Notes on Riemannian Geometry
Differential Geometry MN1

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1 The Levi-Civita Connection

Let M be a manifold and $\mathcal{V}(M)$ denote the real vector space of smooth vector fields on M. A *connection on* M is a bilinear map

$$\nabla\colon \mathcal{V}(M)\times\mathcal{V}(M)\to\mathcal{V}(M),$$

satisfying

$$\begin{aligned}\nabla_{fX}Y &= f\nabla_X Y\\ \nabla_X fY &= X(f)Y + f\nabla_X Y\end{aligned} \tag{†}$$

for all $X, Y \in \mathcal{V}(M)$ and all smooth functions f on M.

Let furthermore, M be a riemannian manifold with riemannian metric $\langle\,,\rangle$. Then the following theorem holds.

1.1 Theorem. *There is a unique connection* ∇ *satisfying*

(a) $X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$,

(b) $\nabla_X Y - \nabla_Y X = [X, Y]$.

The unique connection above is called the *Levi-Civita connection*. A connection satisfying equation (a) above is called *compatible with the metric*. A connection satisfying equation (b) above is called *torsion free*.

Proof. To show uniqueness it is enough to show that $\langle\nabla_X Y, W\rangle$ is determined by (a) and (b). Using (a),

$$\begin{aligned}X\langle Y, W\rangle &= \langle\nabla_X Y, W\rangle + \langle Y, \nabla_X W\rangle,\\ Y\langle X, W\rangle &= \langle\nabla_Y X, W\rangle + \langle X, \nabla_Y W\rangle,\\ W\langle X, Y\rangle &= \langle\nabla_W X, Y\rangle + \langle X, \nabla_W Y\rangle.\end{aligned}$$

Subtracting the third of these equations from the sum of the first two and using (b) gives

$$\begin{aligned}2\langle\nabla_X Y, W\rangle =& X\langle Y, W\rangle + Y\langle X, W\rangle - W\langle X, Y\rangle\\ &+ \langle[X, Y], W\rangle - \langle[X, W], Y\rangle - \langle[Y, W], X\rangle.\end{aligned} \tag{‡}$$

Conversely, if we use this formula to define $\langle \nabla_X Y, W \rangle$ it is straightforward to verify that we get a connection satisfying (a) and (b). $\square$

We want to express our connection in local coordinates, for simplicity we assume that our riemannian manifold is of dimension two. The general case is completely analougus. Thus, let $x = (x^1, x^2)$ be local coordinates on an open subset of M. To simplify the notation we denote the vector field $\frac{\partial}{\partial x^k}$ simply ∂_k, $k = 1, 2$. The metric is then determined by the functions

$$g_{ij}(x) = \langle \partial_i, \partial_j \rangle,$$

and the 2×2-matrix $g(x) = (g_{ij}(x))$ is symmetric and positive.

Using the properties (†) of the connection we see that it is determined by the functions $\Gamma^k_{ij}(x)$, $i, j, k = 1, 2$, defined by

$$\nabla_{\partial_i} \partial_j = \sum_k \Gamma^k_{ij}(x) \partial_k.$$

The functions Γ^k_{ij} are called *Christofel symbols.*

Consider the Levi-Civita connection ∇. Since it is torsion free and $[\partial_i, \partial_j] = 0$ we see that the Christofel symbols of it satisfies the symmetry relation

$$\Gamma^k_{ij} = \Gamma^k_{ji}.$$

Using equation (‡) we can compute the Christofel symbols from the metric. Letting $g(x)^{-1} = (g^{ij}(x))$, we get

$$\Gamma^k_{ij}(x) = \frac{1}{2} \sum_l g^{kl}(x) \left(\partial_i g_{jl}(x) + \partial_j g_{il}(x) - \partial_l g_{ij}(x) \right).$$

2 Parallel translation

Let c be a curve in a Riemannian manifold M, starting at p and ending at q. Let ∇ be the Levi-Civita connection on M. A vector field Y is *parallel along c* if

$$\nabla_{\dot{c}} Y = 0.$$

Let $x = (x^1, x^2)$ be local coordinates on M. Let c be given by $x(t)$, $Y(t) = y^1(t) \frac{\partial}{\partial x^1} + y^2(t) \frac{\partial}{\partial x^2}$ and Γ^k_{ij} be the Christofel symbols, in these coordinates. Then, writing the above equation out, we get

$$\frac{dy^k}{dt} + \sum_{i,j} \dot{x}^i(t) \Gamma^k_{ij}(x(t)) y^j(t), \quad k = 1, 2.$$

Note that this is a system of linear first order ordinary differential equations. It follows that given an initial value $Y_0 \in T_p M$, there exists a unique parallel vector field $Y(t)$ along c with $Y(0) = Y_0$. Moreover, if $\tau_c \colon T_p M \to T_q M$ is the map $\tau_c(Y_0) = Y(1)$ then τ_c is a linear isometry. (The map τ_c takes a vector v

in T_pM to the value at the endpoint of c, of the parallel vector field with initial value v.) The fact that τ_c is an isometry follows from $\dot{c}\langle Y, Y\rangle = 2\langle \nabla_{\dot{c}} Y, Y\rangle = 0$.

Parallel translation is particularly simple along geodesics. Let c be a geodesic with tangent vector T. Then $\nabla_T T = 0$ and if Y is parallel along c then $\nabla_T Y = 0$ and hence,

$$T\langle Y, T\rangle = \langle \nabla_T Y, T\rangle + \langle Y, \nabla_T T\rangle = 0.$$

Thus, to parallel translate a vector along a geodesic we keep its lenght constant and preserve its angle with the tangent vector of the geodesic. In Section 6 we show that using parallel translation along geodesics and a limiting procedure we can parallel translate along any curve.

3 First variation of arc length and geodesics

We shall show that a curve in a riemannian manifold M is a geodesic if and only if it is a stationary curve of the length function on paths in M that connects its end points.

Let $c\colon [a,b] \to M$ be a curve, $c(a) = p$, $c(b) = q$. A *variation of* c is a map $\omega\colon [a,b] \times (-\epsilon, \epsilon) \to M$, $\epsilon > 0$, such that $\omega(t, 0) = c(t)$. Let $L\colon (-\epsilon, \epsilon) \to \mathbb{R}$ be the function

$$L(s) = \int_a^b \langle \frac{\partial}{\partial t}\omega(t,s), \frac{\partial}{\partial t}\omega(t,s)\rangle^{\frac{1}{2}}\, dt.$$

We say that c is a *stationary curve of the length function on paths in* M *connecting* p *to* q if $\frac{dL}{ds}(0) = 0$ for each variation ω of c that satisfies $\omega(a,s) \equiv p$ and $\omega(b,s) \equiv q$. We see that, if c is the shortest path connecting p to q then c is stationary. The converse is not true in general.

Recall that a curve c is a *geodesic* if $\nabla_{\dot{c}}\dot{c} = 0$.

3.1 Theorem. *A curve c, parametrized by arc length, is stationary if and only if it is a geodesic.*

Proof. Let ω be a variation of c. We must compute $\frac{dL}{ds}(0)$. Let T and W be the vector fields $d\omega\left(\frac{\partial}{\partial t}\right)$ and $d\omega\left(\frac{\partial}{\partial s}\right)$, respectively.

$$\begin{aligned}\frac{dL}{ds}(0) &= \frac{\partial}{\partial s}\int_a^b \langle \frac{\partial}{\partial t}\omega(t,s), \frac{\partial}{\partial t}\omega(t,s)\rangle^{\frac{1}{2}}\, dt|_{s=0} = \int_a^b W\langle T, T\rangle^{\frac{1}{2}}\, dt \\ &= \frac{1}{2}\int_a^b \langle T, T\rangle^{-\frac{1}{2}} W\langle T, T\rangle\, dt = \int_a^b \langle T, T\rangle^{-\frac{1}{2}}\langle \nabla_W T, T\rangle\, dt.\end{aligned}$$

Since $[T, W] = d\omega\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right]$, we get $\nabla_T W = \nabla_W T$ and we may rewrite the last expression as

$$\int_a^b \langle T, T\rangle^{-\frac{1}{2}}\langle \nabla_T W, T\rangle\, dt.$$

Since $|\dot{c}| = 1$ we have

$$\frac{dL}{ds}(0) = \int_a^b \langle \nabla_T W, T\rangle\, dt = \int_a^b \left(T\langle W, T\rangle - \langle W, \nabla_T T\rangle\right)\, dt.$$

Integrating the first term we get

$$\frac{dL}{ds}(0) = \langle W, T\rangle|_a^b - \int_a^b \langle W, \nabla_T T\rangle \, dt.$$

This expression is called *the first variation formula.* The theorem follows from it since, the condition that ω fixes endpoints shows that the first term vanishes and the second term vanishes for all W's if and only if $\nabla_T T = 0$. (The last fact is seen by the usual bump function argument.) □

It can be proved that geodesics minimizes length locally. That is, if c is a geodesic and p is a point on c then there is a neighborhood U of p such that for any point q in U on c, the part of c that connects p to q is the shortest curve connecting p to q.

4 The curvature tensor and gaussian curvature

If b and c are two curves connecting p and q we may compare the results of parallel translation along b and c. The difference is measured by first translating along c and then translating along b with the opposite orientation. Doing this infinitesimally we get the curvature tensor

$$R\colon T_pM \times T_pM \times T_pM \to T_pM, \quad (\xi, \eta, \zeta) \mapsto R(\xi, \eta)\zeta,$$

where $R(\xi, \eta)\zeta$ is the vector obtained from translating ζ around the infinitesimal parallelogram with sides ξ and η.

More precisely, let ξ, η and ζ be vectors in T_pM. Let U be a neighborhood of $0 \in T_pM$ and $\phi\colon U \to M$ be a chart with $d\phi_0 = \mathbf{I}$. Choose $\epsilon > 0$ such that the parallelepiped $\Pi_\epsilon = \{t\xi + s\eta \,|\, 0 \le t \le \epsilon, 0 \le t \le \epsilon\}$ is contained in U. Then $\phi(\Pi_\epsilon)$ is a curvilinear parallelepiped in M. Parallel translation along $\partial\phi(\Pi_\epsilon)$, starting in the direction ξ, defines a linear isometry $L_\epsilon\colon T_pM \to T_pM$. As we shall see in the proof below

$$L_\epsilon = \mathbf{I} + \epsilon^2\mathbf{A} + O(\epsilon^3).$$

We define $R(\xi, \eta)\zeta = \mathbf{A}\zeta$. Note that it follows immediately that $R(\xi, \eta)\zeta = -R(\eta, \xi)\zeta$.

As we have seen, do Carmo p. 96,

$$K(p) = \lim_{D \to p} \frac{\varphi(\partial D)}{\text{area}(D)},$$

where $K(p)$ is the Gaussian curvature at p and $\varphi(\partial D)$ is the angle a vector turns when parallel translated around the boundary of the region D. (We have to compute the area of D with the orientation given by the pair (outer normal, tangent vector to ∂D))

Choosing a pair of orthonormal vectors e_1, e_2 in T_pM and parallel translating e_1 around a small parallelepiped Π_ϵ that they span, we can compute the angle as

$$\varphi(\partial\Pi_\epsilon) = \langle L_\epsilon e_1, e_2\rangle.$$

Thus,

$$\begin{aligned} K(p) &= \lim_{\epsilon\to 0} \frac{\varphi(\partial\Pi_\epsilon)}{\mathrm{area}(\Pi_\epsilon)} = \lim_{\epsilon\to 0} \frac{\langle(\mathbf{I} + \epsilon^2\mathbf{A} + O(\epsilon^3))e_1, e_2\rangle}{\mathrm{area}(\Pi_\epsilon)} \\ &= \frac{\epsilon^2\langle\mathbf{A}e_1, e_2\rangle + O(\epsilon^3)}{\epsilon^2 + O(\epsilon^3)} = \langle\mathbf{A}e_1, e_2\rangle. \end{aligned}$$

Hence, the gaussian curvature is computed in terms of the curvature tensor and the metric as

$$K(p) = \langle R(e_1, e_2)e_1, e_2\rangle.$$

4.1 Theorem. *Let ξ, η and ζ be vectors in T_pM. Then*

$$R(\xi,\eta)\zeta = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z - \nabla_{[Y,X]}Z,$$

where X, Y and Z are any vector fields with $X(p) = \xi$, $Y(p) = \eta$ and $Z(p) = \zeta$.

Proof. Introduce coordinates $x = (x^1, x^2)$ around p in M so that p corresponds to 0, ∂_1 to ξ and ∂_2 to η, respectively. Then Π_ϵ corresponds to $\{x \mid 0 \le x^i \le \epsilon, i = 1,2\}$. We next introduce some notation. A vector field $V = v^1(x)\partial_1 + v^2(x)\partial_2$ will be written as the column vector

$$V(x) = \begin{pmatrix} v^1(x) \\ v^2(x) \end{pmatrix}.$$

Let Γ_1 and Γ_2 be the matrices

$$\Gamma_1(x) = \begin{pmatrix} \Gamma^1_{11}(x) & \Gamma^1_{12}(x) \\ \Gamma^2_{11}(x) & \Gamma^2_{12}(x) \end{pmatrix}, \quad \Gamma_2(x) = \begin{pmatrix} \Gamma^1_{21}(x) & \Gamma^1_{22}(x) \\ \Gamma^2_{21}(x) & \Gamma^2_{22}(x) \end{pmatrix}.$$

Then

$$\nabla_{\partial_k} V(x) = \partial_k V(x) + \Gamma_k(x)V(x).$$

Let Z be the field obtained from parallel translating ζ along the first two sides of $\partial\Pi_\epsilon$. Then Z is continuous and satisfies the differential equations

$$\begin{aligned} &\partial_1 Z(t,0) = -\Gamma_1(t,0)Z(t,0), \quad Z(0,0) = \zeta, \\ &\partial_2 Z(\epsilon,s) = -\Gamma_1(\epsilon,s)Z(\epsilon,s). \end{aligned}$$

The initial value for the second equation is given automatically by continuity. That is, the first equation gives us $Z(\epsilon, 0)$.

Using Taylors formula and the above differential equations we shall compute, up to third order in ϵ, the operator L'_ϵ that takes a vector at 0 to the vector at (ϵ,ϵ) that is its parallel translate along the broken curve c',

$$c'(\tau) = \begin{cases} (\tau, 0) & \text{for } 0 \le \tau \le \epsilon \\ (\epsilon, \tau) & \text{for } \epsilon \le \tau \le 2\epsilon. \end{cases}$$

Taylors formula gives

$$Z(\epsilon, 0) = Z(0) + \epsilon \partial_1 Z(0) + \tfrac{\epsilon^2}{2} \partial^2 Z(0) + O(\epsilon^3).$$

Using the differential equation above we see that

$$\begin{aligned}
&\partial_1 Z(0) = -\Gamma_1(0) Z(0), \\
&\partial_1^2 Z(0) = -\partial_1 \Gamma_1(0) Z(0) - \Gamma_1(0) \partial_1 Z(0) = -\partial_1 \Gamma_1(0) Z(0) + \Gamma_1(0)\Gamma_1(0) Z(0).
\end{aligned}$$

Thus,

$$\begin{aligned}
Z(\epsilon, 0) &= \zeta - \epsilon \Gamma_1 \zeta + \tfrac{\epsilon^2}{2} \left(\Gamma_1^2 \zeta - \partial_1 \Gamma_1 \zeta \right) + O(\epsilon^3) \\
&= \Big(\mathbf{I} - \epsilon \Gamma_1 + + \tfrac{\epsilon^2}{2} \left(\Gamma_1^2 - \partial_1 \Gamma_1 \right) + O(\epsilon^3) \Big) \zeta,
\end{aligned}$$

where we use the convention $\Gamma_1 = \Gamma_1(0)$. That is, we drop the argument of a function if it is 0. This simplification of notation will be used throughout this proof.

Similarly, we find that

$$Z(\epsilon, \epsilon) = \Big(\mathbf{I} - \epsilon \Gamma_2(\epsilon, 0) + \tfrac{\epsilon^2}{2} \left(\Gamma_2^2(\epsilon, 0) - \partial_2 \Gamma_2(\epsilon, 0) \right) + O(\epsilon^3) \Big) Z(\epsilon, 0).$$

Taylor expanding the terms containg ϵ we get

$$Z(\epsilon, \epsilon) = \left(\mathbf{I} - \epsilon \Gamma_2 + \epsilon^2 \left(-\partial_1 \Gamma_2 + \tfrac{1}{2} \left(\Gamma_2^2 - \partial_2 \Gamma_2 \right) \right) \right) Z(\epsilon, 0).$$

Substituting our previous expression for $Z(\epsilon, 0)$ into this formula we get the desired L'_ϵ:

$$\begin{aligned}
L'_\epsilon = &\mathbf{I} - \epsilon \left(\Gamma_1 + \Gamma_2 \right) \\
&+ \epsilon^2 \left(\Gamma_2 \Gamma_1 - \partial_1 \Gamma_2 + \tfrac{1}{2} \left(\Gamma_1^2 + \Gamma_2^2 \right) - \tfrac{1}{2} \left(\partial_1 \Gamma_1 + \partial_2 \Gamma_2 \right) \right) + O(\epsilon^3).
\end{aligned}$$

Let Λ_ϵ be the operator that takes a vector at 0 to the vector at (ϵ, ϵ) that is its parallel translate along the broken curve c'',

$$c''(\tau) = \begin{cases} (0, \tau) & \text{for } 0 \leq \tau \leq \epsilon \\ (\tau, \epsilon) & \text{for } \epsilon \leq \tau \leq 2\epsilon. \end{cases}$$

As above, we can compute it up to third order:

$$\begin{aligned}
\Lambda_\epsilon = &\mathbf{I} - \epsilon \left(\Gamma_2 + \Gamma_1 \right) \\
&+ \epsilon^2 \left(\Gamma_1 \Gamma_2 - \partial_2 \Gamma_1 + \tfrac{1}{2} \left(\Gamma_2^2 + \Gamma_1^2 \right) - \tfrac{1}{2} \left(\partial_2 \Gamma_2 + \partial_1 \Gamma_1 \right) \right) + O(\epsilon^3).
\end{aligned}$$

Now, if L''_ϵ is the inverse of Λ_ϵ then L''_ϵ corresponds to parallel translation going along c'' backwards and the operator L_ϵ that corresponds to parallel translation along $\partial \Pi_\epsilon$ satisfies $L_\epsilon = L''_\epsilon L'_\epsilon$. To compute L''_ϵ, consider an operator $\mathbf{I} + \epsilon \mathbf{A}$ where ϵ is small. Then

$$(\mathbf{I} + \epsilon \mathbf{A}) \left(\mathbf{I} - \epsilon \mathbf{A} + \epsilon^2 \mathbf{A}^2 + \cdots + (-1)^k \epsilon^k \mathbf{A}^k + \dots \right) = \mathbf{I}$$

and the series in the second paranthesis converges. The operator Λ_ϵ can be written as $\mathbf{I}+\epsilon(\mathbf{A}_1+\epsilon\mathbf{A}_2+O(\epsilon^2))$. According to the above formula the inverse of such an operator is, up to third order

$$\mathbf{I}-\epsilon\mathbf{A}_1+\epsilon^2\left(\mathbf{A}_1^2-\mathbf{A}_2\right)+O(\epsilon^3).$$

Therefore

$$\begin{aligned}
L''_\epsilon =& \mathbf{I}+\epsilon\left(\Gamma_2+\Gamma_1\right)+\epsilon^2\left(\left(\Gamma_2+\Gamma_1\right)^2-\Gamma_1\Gamma_2\right)\\
&+\epsilon^2\left(\partial_2\Gamma_1-\tfrac{1}{2}\left(\Gamma_2^2+\Gamma_1^2\right)+\tfrac{1}{2}\left(\partial_2\Gamma_2+\partial_1\Gamma_1\right)\right)+O(\epsilon^3)\\
=& \mathbf{I}+\epsilon\left(\Gamma_2+\Gamma_1\right)\\
&+\epsilon^2\left(\Gamma_2\Gamma_1+\partial_2\Gamma_1+\tfrac{1}{2}\left(\Gamma_2^2+\Gamma_1^2\right)+\tfrac{1}{2}\left(\partial_2\Gamma_2+\partial_1\Gamma_1\right)\right)+O(\epsilon^3).
\end{aligned}$$

Finally, we can compute $L_\epsilon=L''_\epsilon L'_\epsilon$,

$$\begin{aligned}
L_\epsilon =& \mathbf{I}+\epsilon\left(\Gamma_2+\Gamma_1-\Gamma_1-\Gamma_2\right)-\epsilon^2\left(\Gamma_2^2+\Gamma_1^2+\Gamma_1\Gamma_2+\Gamma_2\Gamma_1\right)\\
&+\epsilon^2\left(\Gamma_2\Gamma_1+\partial_2\Gamma_1+\tfrac{1}{2}\left(\Gamma_2^2+\Gamma_1^2\right)+\tfrac{1}{2}\left(\partial_2\Gamma_2+\partial_1\Gamma_1\right)\right)\\
&+\epsilon^2\left(\Gamma_2\Gamma_1-\partial_1\Gamma_2+\tfrac{1}{2}\left(\Gamma_1^2+\Gamma_2^2\right)-\tfrac{1}{2}\left(\partial_1\Gamma_1+\partial_2\Gamma_2\right)\right)+O(\epsilon^3)\\
=& \mathbf{I}+\epsilon^2\left(\partial_2\Gamma_1+\Gamma_2\Gamma_1-\partial_1\Gamma_2-\Gamma_1\Gamma_2\right)+O(\epsilon^3).
\end{aligned}$$

Thus, if $\zeta=z^1\partial_1+z^2\partial_2$ and we extend it as a constant vector field $Z(x,y)=z^1\partial_1+z^2\partial_2$ then $\partial_k Z=0$, $k=1,2$ and

$$\begin{aligned}
L_\epsilon\zeta =& \zeta+\epsilon^2\left(\left(\partial_2+\Gamma_2\right)\left(\partial_1+\Gamma_1\right)Z-\left(\partial_1+\Gamma_1\right)\left(\partial_2+\Gamma_2\right)Z\right)+O(\epsilon^3)\\
=& \zeta+\epsilon^2\left(\nabla_{\partial_2}\nabla_{\partial_1}Z-\nabla_{\partial_1}\nabla_{\partial_2}Z\right)+O(\epsilon^3).
\end{aligned}$$

Thus, $R(\xi,\eta)\zeta=\nabla_{\partial_2}\nabla_{\partial_1}Z-\nabla_{\partial_1}\nabla_{\partial_2}Z$ in these coordinates.

Finally, let X and Y be any vector fields extending ξ and η. Using Taylor expansion we can write

$$X(x^1,x^2)=\partial_1+x^1V_1+x^2V_2+O(2),\quad Y(x^1,x^2)=\partial_2+x^1W_1+x^2W_2+O(2),$$

where V_i and W_i are constant vector fields, $i=1,2$. Now,

$$\nabla_Y\nabla_XZ-\nabla_X\nabla_YZ=\nabla_{\partial_2}\nabla_{\partial_1}Z-\nabla_{\partial_1}\nabla_{\partial_2}Z+\nabla_{V_2}Z-\nabla_{W_1}Z+O(1).$$

Note that $V_2-W_1=[Y,X](0)$. Hence, letting $x\to 0$ we obtain

$$R(\xi,\eta)\zeta=\nabla_Y\nabla_XZ-\nabla_X\nabla_YZ-\nabla_{[Y,X]}Z.$$

$\square$

5 Geodesics and the exponential map

The geodesic equation $\nabla_{\dot c}\dot c$ takes the form

$$\frac{d^2x^k}{dt^2}+\sum_{i,j=1}^{2}\Gamma_{ij}^k(x(t))\frac{dx^i}{dt}\frac{dx^j}{dt},\quad k=1,2,$$

in local coordinates $x = (x^1, x^2)$, where $c(t)$ corresponds to $x(t) = (x^1(t), x^2(t))$. Introducing the variables $v^i = \frac{dx^i}{dt}$ this system can be rewritten as a first order system of four equations

$$\begin{cases} \frac{dx^k}{dt} = v^k & k = 1, 2, \\ \frac{dv^k}{dt} + \sum_{i,j=1}^{2} \Gamma_{ij}^k(x(t))v^i(t)v^j(t) & k = 1, 2. \end{cases}$$

Then, by the theorem of existence and uniqueness of ordinary differential equations, we see that, given a point (x_1, v_1), there exist a neighborhood W of it and a number $\epsilon > 0$ such that for any $(x_0, v_0) \in W$ this differential equation has a unique solution $(x(t), v(t))$, defined for $|t| < \epsilon$ and satisfying $(x(0), v(0)) = (x_0, v_0)$. Furthermore, the solution depends smoothly on the initial conditions. Using this we can prove the following.

5.1 Proposition. *For every point p_0 on a Riemannian manifold M there exists a neighborhood U and $\epsilon > 0$ so that: For each $p \in U$ and $\xi \in T_pM$ with length $< \epsilon$ there exists a unique geodesic*

$$\gamma_\xi \colon (-2, 2) \to M,$$

satisfying

$$\gamma_\xi(0) = p, \quad \dot\gamma_\xi(0) = \xi.$$

Proof. By the above there exists a neighborhood U of p_0 and numbers ϵ_1 and δ so that for each $p \in U$ and $\xi \in T_pM$ with $|\xi| \leq \epsilon$ there is a unique geodesic

$$\gamma_\xi \colon (-2\delta, 2\delta) \to M,$$

satisfying the required initial conditions. Now, if a curve $\gamma(t)$ is a geodesic then the curve $\gamma(ct)$, for any constant c is also a geodesic. Finally, if $\epsilon < \epsilon_1\delta$ and $|t| < 2$ then

$$|\frac{\xi}{\delta}| < \epsilon_1 \quad \text{and} \quad |\delta t| < 2\delta.$$

Hence, we can define $\gamma_\xi(t) = \gamma_{\frac{\xi}{\delta}}(\delta t)$. $\square$

Let $\xi \in T_pM$ be a tangent vector and suppose that there exists a geodesic $\gamma\colon [0, 1] \to M$ satisfying

$$\gamma(0) = p, \quad \dot\gamma(0) = \xi.$$

Then the point $\gamma(1) \in M$ will be denoted $\exp_p(\xi)$ and called *the exponential of* ξ. The geodesic can thus be written

$$\gamma(t) = \exp_p(t\xi).$$

Proposition 5.1 says that $\exp_p(\xi)$ is defined provided $|\xi|$ is small enough and that it is smooth. Moreover, its differential at $0 \in T_pM$ is the identity and so, by the implicit function theorem, $\exp_p$ is a diffeomorphism from some open neighborhood of the origin in T_pM onto some open subset around p in M.

We next prove a result known as *the Gauss lemma*, which we shall use to calculate the metric in polar coordinates around $p \in M$. Let T_pM be the tangent space to M at p. Since $T_\xi(T_pM)$ is canonically isomorphic to T_pM each $T_\xi(T_pM)$ inherits an inner product from the Riemannian metric on T_pM. Thus, we can measure angles between vectors in $T_\xi(T_pM)$ for any $\xi \in T_pM$.

5.2 Lemma. *If $\rho(t) = t\xi$ is a ray through the origin in T_pM and η is a vector in $T_{\rho(t)}(T_pM)$ that is perpendicular to $\dot{\rho}(t)$, then $d\exp(\eta)$ is perpendicular to $d\exp(\dot{\rho}(t))$.*

Proof. Let $c(s)$ be a curve in T_pM such that $c(0) = \xi$, $\dot{c}(0) = \eta$ and such that every point of c is at the same distance from the origin in T_pM. Let $\omega(t,s)$ be the rectangle in M defined by

$$\omega(t,s) = \exp(\rho_s(t)),$$

where $\rho_s(t)$ is the ray from 0 to $c(s)$ in T_pM. From the definition of exp we know that the lenghts of the curves $t \mapsto \omega(t,s)$ are independent of s. Also, $t \mapsto \omega(t,0)$ is a geodesic, and

$$\frac{\partial\omega}{\partial s}(0,0) = 0, \quad \frac{\partial\omega}{\partial s}(t,0) = d\exp(\eta).$$

The first variation formula gives

$$0 = \frac{\partial}{\partial s}L(\exp(\rho_s))\Big|_{s=0} = \langle\frac{\partial\omega}{\partial s}(\tau,0), \dot{\gamma}_\xi(\tau)\rangle\Big|_0^t = \langle d\exp(\eta), d\exp(\dot{\rho}(t))\rangle.$$

□

Let $p \in M$ be a point on a Riemannian surface. We can introduce polar coordinates (r,θ) on a neighborhood of the origin in T_pM: Choose an orthonormal basis e_1, e_2 and let

$$(r,\theta) \mapsto r(\cos(\theta)e_1 + \sin(\theta)e_2).$$

Via the eponential map get coordinates on a neighborhood of p,

$$\phi(r,\theta) = \exp_p\left(r(\cos(\theta)e_1 + \sin(\theta)e_2)\right).$$

By the Gauss lemma we get,

$$\begin{aligned} g_{11} &= \left\langle d\phi\frac{\partial}{\partial r}, d\phi\frac{\partial}{\partial r}\right\rangle = 1, \\ g_{12} = g_{21} &= \left\langle d\phi\frac{\partial}{\partial r}, d\phi\frac{\partial}{\partial \theta}\right\rangle = 0, \\ g_{22} &= \left\langle d\phi\frac{\partial}{\partial \theta}, d\phi\frac{\partial}{\partial \theta}\right\rangle. \end{aligned}$$

Thus, the metric can be written as

$$ds^2 = dr^2 + g_{22}(r,\theta)d\theta^2,$$

in these coordinates. To conform with our notation used for Christofel symbols, let $r = x^1$ and $\theta = x^2$. Then,

$$g^{11} = 1, \quad g^{12} = 0, \quad g^{22} = \frac{1}{g_{22}}$$

and the formulas for Christofel symbols gives

$$\begin{aligned}
\Gamma^1_{12} &= \Gamma^1_{21} = \Gamma^1_{11} = \Gamma^2_{11} = 0 \\
\Gamma^1_{22} &= -\frac{\partial_1 g_{22}}{2} \\
\Gamma^2_{12} &= -\Gamma^2_{21} = \frac{\partial_1 g_{22}}{2 g_{22}}.
\end{aligned}$$

Using the curvature tensor to express the Gaussian curvature we get

$$K(x^1, x^2) = -\frac{\partial_1 \partial_1 \sqrt{g_{22}}}{\sqrt{g_{22}}}.$$

The last equation can be seen as a differential equation for $E = \sqrt{g_{22}}$,

$$\partial_1 \partial_1 E(x) + K(x) E(x) = 0.$$

6 Jacobi fields

Let $c = c(t)$, 0 be a unit speed geodesic on M. A vector field $Y(t)$ along c is a *Jacobi field* if

$$\langle \dot{c}(t), Y(t) \rangle = 0 \quad \text{and } \nabla_{\dot{c}} \nabla_{\dot{c}} Y(t) + K(c(t)) Y(t) = 0,$$

where K is the Gaussian curvature. Let $e_1(t), e_2(t)$ be a Frenet frame along c. That is, $e_1(t) = \dot{c}(t)$ and $e_2(t)$ is the parallel translate along c of a unit vector at $c(0)$, orthogonal to $e_1(0)$. We can then write $Y(t) = y(t) e_2(t)$ for some smooth function y. The Jacobi equation above can be written as

$$\ddot{y}(t) + K(c(t)) y(t) = 0,$$

since $\nabla_{\dot{c}} e_2 = 0$. From the existence and uniqueness theorem for ordinary differential equations it follows that there exists a unique Jacobi field $Y(t) = y(t) e_2(t)$ satisfying the inital conditions $y(0) = a_0 e_2(0)$ and $\nabla_{\dot{c}} Y(0) = \dot{y}(0) e_2(0) = a_1 e_2(0)$, for any $a_0, a_1 \in \mathbb{R}$.

We next show how to produce a Jacobi field.

6.1 Lemma. *Let $c(t)$ be a unit speed geodesic with $c(0) = p \in M$. Let $\rho(t)$ denote the segment $t\dot{c}(0)$, $0 \leq t \leq a$ in T_pM. Let $A \in T_pM$ be a vector orthogonal to $\dot{c}(0)$. Then*

$$Y(t) = (d \exp_p)_{\rho(t)}(tA) \in T_{c(t)} M$$

is a Jacobi field along $c(t)$. Moreover, $Y(t)$ satisfies the initial conditions $Y(0) = 0$ and $\nabla_{\dot{c}} Y(0) = A$. Since a Jacobi field is uniquely determined by the initial conditions, every Jacobi field with $Y(0) = 0$ arises in this way.

Proof. Without loss of generality we may assume $A \neq 0$. The solutions of the Jacobi equation forms a vector space, therefore we may assume $|A| = 1$.

Consider the orthonormal basis $\{e_1(p), e_2(p)\} = \{\dot{c}(0), A\}$ in T_pM. For sufficiently small $\delta, \epsilon > 0$, we will define $\phi\colon [0, a+\delta) \times (-\epsilon, \epsilon) \to M$ by

$$\phi(r, \theta) = \exp_p(r\cos(\theta)e_1 + r\sin(\theta)e_2).$$

For sufficiently small r this is a polar coordinate system centered at p. We need to show that there exists $\delta > 0$ and $\epsilon > 0$ so that ϕ is defined. Notice that $\phi(t, 0)$ is defined for $t \in [0, a]$. In fact $\phi(t, 0) = c(t)$. Moreover, $\phi([0, a], 0)$ is compact. If $U \subset T_pM$ is the domain of definition of $\exp_p$ then $\phi([0, a], 0)$ lies in $\exp_p(U)$, an open set (why?). The existence of δ and ϵ now follow from the compactness of $\phi([0, a], 0)$.

Let $e_1(t), e_2(t)$ be a Frenet frame along $c(t)$, with $e_1(0) = e_1(p)$ and $e_2(0) = e_2(p)$. Now, $Y(t) = \frac{\partial\phi}{\partial\theta}(t, 0)$, so if we write $Y(t) = y(t)e_2(t)$, then $y^2(t) = |Y(t)|^2$. Hence, wherever ϕ is a coordinate system its metric must have $g_{22}(t, 0) = y^2(t)$.

Using the implicit function theorem we see that ϕ is a coordinate system in a neighborhood of any point $(t, 0)$ where $y(t) \neq 0$, that is, where $Y(t) \neq 0$. On such a neighborhood we have geodesic polar coordinates and we can use the formula

$$K = -\frac{\partial_1\partial_1\sqrt{g_{22}}}{\sqrt{g_{22}}},$$

for the Gaussion curvature, where we consider $(r, \theta) = (x^1, x^2)$. Since $g_{22}(t, 0) = y^2(t)$, the above formula implies that, at least for $t \in [0, a+\delta)$ where $y(t) \neq 0$,

$$\ddot{y}(t) + K(c(t))y(t) = 0.$$

We need to take care about the points where $y(t) = 0$. Such points must be isolated, for if t_0 were a nonisolated point of this set then $Y(t_0)$ would be the unique Jacobi field with $Y(t_0) = \nabla_{\dot{c}}Y(t_0) = 0$. Hence, $Y(t) \equiv 0$, contradicting $\nabla_{\dot{c}}Y(0) = A \neq 0$. Thus, $y(t)$ is defined and smooth for all t and satisfies the differential equation except at isolated points. It follows by continuity that the differential equation is satisfied everywhere.

Finally, in the polar coordinates used $Y(t)$ corresponds to the vector field $t\partial_2$, $\partial_2(0)$ to A and $\partial_1(0)$ to $\dot{c}(0)$. Hence, $\nabla_{\dot{c}}Y(0)$ corresponds to

$$\lim_{t\to 0} \nabla_{\partial_1} t\partial_2 = \lim_{t\to 0}(\partial_2 + t\nabla_{\partial_1}\partial_2) = \partial_2(0).$$

That is, $\nabla_{\dot{c}}Y(0) = A$. □

Using the Jacobi equation we get the following Taylor expansion for a Jacobi field $Y(t) = y(t)e_2(t)$, with $y(0) = 0$ and $\dot{y}(0) = 1$, along $c(t)$, $c(0) = p$:

$$y(t) = t - K(p)\frac{t^3}{6} + O(t^4).$$

6.2 Proposition. *Let $L(r)$ be the length of the geodesic circle of radius r at p. Then, for small r,*

$$L(r) = 2\pi r - 2\pi K(p)\frac{r^3}{6} + O(r^4).$$

Let $A(r)$ be the area of the geodesic disk of radius r at $p \in M$. Then, for small r,

$$A(r) = \pi r^2 - \pi K(p)\frac{r^4}{12} + O(r^5).$$

As an immediate corollary we get two geometric ways to compute the Gaussian curvature: It measures the deviations of the geometric functins $L(r)$ and $A(r)$ from the corresponding Euclidean quantities.

6.3 Corollary.

$$K(p) = \lim_{r\to 0} \frac{2\pi r - L(r)}{r^3} \cdot \frac{3}{\pi} = \lim_{r\to 0} \frac{\pi r^2 - A(r)}{r^4} \cdot \frac{12}{\pi} =$$

□

Proof. Let e_1, e_2 be an orthonormal basis in T_pM. The unit circle in T_pM is $b(s) = \cos(s)e_1 + \sin(s)e_2$. The geodesic circle c_r of radius r can be expressed as $\exp_p(rb(s))$, $0 \le s \le 2\pi$. The value of $\dot{c}_r = (d\exp_p)_{rb(s)}(r\dot{b}(s))$ can be interpreted as the value at $t = r$ of the unique Jacobi field $Y(t,s) = (d\exp_p)_{tb(s)}(t\dot{b}(s))$ along $\exp_p(tb(s))$, $0 \le t \le r$. Since $|\dot{b}(s)| = 1$, we have

$$|Y(r,s)| = r - K(p)\frac{r^3}{6} + O(r^4),$$

for r small. Thus,

$$L(c_r) = \int_0^{2\pi} |Y(r,s)|\, ds = \int_0^{2\pi} \left(r - K(p)\frac{r^3}{6} + O(r^4)\right)\, ds$$

and

$$A(r) = \int_0^r \int_0^{2\pi} |Y(r,s)|\, ds\, dt = \int_0^{2\pi} \left(r - K(p)\frac{r^3}{6} + O(r^4)\right)\, ds\, dt.$$

This proves the proposition. □

To understand the last computation properly, recall that the area form at a point p on a surface was defined as follows. Pick an orthonormal frame e_1, e_2 around p. Let ω_1, ω_2 be the dual coframe. That is, $\omega_i(e_j) = \delta_{ij}$. Then $\omega_1 \wedge \omega_2$ is the area from of M at p.

Performing the nessecary computations we see that given coordinates $x = (x^1, x^2)$ around p the volume form is

$$\sqrt{g(x)}\, dx^1 \wedge dx^2,$$

where $g(x) = \det(g_{ij}(x))$.

We end this section by describing parallel translation in a more geometric way. First note that parallel translation along a unit speed geodesic is very simple: Just keep the length of the vector and the angle between it and the tangent vector of the geodesic constant.

Let $c\colon [0,a] \to M$ be a curve of length l. Subdivide c into N parts of equal length. If N is large enough, each part is contained in some geodesic disk of radius $\frac{l}{N}$ and we can connect the points of subdivison with geodesic segments contained in the same geodesic disks. Let A be a vector in $T_{c(0)}M$ and p_i be the points of subdivision. If we parallel translate A along c we get a vectors $A(p_i) \in T_{p_i}M$. If we parallel translate it along the geodesic segments γ_i from p_i to p_{i+1} we get vectors $A'(p_i)$. Now, the difference of parallel translation along γ_i and the part of c between p_i and p_{i+1} is the integral of the curvature over the region that these curves bound and hence is $O(\frac{1}{N^2})$, since the curvature is bounded on some compact neighborhood of c and the region is contained in a geodesic disk of radius $\frac{l}{N}$. Thus,

$$A(c(a)) - A'(c(a)) = N \cdot O\left(\frac{1}{N^2}\right) = O\left(\frac{1}{N}\right).$$

Hence, as $N \to \infty$ $A'(c(a)) \to A(c(a))$.

That is, to parallel translate along a curve we can approximate it with small geodesic segments parallel translate along these and take the limit as the approximation gets better and better.