proof of Theorem 1. The first statement is Lemma 1. The estimates in (a) and (b) follow from Lemmas 2 and 3. The last statement is Lemma 5. □

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