NOTES FOR THE FMB-COURSE
PDE AND RIEMANNIAN GEOMETRY
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1. Manifolds

A subset $M \subset \mathbb{R}^N$ is a n-dimensional submanifold if in a neighborhood U of every point $p \in M$ there exists $N-n$ smooth functions $f_j \colon U \to \mathbb{R}$, $j = 1, \dots, N-n$ such that

$$U \cap M = \{x \in U \colon f_1(x) = \cdots = f_{N-n}(x) = 0\},$$

and such that the vectors $\nabla f_1(x), \dots, \nabla f_{N-n}(x)$ are linearly independent for $x \in U$.

More abstractly, a set M is given the structure of a *differential n-dimensional manifold* if it is provided by a finite or countable collection of *compatible charts* (see below), so that every point in M is represented in at least one chart.

A chart is an open set $U \subset \mathbb{R}^n$ together with a $1-1$ mapping $\phi \colon U \to \phi(U) \subset M$. If p and p' in two charts U and U' have the same image in M then p and p' have neighborhoods $V \subset U$ and $V' \subset U'$ with the same image in M. In this way we obtain a mapping ${\phi'}^{-1}\phi \colon V' \to V$. This is a mapping between open subsets of Euclidean space. The charts are *compatible* is these functions are differentiable.

An *atlas* is a union of compatible charts. Two atlases are equivalent if their union is an atlas. A *differential structure* on M is a class of equivalent atlases.

A *neighborhood* of a point p in a manifold M is the image under a mapping $\phi \colon U \to M$ of a neighborhood of the representative of p in the chart U.

We assume that any two distinct points in M have non-intersecting neighborhoods.

It can be shown that any n-manifold can be embedded into some Euclidean space (in fact it can be embedded into $\mathbb{R}^{2n}$) so that the first definition is no less general than the second.

Excercise 1.1. Equip a submanifold with the structure of a manifold as defined above.

2. Tangent spaces

If $M \subset \mathbb{R}^N$ is a manifold then the tangent space T_pM of M at $p \in M$ is the space of vectors tangent to M at p, or in other words the orthogonal complement of the vectors $\nabla f_1(p), \dots, \nabla f_{N-n}(p)$.

We can also define tangent vectors directly as tangent vectors of curves in M,

$$\dot\gamma(0) = \lim_{t \to 0} \frac{1}{t}\Big(\gamma(t) - \gamma(0)\Big); \quad \gamma(0) = p,\, \gamma(t) \in M.$$

The latter definition makes sense also for non-embedded manifolds: two curves in $\gamma(t)$ and $\eta(t)$ in M are called equivalent at p if $\gamma(0) = \eta(0) = p$ and

$$\lim_{t\to 0} \frac{1}{t}\Big(\gamma(t) - \eta(t)\Big) = 0$$

in some (and hence any, why?) chart. We may now define a tangent vector of M at p as an equivalence class of curves at p and the collection of tangent vectors as the tangent space T_pM.

We note that T_pM is a vector space and that if $\phi\colon U \to M$ is a chart with coordinates $x = (x^1, \dots, x^n)$ and $\phi(0) = p$ then the vectors

$$\frac{\partial}{\partial x^j} = \partial_j = \frac{d}{dt}(0, \dots, t, \dots, \dots, 0),$$

where t is in the j^{th} position, form a basis of T_pM.

3. The tangent bundle

The union of tangent spaces to M at various points

$$\bigcup_{p\in M} T_pM$$

has a natural differential manifold structure of dimension $2\dim(M)$. This manifold is called the tangent bundle of M and denoted TM. Note that there is natural map $\pi\colon TM \to M$ which maps a tangent vector v at a point p to the point p and that $\pi^{-1}(p) = T_pM$.

If $M \subset \mathbb{R}^N$ is a manifold with local defining functions $f_1, \dots, f_{N-n}$ in U around $p \in M$ then $TM \cap U \times \mathbb{R}^N \subset \mathbb{R}^N \times \mathbb{R}^N$ is given by the equations (in coordinates $(x, v) \in U \times \mathbb{R}^N$)

$$f_j(x) = 0, \quad \nabla f_j(x) \bullet v = 0, \quad j = 1, \dots, N-n,$$

where $\bullet$ denotes the dot product on $\mathbb{R}^N$.

Excercise 3.1. Produce from an atlas on a manifold M an atlas on TM.

4. Riemannian manifolds

If $M \subset \mathbb{R}^N$ is an embedded manifold then the metric in $\mathbb{R}^N$ allows us to measure the lengths of curves, angles between vectors, volumes, etc. All of these quantities are expressed by means of lengths of tangent vectors. That is by the positive definite quadratic form given on each tangent space T_pM,

$$T_pM \times T_pM \to \mathbb{R}, \quad v, w \mapsto \langle v, w\rangle = v \bullet w.$$

Excercise 4.1. Let γ be a smooth curve in M. Express the length of γ in terms of the above quadratic form.

In general a differential manifold with a smoothly varying positive definite quadratic form $\langle\,,\rangle$ on its tangent spaces is called a *Riemannian manifold.* In local coordinates $x = (x^1, \dots, x^n)$ around $p \in M$ the metric can be written as

$$ds^2 = \sum_{i,j} g_{ij}(x) dx^i dx^j,$$

where

$$g_{ij}(x) = \langle \partial_i, \partial_j\rangle.$$

In the sequel we shall employ the *Einstein summation convention: repeated indices are summed over* to simplify expressions as the one above and write simply

$$ds^2 = g_{ij}(x)dx^i dx^j.$$

Excercise 4.2. Show that any manifold can be equipped with a Riemannian metric.

5. Maps between manifolds and differentials

A map $f\colon M \to N$ between manifolds is called differentiable if in local coordinates on M and N it is given by a differentiable function.

The *differential* of a differentiable map $f\colon M \to N$ is the linear map on tangent spaces

$$df_p\colon T_pM \to T_{f(p)}N$$

which is defined as follows. Let $v \in T_pM$ and represent v by a curve $\gamma(t)$, $\gamma(0) = p$, and $\dot\gamma(0) = v$. Then

$$df_p v = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)).$$

Excercise 5.1. Let $x = (x^1, \dots, x^m)$ and $y = (y^1, \dots, y^n)$ be local coordinates on M around p and on N around $f(p)$, respectively. Assume that in these coordinates

$$f(x) = (y^1(x), \dots, y^n(x)).$$

Show that the differential of f at x is the map

$$\frac{\partial}{\partial x^i} \mapsto \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y_j},$$

and conclude that *df* may be viewed as a differentiable map

$$df\colon TM \to TN.$$

Excercise 5.2. If $f\colon M \to N$ and $g\colon N \to K$ are differentiable show $d(f \circ g) = df \circ dg$.

6. Tensor bundles

Associated to the tangent bundle TM of a manifold M are other so called tensor bundles. The simplest is the cotangent bundle T^*M which as a set is

$$T^*M = \bigcup_{p\in M} (T_pM)^*,$$

where $(T_pM)^*$ is the dual space of T_pM. As the tangent bundle, the cotangent bundle comes equipped with a natural differential structure induced from the differential structure on M and with a projection $\pi\colon T^*M \to M$. Namely, if $x = (x^1, \dots, x^n)$ are local coordinates on M then (x, ξ), where $\xi = (\xi_1, \dots, \xi_n)$, are local coordinates on T^*M:

$$(x, \xi) \mapsto \sum_{j=1}^{n} \xi_j \, dx^j,$$

where dx_i is the differential of the function x_j, $dx^j(\partial_k) = \delta^j_k$ (δ^j_k is the Kronecker delta $\delta^k_j = 1$ if $j = k$ and $\delta^k_j = 0$ otherwise).

Similarly, any natural operation on vector spaces gives rise to other bundles. One example is the bundle of *tensors of type* (j,k) T^j_kM which as a set is

$$T^j_kM = \bigcup_{p\in M} \underbrace{T_pM\otimes\cdots\otimes T_pM}_{j} \otimes \underbrace{(T_pM)^*\otimes\cdots\otimes(T_pM)^*}_{k}.$$

In local coordinates, elements in T^j_kM are written as

$$a^{t_1,\dots,t_j}_{s_1,\dots,s_k}\partial_{t_1}\otimes\cdots\otimes\partial_{t_j}\otimes dx^{s_1}\otimes dx^{s_k}.$$

Another example is the bundle of *exterior differential forms* Λ^kT^*M which as a set is

$$\Lambda^kT^*M = \bigcup_{p\in M}\Lambda^k(T_pM)^*.$$

In local coordinates, elements in Λ^kT^*M have the form

$$\sum_{i_1<i_2<\cdots<i_k} a_{i_1,\dots,i_k}\, dx^{i^1}\wedge\cdots\wedge dx^{i_k},$$

where $dx^{i_1}\wedge\cdots\wedge dx^{i_k}\colon (T_pM)^k\to\mathbb{R}$ is the multilinear alternating map

$$dx^{i_1}\wedge\cdots\wedge dx^{i_k}(v_1,\dots,v_k) = \det\left(dx^{i_m}(v_n)\right).$$

Excercise 6.1. Determine the dimensions of the manifolds T^j_kM and Λ^kM in terms of $\dim(M)$.

A (j,k)*-tensor field* on a manifold M is a smooth function

$$\omega\colon M\to T^j_kM$$

such that $\pi\circ\omega=\mathrm{id}$. Similarly we speak of *vector fields* and *fields of exterior forms* which are also called *differential forms.*

An n-manifold M is called *orientable* if Λ^nT^*M admits a non-zero section.

Excercise 6.2. What is a Riemannian metric on M in the terminology above.

Excercise 6.3. Show that a manifold M^n is orientable if and only if it has an atlas such that the Jacobians of all coordinate changes are positive.

Excercise 6.4. What does a tensor field (vector field, differential form) look like in local coordinates? How does it transform under coordinate changes?

7. Examples of manifolds

Excercise 7.1. Show that the n-sphere $S^n=\{x\in\mathbb{R}^{n+1}: |x|=1\}$ is a differentiable manifold.

Excercise 7.2. Show that the set of hyperplanes through the origin in $\mathbb{R}^{n+1}$ has a natural structure as a differential manifold. This manifold is called *real projective n-space* and denoted $\mathbb{R}P^n$. Show also that there is a natural $2-1$ map $S^n\to\mathbb{R}P^n$. For which n is $\mathbb{R}P^n$ orientable?

Excercise 7.3. The space of possible configurations of a mechanical system is called its *configuration space.* Show that the configuration space of a rigid triangle with one corner fixed at the origin in $\mathbb{R}^3$ is the space of 3×3-orthogonal matrices of determinant 1. This space is called $SO(3)$. Show that $SO(3)$ is 3-dimensional and in fact diffeomorphic to $\mathbb{R}P^3$.

8. The gradient

Let M be a manifold with a Riemannian metric $g = \langle\, , \rangle$. If $f\colon M \to \mathbb{R}$ is a function then the gradient of f is the vector field $\nabla f\colon M \to TM$ defined by the equation

$$df_p(v) = \langle \nabla f(p), v\rangle\,, \qquad \text{for all } v \in T_pM. \tag{1}$$

In local coordinates we have

$$\nabla f(x) = g^{ij}(x)\frac{\partial f}{\partial x^i}\partial_j, \tag{2}$$

where $g^{ij}(x)$ is the inverse of the metric tensor $g_{ij}(x)$.

9. The divergence

If $X\colon M \to TM$ is a vector field on M then the divergence $\operatorname{div} X$ of X is the function which measures the rate of change of the volume form ω induced by g under the flow generated by X. In other words it is defined by the equation

$$L_X\omega = (\operatorname{div} V)\omega. \tag{3}$$

Recall (see Taylor p. 68) the formula $L_X\alpha = i_X \circ d\alpha + d(i_X \circ \alpha)$ for the Lie derivative, where i_X denotes contraction with X. Since $d\omega = 0$ (why?). We find

$$(\operatorname{div} X)\omega = d(i_X \circ \omega).$$

In local coordinates $x = (x^1, \ldots, x^n)$ the volume element is

$$\omega(x) = g^{\frac{1}{2}}dx^1 \wedge \cdots \wedge dx^n,$$

where $g = g(x) = \det(g_{ij}(x))$. Thus if $X = X^j\partial_j$ we find

$$i_X \circ \omega = \sum_j (-1)^{j-1}X^j g^{\frac{1}{2}}dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge dx^n.$$

Hence

$$d(i_X \circ \omega) = \partial_j(g^{\frac{1}{2}}X^j)dx^1 \wedge \cdots \wedge dx^n$$

and thus

$$\operatorname{div} X = g^{-\frac{1}{2}}\partial_j(g^{\frac{1}{2}}X^j). \tag{4}$$

Theorem 1. *Let M be a compact oriented Riemannian manifold with boundary ∂M. Let dV and dS denote the volume forms on M and ∂M, respectively. If X is a vector field on M then*

$$\int_M (\operatorname{div} X)\, dV = \int_{\partial M} \langle X, \nu\rangle\, dS,$$

where ν is the outer unit normal vector field along ∂M.

Proof. Add a collar to ∂M and extend the metric to this collar. We obtain in this way a new manifold $\hat{M}$. If ϕ_t denotes the time t flow of X we have for small t

$$\operatorname{vol}(\phi_t(M)) - \operatorname{vol}(M) = t\int_{\partial M} \langle X, \nu\rangle\, dS + \mathcal{O}(t^2).$$

Dividing by t and taking the limit as $t \to 0$ the right hand side approaches

$$\frac{d}{dt}\int_{\phi_t(M)} dV = \int_M L_X(\omega) = \int_M (\operatorname{div} X)\, dV$$

and the left hand side

$$\int_{\partial M} \langle X, \nu \rangle \, dS.$$

□

Corollary 2. *If M is closed, if X is a vector field on M and if u is a function then*

$$\int_M (u(\operatorname{div} X) + \langle \nabla u, X \rangle) \, dV = 0.$$

Proof. $\operatorname{div}(uX) = u(\operatorname{div} X) + \langle \nabla u, X \rangle$. □

10. The Laplacian and its spectrum

Let (M, g) be a Riemannian manifold. The Laplace operator on M is the operator on functions $f\colon M \to \mathbb{R}$ given by

$$\Delta f = \operatorname{div} \nabla f.$$

Applying (2) and (4) we find the following local coordinate expression

$$\Delta f = g^{-\frac{1}{2}} \partial j (g^{\frac{1}{2}} g^{ij} \frac{\partial f}{\partial x^i}) = \Big(g^{ij} \partial_j \partial_i + g^{-\frac{1}{2}} \partial_j (g^{\frac{1}{2}} g^{ij}) \partial_i \Big) f. \tag{5}$$

In particular, we find that Δ is a second order elliptic differential operator.

Theorem 3. *Let M be a compact closed Riemannian n-manifold. Then there exists a complete orthonormal basis $\{u_k\}_{k=1}^{\infty}$ of $L^2(M)$ such that*

$$\Delta u_k + \lambda_k u_k = 0, \quad \textit{for all } k,$$

where u_k is smooth and where $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$. In fact, for some constant $c > 0$ depending only on M,

$$\lambda_k \geq c k^{4/n(n+6)}, \quad \textit{for all } k. \tag{6}$$

Proof. Note first that if $u \in L^2_s$ for some $s \in \mathbb{Z}$ and if $-\Delta u = \lambda u$ then, by the elliptic estimate, u is smooth and by Corollary 2,

$$\lambda \int_M u^2 \, d\mathrm{vol} = \int_M -(\Delta u) u \, d\mathrm{vol} = \int_M \langle \nabla u, \nabla u \rangle d\mathrm{vol} \geq 0.$$

Hence $\lambda \geq 0$ and $\lambda > 0$ unless u is (locally) constant. The fact that each eigenspace of $-\Delta$ is finite dimensional follows from the ellipticity of $\Delta + \lambda \,\mathrm{id}$. We also note that if u_λ and u_ν are eigenfunctions of different eigenvalues then

$$(\lambda - \nu) \int_M u_\lambda u_\nu \, d\mathrm{vol} = \int_M u_\lambda (\Delta u_\nu) - (\Delta u_\lambda) u_\nu \, d\mathrm{vol} = 0,$$

Hence $u_\lambda \perp u_\nu$.

We now turn to (6). Let E_λ denote the eigenspace corresponding to λ. Given $\epsilon > 0$, we say that a subset $A \subset M$ is *ϵ-dense* if for each $x \in M$ there is a point $a \in A$ with $\operatorname{dist}(a, x) < \epsilon$. For each ϵ let $N(\epsilon)$ be the minimal possible number of elements in an ϵ-dense subset. It is then clear that there exists some constant C such that

$$N(\epsilon) \leq C \epsilon^n, \qquad \text{for all } \epsilon. \tag{7}$$

Let $u \in E_\lambda$ and note that for every p, $(-\Delta)^p u = \lambda^p u$. Hence by the elliptic estimate there exists a constant C_p such that

$$\|u\|_{2p} \leq C_p (\|u\|_0 + \|(-\Delta)^p u\|_0).$$

Set $E(\Lambda) = \bigoplus_{\lambda<\Lambda} E_\lambda$ and write $u \in E(\Lambda)$ as $u = \sum a_\lambda u_\lambda$, where $\|u_\lambda\|_0 = 1$. Then $(-\Delta)^p u = \sum a_\lambda \lambda^p u_\lambda$ and since $u_\lambda \perp u_\nu$, we have that $\|(-\Delta)u\|_0^2 = \sum |a_\lambda|^2|\lambda|^{2p} \leq \Lambda^{2p}\sum |a_k|^2 = \Lambda^{2p}\|u\|_0^2$. Consequently,

$$\|u\|_{2p} \leq C_p(1+\Lambda^p)\|u\|_0,$$

for all $u \in E(\Lambda)$. If we assume $2p > \frac{n}{2}+1$ then the Sobolev embedding theorem gives a constant C_p' such that

$$\sup_M |\nabla u| \leq C_p'(1+\Lambda^p)\|u\|_0, \tag{8}$$

for all $u \in E(\Lambda)$.

Suppose for a given $\epsilon > 0$ we have $d(\Lambda) = \dim(E(\Lambda)) > N(\epsilon)$. Then we find an element $u \in E(\Lambda)$ with $\|u\|_0 = 1$ and with $u = 0$ on an ϵ-dense subset. It then follows from (8) that

$$\sup_M |u| \leq \epsilon C_p'(1+\Lambda^p).$$

However, this is impossible if $\epsilon C_p'(1+\Lambda^p)\operatorname{vol}(M)^{\frac{1}{2}} < 1$ since $\int_M |u|^2\, d\mathrm{vol} = 1$. We conclude $d(\Lambda) < N(\epsilon_\Lambda)$ where $\epsilon_\Lambda^{-1} = 2C_p'(1+\Lambda^p)\operatorname{vol}(M)^{\frac{1}{2}} < C_p''\Lambda^p$. Consequently, by (7), we have

$$d(\Lambda) \leq C\epsilon_\Lambda^{-n} \leq c\Lambda^{np}.$$

Taking $p = \frac{n+6}{4}$ completes the proof of (6).

It remains to prove that $\bigoplus_\lambda E_\lambda$ gives a direct Hilbert space decomposition of $L^2(M)$. Let V_s denote the closure in $L_s^2(M)$ of the space of finite linear combinations of eigenfunctions of $-\Delta$.

We must show $V_0^\perp = \{0\}$. Assume not and consider

$$\mu = \inf\left\{(-\Delta u, u) = (\nabla u, \nabla u) \colon u \in V_0^\perp, \|u\|_0 = 1\right\}.$$

Note that $V_1^\perp$ is dense in $V_0^\perp$ and hence $\mu < \infty$. (This density follows from the fact that L_1^2-orthogonality implies L^2-orthogonality, V_1 is dense in V_0 and L_1^2 is dense in L_0^2.)

Choose a sequence $\{u_k\}_{k=1}^\infty \subset V_0^\perp$ with $\|u\|_0 = 1$ and $\lim_k(-\Delta u_k, u_k) = \mu$. Since $\|u_k\|_0^2 + \|\nabla u_k\|_0^2$ is uniformly bounded the Sobolev embedding theorem implies that there exists a subsequence converging in L_0^2. Let u be its limit. We claim

$$(-(\Delta+\mu)u, v)_0 = (u, -(\Delta+\mu)v)_0 = 0, \tag{9}$$

for all $v \in V_2^\perp$. If not then there exists $v \in V_2^\perp$ with $\|v\|_0 = 1$ and $(-(\Delta+\mu)u, v) = -\alpha < 0$. For $t \in \mathbb{R}$ consider $u(t) = u + tv$ and note that

$$0 \leq \frac{(-(\Delta+\mu)u(t), u(t))}{\|u(t)\|_0^2} = \frac{-2\alpha t + \mathcal{O}(t^2)}{1 + 2t(u,v) + t^2} < 0,$$

provided $|t|$ is sufficiently small. This shows that $-(\Delta+\mu)u$ is L^2-orthogonal to $V_2^\perp$. Moreover, $-\mu u$ and $-\Delta u$ are both orthogonal to V_2. Hence $-\Delta u = \mu u$ which together with $\|u\|_0 = 1$ contradicts $u \in V_0^\perp$. Thus $V_0^\perp = \{0\}$. □

11. The heat equation and the maximum principle

The heat equation on a closed Riemannian manifold M is

$$(\partial_t - \Delta)u = 0, \tag{10}$$

for $u \colon M \times (0,\infty) \to \mathbb{R}$.

Solutions of the heat equation abides a maximum principle.

Theorem 4. *If $u \in C([0,T) \times M) \cap C^2((0,T) \times M)$ solves (10) in $(0,a) \times M$ then*

$$\sup_{[0,T)\times M} u(t,x) = \max_M u(x,0).$$

Proof. Let $\sup_M u(x,0) = m$. For small $\epsilon, \delta > 0$ let $u_{\epsilon,\delta} = u(t,x) - \delta - \epsilon t$. We claim that $u_{\epsilon,\delta}(t,x) < m$ for $(t,x) \in M \times [0,T)$. Assume not then there exists a smallest $t = t_0$ and $x_0 \in M$ such that $u(t,x_0) = m$. Then $\partial_t u_{\epsilon,\delta}(t_0,x_0) \geq 0$ and $\Delta u_{\epsilon,\delta}(t_0,x_0) \leq 0$. On the other hand $\partial_t u_{\epsilon,\delta} = \Delta u_{\epsilon,\delta} - \epsilon$, so this yields a contradiction. Since ϵ, δ were arbitrary the theorem follows. □

Excercise 11.1. Show that the solution to the problem on $[0,\infty) \times M$

$$\begin{cases} \partial_t u = \Delta u, \\ u(x,0) = f(x), \end{cases} \tag{11}$$

is unique.

12. The heat kernel

We now turn to the problem of solving (11). Assume that $f(x) = u(x,0) \in L^2(M)$. Then f has a Fourier expansion in terms of orthonormal eigenfunctions $\{\phi_k\}$ of $-\Delta$

$$f = \sum_k c_k \phi_k,$$

since $\Delta\phi_k = -\lambda_k\phi_k$ we see that the solution of (11) formally equals

$$u(t,x) = \sum_k e^{-\lambda_k t} c_k \phi_k(t,x).$$

Noting that $c_k = \int_M f\phi_k \, d\text{vol}$, we find that

$$u(x,t) = \int_M e(t,x,y) f(y) \, d\text{vol}_y,$$

where

$$e(t,x,y) = \sum_k e^{-\lambda_k t}\phi_k(x)\phi_k(y).$$

The function $e(t,x,y)$ is called the *heat kernel on M* and we have

Theorem 5. *For any $r \geq 0$ and any closed interval $I \subset (0,\infty)$ the series*

$$\sum_{k=1}^{\infty} e^{-\lambda_k t}\phi_k(x)\phi_k(y)$$

converges uniformly in the C^r-topology on $I \times M \times M$. In particular, $e(t,x,y)$ is smooth on $(0,\infty) \times M \times M$.

Proof. Fix a positive integer s with $2s > \frac{n}{2} + r$. By the Sobolev embedding theorem and the elliptic estimate there exists constants c and c' such that

$$\|\phi_k\|_{C^r} \leq c(\|\phi_k\|_0 + \|\Delta^s\phi_k\|_0) = c(1 + \lambda_k^s),$$

for all k. By Theorem 3, λ_k satisfies

$$\lambda_k \geq k^{\gamma},$$

where $\gamma = 4/n(n+6)$. This implies

$$e^{-\lambda_k t}\lambda_k^s \leq (e^{-k^{\gamma} t})(k^{s\gamma}),$$

for all $k > (s/t)^{\frac{1}{\gamma}}$. The uniform convergence then follows from the convergence of the integral $\int_1^\infty e^{tx} x^s \, dx$. $\square$

Theorem 6. *If $f \in L^2(M)$, then the function*

$$u(t,x) = \int_M e(t,x,y) f(y) \, d\mathrm{vol}_y$$

is C^∞ on $(0,\infty) \times M$ and satisfies the heat equation.

Proof. The C^∞-statement follows immediately from the smoothness of $e(t,x,y)$. The function u satisfies the heat equation by the calculations above which are rigorous after Theorem 5. $\square$

The *trace* of the heat kernel $e(t,x,y)$ is the function

$$\mathrm{tr}(e^{t\Delta}) = \int_M e(t,x,x) \, d\mathrm{vol}_x = \sum_{k=1}^\infty e^{-\lambda_k t},$$

which is defined and analytic for all $t > 0$.

The heat kernel can also be characterized as follows.

Lemma 7. *If $e(t,x,y) \in C^2((0,\infty) \times M \times M)$ satisfies*

$$(\partial_t - \Delta_y) e = 0,$$

and

$$\lim_{t \to 0} \int_M e(t,x,y) f(y) \, d\mathrm{vol}_y = f(x),$$

for all $f \in L^2(M)$. Then

$$e(t,x,y) = \sum_i e^{-\lambda_i t} \phi_i(x) \phi_i(y).$$

In particular, $e(t,x,y)$ is smooth and equals the heat kernel.

Proof. Fix t, x then

$$e(t,x,y) = \sum_i f(t,x) \phi_i(y)$$

in L^2, where $f(t,x) = \int_M e(t,x,y)\phi_i(y) \, d\mathrm{vol}_y$. Thus

$$\partial_t f(t,x) = \int_M \Delta_y e(t,x,y) \phi_i(y) \, d\mathrm{vol}_y = -\int_M e(t,x,y) \Delta_y \phi_i(y) \, d\mathrm{vol}_y = -\lambda_i f_i(x,t).$$

Hence $f_i(x,t) = e^{-\lambda_i t} k_i(x)$. On the other hand we have

$$\begin{aligned} \phi_i(x) &= \lim_{t \to 0} \int_M e(t,x,y) \phi_i(y) \, d\mathrm{vol}_y \\ &= \lim_{t \to 0} e^{-\lambda_i t} k_i(x) = k_i(x). \end{aligned}$$

Thus

$$e(t,x,y) = \sum_i e^{-\lambda_i t} \phi_i(x) \phi_i(y),$$

in $L^2(M)$ in the y-variable for fixed (t,x). Since both sides in the equation are continuous we find that equality holds at every (t,x,y). $\square$

13. Asymptotics of the heat kernel

In this section we give an alternative construction of the heat kernel which is better suited to determine its asymptotics. We will be concerned with its short time asymptotics which measures in which way the heat kernel approaches the Dirac delta-function as $t \to 0$. As we shall see, the rate depends on the geometry of the manifold.

13.A. **Construction of a parametrix.** A *parametrix* for the heat operator $\partial_t - \Delta$ on M is a C^∞-function $H\colon (0,\infty) \times M \times M \to \mathbb{R}$ such that $(\partial_t - \Delta_y)H(t,x,y) \in C^0([0,\infty) \times M \times M)$ and

$$\lim_{t\to 0} \int_M H(t,x,y) f(y)\, d\mathrm{vol}_y = f(x).$$

Before starting the construction of a parametrix we present two lemmas.

Lemma 8. *For functions f and g on M,*

$$\Delta fg = (\Delta f)g + f(\Delta g) + 2\langle \nabla f, \nabla g\rangle.$$

Proof. Note that $\nabla fg = f(\nabla g) + g(\nabla f)$ and apply the proof of Corollary 2. □

Lemma 9. *Let M be Riemannian manifold and let $p \in M$. Let $(r,\theta) \in [0,\epsilon) \times S^{n-1}$ be polar coordinates around p. Let $f = f(r)$ be a function around $p = \exp_p(0)$ which is independent of $\theta \in S^{n-1}$ and let $D = D(r,\theta) = \det(d\exp_p(r,\theta))$ then*

$$\Delta f(r,\theta) = \partial_r^2 f + \Big(\frac{n-1}{r} + \frac{\partial_r D}{D}\Big).$$

Proof. If $r = x_1$ and $\theta = (x^2, \dots, x^n)$ then

$$g_{1j}(x) = g^{1j}(x) = \delta_{1j}.$$

Since f is independent of θ we find from (5) that

$$\Delta f = g^{-\frac{1}{2}} \partial_1 (g^{\frac{1}{2}} \partial_1 f) = \partial_1^2 f + g^{-\frac{1}{2}}(\partial_1 g^{\frac{1}{2}})\partial_1 f. \tag{12}$$

It remains to compute $g^{\frac{1}{2}}$. Let $e_2, \dots, e_n$ be an orthonormal basis of the tangent space of $rS^{n-1} \subset T_pM$ at $r\theta \in T_pM$. Linear translation of these vectors to $0 \in T_pM$ gives an orthonormal basis in T_pM. Let $X_2, \dots, X_n$ be the vector fields obtained by parallel translation of $e_2, \dots, e_n$ along the radial geodesic from p to $q = \exp_p(r\theta)$. If

$$\hat{Z}_j = \frac{d}{ds}\Big|_{s=0} (\exp_p(r\partial_1 + s\partial_j)) = \frac{d}{ds}\Big|_{s=0} (\exp_p(r(\partial_1 + \frac{s}{r}\partial_j)))$$

then $D = \det(\langle \hat{Z}_i, X_j\rangle)$ but the image of ∂_j at $r\theta$ is given by

$$Z_j = \frac{d}{ds}\Big|_{s=0} \exp_p(r(\partial_1 + s\partial_j)) = r\hat{Z}_j.$$

Hence $g = \det(\langle Z_i, X_j\rangle \langle X_j, Z_k\rangle)$ and $g^{\frac{1}{2}} = r^{n-1} D$. □

We now turn to the construction of the parametrix for the heat operator on M. Fix $\epsilon > 0$ such that the exponential map $\exp_x$ for any $x \in M$ is a diffeomorphism of the ball $B_\epsilon(0) \subset T_xM$ onto a neighborhood V_x of x. For $y \in V_x$ let $r(x,y) < \epsilon$ be the lenght of the radial geodesic connecting x and y.

Let

$$U_\epsilon = \{(x,y) \in M \times M : y \in V_x\}.$$

Then U_ϵ is a neighborhood of the diagonal in $M \times M$. In analogy with the heat kernel on $\mathbb{R}^n$, define $G\colon U_\epsilon \to \mathbb{R}$ by

$$G(t,x,y) = (4\pi t)^{-\frac{n}{2}} e^{\frac{r^2(x,y)}{4t}}.$$

Then G is smooth on U_ϵ.

Fix $k \in \mathbb{Z}_{\geq 0}$ and set

$$S_k = S_k(t,x,y) = G(t,x,y)\left(u_0(x,y) + u_1(x,y)t + \cdots + u_k(x,y)t^k\right),$$

where the u_i are smooth functions to be determined in such a way that $(\partial_t - \Delta_y)S_k$ vanishes to order $k - \frac{n}{2} - 1$ in t. Now,

$$\begin{aligned}\partial_t S_k = G\Big(&\big(-\frac{n}{2t} + \frac{r^2}{4t^2}\big)(u_0(x,y) + u_1(x,y)t + \cdots + u_k(x,y)t^k)\\ &+ (u_1(x,y) + \cdots + ku_k(x,y)t^{k-1}\Big)\end{aligned}$$

and

$$\begin{aligned}\Delta_y S =& (\Delta G)\left(u_0(x,y) + u_1(x,y)t + \cdots + u_k(x,y)t^k\right)\\ &+ 2\left\langle \nabla G, \nabla\left(u_0(x,y) + u_1(x,y)t + \cdots + u_k(x,y)t^k\right)\right\rangle\\ &+ G\,\Delta\big(u_0(x,y) + u_1(x,y)t + \cdots + u_k(x,y)t^k\big).\end{aligned}$$

Also, since G is a function of r only,

$$\Delta G = \frac{\partial^2 G}{\partial r^2} + \frac{\partial G}{\partial r}\Big(\frac{D'}{D} + \frac{n-1}{r}\Big) = \Big(\frac{r^2}{4t} - \frac{n}{2t}\Big)G - \frac{r}{2t}\frac{D'}{D}G.$$

Moreover,

$$\begin{aligned}\big\langle \nabla G, \nabla(u_0 + \cdots + u_k t^k)\big\rangle &= \frac{\partial G}{\partial r}\left(\frac{\partial u_0}{\partial r} + \cdots + \frac{\partial u_k}{\partial r}t^k\right)\\ &= \frac{r}{2t}\left(\frac{\partial u_0}{\partial r} + \cdots + \frac{\partial u_k}{\partial r}t^k\right)G.\end{aligned}$$

We find

$$\begin{aligned}(\partial_t - \Delta_y)S_k = G\Big(&u_1 + \cdots + kt^{k-1}u_k + \frac{r}{2t}\frac{D'}{D}\left(u_0 + \cdots + t^k u_k\right)\\ &+ \frac{r}{t}\left(\frac{\partial u_0}{\partial r} + \cdots + t^k\frac{\partial u_k}{\partial r}\right) + \Delta_y u_0 + \cdots + t^k \Delta_y u_k\Big)\end{aligned}$$

In order to make $(\partial_t - \Delta_y)S_k$ vanish to the desired order we require the so called *transport-equations*

$$\begin{aligned}r\frac{\partial u_0}{\partial r} + \frac{r}{2}\frac{D'}{D}u_0 &= 0,\\ r\frac{\partial u_i}{\partial r} + \Big(\frac{r}{2}\frac{D'}{D} + i\Big)u_i + \Delta u_{i-1} &= 0, \quad i = 1, \ldots, k\end{aligned}$$

to hold. The first equation reduces to

$$\frac{\partial \log u_0}{\partial r} = -\frac{1}{2}\frac{\partial \log D}{\partial r},$$

and hence $u_0 = k(\theta)D^{-1/2}$, where $k(\theta)$ is an arbitrary function of $\theta \in S^{n-1}$. In order to have u_0 defined at the origin $r = 0$ we take $k(\theta)$ constantly equal to 1 and get

$$u_0(x,y) = \left(D(\exp_x^{-1}(y))\right)^{-\frac{1}{2}}.$$

In particular,

$$u_0(x,x) = 1.$$

To find the other u_i we use the method of variation of parameters: the homogenous equation

$$r\frac{\partial u}{\partial r} + \Big(\frac{r}{2}\frac{D'}{D} + i\Big)u = 0.$$

has a solution on the form $kr^{-i}D^{-\frac{1}{2}}$ with $k = k(\theta)$. We then look for u_i on the form $u_i = k(r)r^{-i}D^{-\frac{1}{2}}$ and find that k must satisfy

$$\frac{\partial k}{\partial r} = -D^{-\frac{1}{2}}(\Delta u_{i-1})r^{i-1}$$

and it follows by integration in the r-direction (keeping $\theta \in S^{n-1}$ fixed) that

$$u_i(x, r\theta) = u_i(x,y) = -r^{-i}D^{-\frac{1}{2}}(x,y)\int_0^r D^{\frac{1}{2}}(x,s\theta)\Delta_y u_{i-1}(x,s\theta)s^{i-1}\,ds. \tag{13}$$

In conclusion, with u_i, $i = 0,\dots,k$ defined as above we have

$$(\partial_t - \Delta_y)S_k = (4\pi t)^{-\frac{n}{2}}e^{-\frac{r^2}{4t}}t^k\Delta_y u_k.$$

An easy induction using (13) shows that $u_i \in C^\infty(U_\epsilon)$. We extend the function S_k to a smooth function on all of $(0,\infty)\times M\times M$ by choosing a bump function $\eta\colon M\times M \to [0,1]$ such that $\eta = 1$ on $U_{\epsilon/2}$ and $\eta = 0$ outside U_ϵ. Now set $H_k = \eta S_k \in C^\infty((0,\infty)\times M\times M)$.

Lemma 10. *H_k is a parametrix for $(\partial_t - \Delta)$ if $k > \frac{n}{2}$. In fact, $(\partial_t - \Delta)H_k \in C^l([0,\infty)\times M\times M)$ if $k > \frac{n}{2} + l$.*

Proof. To see that $(\partial_t - \Delta_y)H_k$ extends to $t = 0$, first note that $H_k = 0$ on $(0,\infty)\times M\times(M - U_\epsilon)$ and hence $(\partial_t - \Delta_y)H_k$ trivially extends on this region. On $(0,\infty)\times U_{\epsilon/2}$,

$$(\partial_t - \Delta_y)H_k = (\partial_t - \Delta_y)S_k = (4\pi t)^{-\frac{n}{2}}e^{-\frac{r^2}{4t}}t^k\Delta_y u_k \to 0,$$

as $t \to 0$, and $(\partial_t - \Delta_y)H_k$ extends by 0 to a C^l-function if $k > \frac{n}{2} + l$. Finally, on $U_\epsilon - U_{\epsilon/2}$, $H_k = \eta S_k$ and

$$\begin{aligned}(\partial_t - \Delta_y)H_k &= \eta(\partial_t - \Delta_y)S_k - 2\langle\nabla\eta, \nabla S_k\rangle + S_k\Delta_y\eta \\ &= (4\pi t)^{-\frac{n}{2}}e^{-\frac{r^2}{4t}}\phi(x,y,t),\end{aligned}$$

where $\phi(x,y,t) \in C^\infty((0,\infty)\times M\times M)$ has at most a pole of order 1 at $t = 0$. Since $r > \frac{\epsilon}{2}$, $(\partial_t - \Delta_y)H_k$ extends by 0.

To finish the proof we must show

$$\lim_{t\to 0}\int_M (4\pi t)^{-\frac{n}{2}}e^{-\frac{r^2}{4t}}\eta(x,y)\left(u_0(x,y) + \dots + t^k u_k(x,y)\right)f(y)\,d\mathrm{vol}_y = f(x). \tag{14}$$

To this end we note

$$\begin{aligned}
&\lim_{t\to 0}\int_M (4\pi t)^{-\frac{n}{2}}\eta(x,y)e^{-\frac{r^2}{4t}}u_i(x,y)f(y)\,d\mathrm{vol}_y\\
&\quad=\lim_{t\to 0}\int_{B(x;\frac{\epsilon}{2})}(4\pi t)^{-\frac{n}{2}}\eta(x,y)e^{-\frac{r^2}{4t}}u_i(x,y)f(y)\,d\mathrm{vol}_y\\
&\qquad+\lim_{t\to 0}\int_{M-B(x;\frac{\epsilon}{2})}(4\pi t)^{-\frac{n}{2}}\eta(x,y)e^{-\frac{r^2}{4t}}u_i(x,y)f(y)\,d\mathrm{vol}_y.
\end{aligned}$$

Here the second integral on the right hand side goes to 0 as $t\to 0$ since on its domain $r\geq\frac{\epsilon}{2}$. The first integral becomes in polar coordiantes around x

$$\begin{aligned}
&\int_{B(0;\frac{\epsilon}{2})\subset T_xM}(4\pi t)^{-\frac{n}{2}}e^{-\frac{r^2}{4t}}u_i(x,\exp_x v)f(\exp_x v)\,D(v)\,dv^1\dots dv^n\\
&=\int_{T_xM\approx\mathbb{R}^n}\left[(4\pi t)^{-\frac{n}{2}}e^{-\frac{r^2}{4t}}\right]u_i(x,\exp_x v)f(\exp_x v)\,D(v)\,dv^1\dots dv^n,
\end{aligned}$$

where u_i is extended by 0 off $B(0;\frac{\epsilon}{2})$. Since the expression in square brackets in the last integral is the usual heat kernel on $\mathbb{R}^n$ we find that this integral approaches

$$u_i(x,\exp_x 0)f(\exp_x 0)D(0)=u_i(x,x)f(x).$$

Since $u_0(x,x)=1$, we conclude that (14) holds. □

13.B. **Construction of the heat kernel.** For $A,B\in C^0([0,\infty)\times M\times M)$ define

$$(A*B)(x,y,t)=\int_0^t d\theta\int_M A(\theta,x,q)B(t-\theta,q,y)\,d\mathrm{vol}_q\in C^0([0,\infty)\times M\times M).$$

Write $A^{*1}=A$ and

$$A^{*k}=\underbrace{A*\cdots*A}_{k}.$$

Lemma 11. *Set $K_k=(\partial_t-\Delta_y)H_k$. Then $Q_k=\sum_{\lambda=1}^{\infty}(-1)^{\lambda+1}K_k^{*\lambda}$ exists and lies in $C^l([0,\infty)\times M\times M)$ if $k>\frac{n}{2}+l$. Moreover, given $T>0$, there exists $C=C(T)$ such that $|Q_k(t,x,y)|\leq Ct^{k-\frac{n}{2}}$ for all $t\in[0,T]$.*

Proof. Writing $K_k=(\partial_t-\Delta_y)(\eta S_k)$ and differentiating we get

$$|K_k|\leq A(T)t^{k-\frac{n}{2}}\leq A(T)T^{k-\frac{n}{2}}=B,$$

for some constants $A=A(T)$ and B. We show that

$$|K_k^{*\lambda}|\leq\frac{AB^{\lambda-1}\,\mathrm{vol}(M)^{\lambda-1}t^{\lambda-1}}{t}^{k-\frac{n}{2}}(k-\frac{n}{2}+1)(k-\frac{n}{2}+2)\dots(k-\frac{n}{2}+\lambda-1). \tag{15}$$

The case $\lambda=1$ is done above provided we adjust A. Assuming (15) for $\lambda-1$ we have

$$\begin{aligned}
|K_k^{*\lambda}|&\leq\int_0^t d\theta\int_M|K_k^{*(\lambda-1)}(\theta,x,q)||K_k(t-\theta,q,y)|\,d\mathrm{vol}_q\\
&\leq\int_0^t d\theta\int_M\frac{AB^{\lambda-2}\,\mathrm{vol}(M)^{\lambda-2}\theta^{k-\frac{n}{2}+\lambda-2}}{(k-\frac{n}{2}+1)(k-\frac{n}{2}+2)\dots(k-\frac{n}{2}+\lambda-2)}B\,d\mathrm{vol}_q\\
&\leq\frac{AB^{\lambda-1}\,\mathrm{vol}(M)^{\lambda-1}}{(k-\frac{n}{2}+1)(k-\frac{n}{2}+2)\dots(k-\frac{n}{2}+\lambda-2)}\int_0^t\theta^{k-\frac{n}{2}+\lambda-2}\,d\theta,
\end{aligned}$$

and evaluating the last integral proves (15).

The right hand side of (15) is bounded by a constant times

$$t^{k-\frac{n}{2}}\frac{(B\,\mathrm{vol}(M)t)^{\lambda-1}}{\Gamma(k-\frac{n}{2}+\lambda-2)}, \tag{16}$$

so the ratio test implies that $\sum_{\lambda=1}^{\infty}|K^{*\lambda}|$ converges. This shows that Q_k is continuous if $k>\frac{n}{2}$. The continuity of the derivatives of Q_k can be handled in a similar manner. The estimate on $|Q_k|$ follows immediately from (16). $\square$

Lemma 12. *If $P\in C^l([0,\infty)\times M\times M)$ and $k>\frac{n}{2}+l$ then $P*H_k\in C^l((0,\infty)\times M\times M)$. Moreover, $(\partial_t+\Delta_y)P*H_k=P+P*K_k$ if $k>2+\frac{n}{2}$.*

Proof. The first statement is left to the reader. For the second statement we compute

$$\begin{aligned}(\partial_t-\Delta_y)(P*H_k)=&(\partial_t-\Delta_y)\int_0^t d\theta\int_M P(\theta,x,q)H_k(t-\theta,q,y)\,d\mathrm{vol}_q\\ =&\lim_{s\to t}\int_M P(s,x,q)H_k(t-s,q,y)\,d\mathrm{vol}_q\\ &+\int_0^t d\theta\int_M P(\theta,x,q)(\partial_t-\Delta_y)H_k(t-\theta,q,y)\,d\mathrm{vol}_q\\ =&P(t,x,y)+P*K_k.\end{aligned}$$

$\square$

Theorem 13. *Set $e(t,x,y)=H_k(t,x,y)+Q_k*H_k(t,x,y)$. Then $e\in C^\infty((0,\infty)\times M\times M)$ is independent of k if $k>\frac{n}{2}+2$ and is the heat kernel.*

Proof. We know that e is a C^2-function. Moreover,

$$\begin{aligned}(\partial_t+\Delta_y)e(t,x,y)&=(\partial_t-\Delta_y)(H_k-Q_k*H_k)\\ &=K_k-Q_k-Q_k*K_k\\ &=K_k-\sum_{\lambda=1}^{\infty}(-1)^{\lambda+1}K_k^{*\lambda}-\sum_{\lambda=1}^{\infty}(-1)^{\lambda+1}K_k^{*\lambda}*K_k\\ &=0.\end{aligned}$$

Also,

$$\begin{aligned}&\lim_{t\to 0}\int_M e(t,x,y)f(y)\,d\mathrm{vol}_q\\ &=\lim_{t\to 0}\left(\int_M H_k(t,x,y)f(y)\,d\mathrm{vol}_y-\int_M (Q_k*H_k)(t,x,y)f(y)\,d\mathrm{vol}_y\right)\\ &=f(x)-\lim_{t\to 0}\int_M (Q_k*H_k)(t,x,y)f(y)\,d\mathrm{vol}_y.\end{aligned}$$

Now $R_k=t^{\frac{n}{2}-k}Q_k$ is bounded for all t in some finite interval so by Lemma 10 the last term equals 0 for $k>\frac{n}{2}$.

Applying Lemma 7, we see that $e(t,x,y)$ is the heat kernel. $\square$

13.C. **Asymptotics.** We write $A(t) \sim \sum_{k=k_0}^{\infty} b_k t^k$ if for all $N \geq k_0$,

$$\lim_{t \to 0} \frac{A(t) - \sum_{k=k_0}^{N} b_k t^k}{t^N} = 0,$$

and say that $\sum_{k=k_0}^{\infty} b_k t^k$ is an asymptotic expansion of $A(t)$.

Lemma 14. *The trace density $e(t,x,x)$ of the heat kernel has the asymptotic expansion*

$$e(t,x,x) \sim (4\pi t)^{-\frac{n}{2}} \sum_{k=0}^{\infty} u_k(x,x) t^k.$$

Proof. We know

$$e(t,x,x) = (H_k - Q_k * H_k)(t,x,x),$$

for any $k \gg 0$. Since $|Q_k| \leq C t^{k-\frac{n}{2}}$, we have $(4\pi t)^{\frac{n}{2}} |Q_k * H_k| \leq C t^{k+1}$. As $(4\pi t)^{\frac{n}{2}} H_k(t,x,x) = (u_0(x,x) + \cdots + t^k u_k(x,x))$, we find

$$(4\pi t)^{\frac{n}{2}} e(t,x,x) = u_0(x,x) + \cdots + t^k u_k(x,x) + R(t,x),$$

where $\lim_{t \to 0} \frac{R(t,x)}{t^k} \to 0$. □

Theorem 15. *Let $\{\lambda_i\}$ be the spectrum of the Laplace operator on (M,g). Then the trace of the heat kernel satisfies*

$$\sum_{i=0} e^{-\lambda_i t} \sim (4\pi t)^{-\frac{n}{2}} \sum_{k=0}^{\infty} a_k t^k,$$

with

$$a_k = \int_M u_k(x,x) \, d\mathrm{vol}_x$$

Proof.

$$\begin{aligned} \sum_i e^{-\lambda_i t} &= \int_M e(t,x,x) \, d\mathrm{vol}_x \\ &\sim \int_M (4\pi t)^{-\frac{n}{2}} \sum_k u_k(x,x) t^k \, d\mathrm{vol}_x \\ &\sim (4\pi t)^{-\frac{n}{2}} \sum_k \left(\int_M u_k(x,x) \, d\mathrm{vol}_x \right) t^k. \end{aligned}$$

□

Two Riemannian manifolds are called *isospectral* if the eigenvalues (with multiplicities) of their respective Laplace operators agree.

Corollary 16. *Let M and N be closed isospectral Riemannian manifolds. Then* $\dim(M) = \dim(N)$ *and* $\mathrm{vol}(M) = \mathrm{vol}(N)$.

Proof. With $\dim(M) = m$ and $\dim(N) = n$ we have

$$\begin{aligned}
&(4\pi t)^{-\frac{m}{2}} \sum_{k=0}^{\infty} \left(\int_M u_k^M(x,x)\, d\mathrm{vol}_x \right) t^k \\
&\sim \sum_{k=0}^{\infty} e^{-\lambda_i t} \\
&\sim (4\pi t)^{-\frac{n}{2}} \sum_{k=0}^{\infty} \left(\int_N u_k^N(x,x)\, d\mathrm{vol}_x \right) t^k,
\end{aligned}$$

and thus $m = n$. Hence

$$\begin{aligned}
&(4\pi t)^{-\frac{m}{2}} \left[\int_M u_0^M(x,x)\, d\mathrm{vol}_x - \int_N u_0^N(x,x)\, d\mathrm{vol}_x \right] \\
&\sim (4\pi t)^{-\frac{m}{2}} \sum_{k=1}^{\infty} \left(\int_M u_0^M(x,x)\, d\mathrm{vol}_x - \int_N u_0^N(x,x)\, d\mathrm{vol}_x \right) t^k.
\end{aligned}$$

We conclude

$$\int_M u_0^M(x,x)\, d\mathrm{vol}_x = \int_M u_0^N(x,x)\, dvol_x,$$

and since $u_0^M(x,x) = 1 = u_0^N(x,x)$ we find $\mathrm{vol}(M) = \mathrm{vol}(N)$. □

13.D. **Some further remarks.** By iterating the argument in the proof of Corollary 16, one shows that if M and N are isospectral Riemannian manifolds then

$$\int_M u_k^M(x,x)\, d\mathrm{vol}_x = \int_N u_k^N(x,x)\, d\mathrm{vol}_x,$$

for all k. The functions $u_k(x,x)$ are therefore of great interest. In general, $u_k(x,x)$, $x \in M$ is expressed as a polynomial, which depend only on $\dim(M)$, in the curvature tensor R and its covariant derivatives of orders $\leq 2k - 4$.

The interested reader is invited to prove the following result using Riemann normal coordinates and Equation (13)

Theorem 17. *For any Riemannian manifold*

$$u_1(x,x) = \frac{1}{6} s(x),$$

where s is the scalar curvature of M.

Using this result we demonstrate

Theorem 18. *If M and M are closed isospectral surfaces then they are diffeomorphic.*

Proof. Immediate from the classification of surfaces, the Gauss-Bonnet theorem, and Theorem 17 □

It can be shown that the asymptotic expansion in Theorem 15 gives (see the first problem in Section 14 for a special case)

Theorem 19. *If M is a compact Riemannian manifold and if $N(\lambda)$ denotes the number of eigenvalues counted with multiplicities of $-\Delta$ on M which are $\leq \lambda$ then*

$$N(\lambda) \sim \frac{\omega_n \,\mathrm{vol}(M) \lambda^{\frac{n}{2}}}{(2\pi)^n},$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

14. Problems

To pass the course solutions to the following problems should be handed in.

The Laplacian on the n-torus. Let T^n be the flat cubical n-dimensional torus $\mathbb{R}^n/2\pi\mathbb{Z}^n \approx S^1 \times \cdots \times S^1$ with metric

$$ds^2 = d\theta_1^2 + d\theta_2^2 + \cdots + d\theta_n^2.$$

Find the spectrum and eigenfunctions of $-\Delta$ on T^n (for example using the discrete Fourier transform and viewing the Laplacian as an operator on complex valued functions). Use the results to demonstrate: If $N(\lambda)$ denotes the number of eigenvalues counted with multiplicities of $-\Delta$ which are $\leq \lambda$ and if ω_n denotes the volume of the unit ball in $\mathbb{R}^n$ then

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\omega_n \lambda^{n/2}} = 1.$$

Long-term behavior of heat. Let M be a closed Riemannian manifold and let $u(t,x)$ be a solution to the heat equation. Let c_0 denote the mean value of $u(x,0)$,

$$c_0 = \frac{1}{\mathrm{vol}(M)} \int_M u(0,x)\, d\mathrm{vol}_x.$$

Show that

$$\lim_{t\to\infty} \int_M (u(t,x) - c_0)^2\, d\mathrm{vol}_x = 0.$$

Interpret this result in physical terms. (For extra credit, show that

$$\int_M (u(t,x) - c_0)^2\, d\mathrm{vol}_x = \mathcal{O}(e^{-\mu t}),$$

as $t \to \infty$ for some $\mu > 0$ and determine the largest possible μ for which this holds.)