REGULAR HOMOTOPY AND VASSILIEV INVARIANTS OF GENERIC IMMERSIONS $S^k \to \mathbb{R}^{2k-1}$, $k \geq 4$

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ABSTRACT

The regular homotopy class of a generic immersion $S^k \to \mathbb{R}^{2k-1}$ is calculated in terms of its self intersection manifold with natural additional structures.

There is a natural notion of Vassiliev invariants of generic immersions. These may take values in any Abelian group G. It is proved that, for any m, the group of m^{th} order G-valued invariants modulo invariants of lower order is isomorphic to G and that the Vassiliev invariants are not sufficient to separate generic immersions, which can not be obtained from each other by a regular homotopy through generic immersions.

Keywords: Immersions, Vassiliev Invariants, Nonseparation, Regular Homotopy.

1 Introduction

1.1 Formulas for regular homotopy classes of generic immersions. The set of immersions $S^k \to \mathbb{R}^{k+n}$ is an open subset of the space of r times differentiable maps with the C^r-topology, $r \geq 1$. The *space of immersions* is this set with the subspace topology. If two immersions belong to the same path component of the space of immersions they are called *regularly homotopic*.

Whitney [1] classified immersed plane curves up to regular homotopy: Two curves are regularly homotopic if and only if they have the same tangential degree. He gave a formula for the tangential degree of a generic plane curve in terms of its double points.

Smale [2] classified immersions $S^k \to \mathbb{R}^{k+n}$ up to regular homotopy: Comparing any immersion f to the standard embedding $S^k \subset \mathbb{R}^{k+1} \subset \mathbb{R}^{k+n}$, we can associate to it its Smale invariant $\Omega(f) \in \pi_k(V_{k+n,k})$, the k^{th} homotopy group of the Stiefel manifold of k-frames in $(k+n)$-space. The invariant Ω is the higher dimensional analogue of the tangential degree of a plane curve. It gives a one-one correspondence between the regular homotopy classes of immersions $S^k \to \mathbb{R}^{k+n}$ and the elements of the group $\pi_k(V_{k+n,k})$.

Smale generalized the formula of Whitney: If $f\colon S^k \to \mathbb{R}^{2k}$, $k \geq 2$ is a generic immersion then $\Omega(f)$ can be computed geometrically: Count the double points of f (modulo 2 if k is odd, with signs if k is even).

We shall give a similar formula for the regular homotopy class of a generic immersion $S^k \to \mathbb{R}^{2k-1}$, $k \geq 4$. It expresses the Smale invariant in terms of the geometry of the self intersection.

The self intersection M_f of a generic immersion $f\colon S^k \to \mathbb{R}^{2k-1}$ contains only transverse double points. Hence, M_f is a collection of circles. If c is a component of M_f such that $f^{-1}(c)$ is disconnected then there is an induced spin structure on c ([4], Proposition 2.10). There are two spin structures on the circle. The trivial one which is spin cobordant to zero and the nontrivial (Lie group) one. Let $T(f)$ and $N(f)$ denote the number of components of M_f with trivial and nontrivial spin structure, respectively.

If $f^{-1}(c)$ is connected (this can happen only if k is odd) then a framing of its normal bundle induces a twist framing of the normal bundle of c. There are two twist framings up to isotopy. We call them right and left respectively. Let $R(f)$ and $L(f)$ denote the number of components with right and left twist framing, respectively.

Theorem 1. *Let $k \geq 4$.*

(i) *If $k = 4s$, let $r(f) = 0 \in \{0\}$,*

(ii) *if $k = 4s+1$, let*

$$r(f) = (1,1)\cdot N(f) + (1,0)\cdot R(f) + (0,1)\cdot L(f) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2,$$

(iii) *if $k = 4s+2$, let*

$$r(f) = N(f) \mod 2 \in \mathbb{Z}_2,$$

(iv) *if $k = 4s+3$, let*

$$r(f) = 2N(f) + R(f) - L(f) \mod 4 \in \mathbb{Z}_4.$$

Denote by I_k the groups above where r takes values. Then there exist bijections $\phi\colon \pi_k(V_{2k-1,k}) \to I_k$ such that $r(f) = \phi(\Omega(f))$, for any generic immersion f.

Theorem 1 is proved in Section 6.

1.2 Vassiliev invariants and their insufficiency. Let $\mathcal{F}_k$ denote the space of immersions $f\colon S^k \to \mathbb{R}^{2k-1}$. The set of generic immersions (see Section 2.1) is an open dense subset of $\mathcal{F}_k$. For an Abelian group G, a G-invariant of generic immersions is a G-valued locally constant function on the space of generic immersions.

Let $\Sigma_k \subset \mathcal{F}_k$ denote the subset of all nongeneric immersions. Then there is an isomorphism from the group of G-invariants onto $H^0(\mathcal{F}_k - \Sigma_k; G)$.

The subspace $\Sigma_k \subset \mathcal{F}_k$ will be called the *discriminant*. The subset of Σ_k consisting of all immersions with a single self tangency point (see Definition 2.2.2) will be denoted Σ_k^1. It is a dense open subset of Σ_k and a smooth codimension one submanifold of $\mathcal{F}_k$.

We can associate a jump to any G-invariant of generic immersions: Since Σ_k^1 is a smooth submanifold of codimension one, it locally separates $\mathcal{F}_k$. Given an immersion $f_0 \in \Sigma_k^1$ and an invariant of generic immersions, we define the jump of this invariant at f_0 to be the difference of the values of the invariant at two generic immersions which are close to f_0 and are on opposite sides of Σ_k^1. Such a difference is defined only up to sign.

In order to make the jump well defined we make a choice of a positive and a negative part of a small neighborhood of f_0, cut into two parts by Σ_k^1. A coherent choice like that for all $f_0 \in \Sigma_k^1$ is called a *coorientation* of Σ_k.

If we compute the jump using the signs coming from a coorientation then the jump will be a locally constant G-valued function on Σ_k^1. Thus, if Σ_k is cooriented then any invariant $a \in H^0(\mathcal{F}_k - \Sigma_k; G)$ has a jump $\nabla a \in H^0(\Sigma_k^1; G)$.

If the coorientation of Σ_k satisfies some additional conditions then the above construction can be iterated to give a m^{th} jump $\nabla^m a \in H^0(\Sigma_k^m; G)$ of a, where Σ_k^m consists of all immersions with m distinct self tangency points. (If a coorientation satisfies these additional conditions, we call it *continuous*, see Definition 7.6.2).

We define the m^{th} *order Vassiliev* G-invariants to be the G-invariants a that satisfies $\nabla^{m+1} a \equiv 0$. This definition is similar to the definition of Vassiliev invariants of knots in 3-space given by Arnold in [3].

The Vassiliev invariants form an increasing sequence of subgroups of the group of all invariants. If V_m denotes the group of m^{th} order Vassiliev G-invariants, we have

$$V_0 \subset V_1 \subset \cdots \subset V_m \subset \cdots \subset H^0(\mathcal{F}_k - \Sigma_k; G).$$

The cohomology of $\mathcal{F}_k - \Sigma_k$ splits naturally into a sum over the path components of $\mathcal{F}_k$, which are described by Theorem 1. So does V_m. Let V_m' denote a summand of V_m corresponding to a path component of $\mathcal{F}_k$.

The following theorem describes the structure of the filtered group of Vassiliev invariants.

Theorem 2. *If $k \geq 4$ then $V_m'/V_{m-1}' \cong G$, for all m.*

Theorem 2 is proved in Section 9.5.

If $k = \dim(S^k)$ is divisible by 4 then Σ_k does not admit an continuous coorientation (see Proposition 7.6.3). Therefore, we have to restrict the theory of Vassiliev G-invariants to $G = \mathbb{Z}_2$ in this case.

The Vassiliev invariants can be explicitly computed. The following theorem shows how to do that in case of $\mathbb{Z}$-invariants ($\mathbb{Z}_2$-invariants if $k = 4s$).

Theorem 3. *Let $f \in \mathcal{F}_k - \Sigma_k$.*

(i) *If $k = 4s + d$, $d = 1, 2, 3$, let*

$$v_m(f) = \binom{T(f) + N(f) + R(f) + L(f)}{m},$$

(ii) *if $k = 4s$, let*

$$v_m(f) = \binom{T(f)}{m} \mod 2,$$

where $\binom{l}{j}$ is the binomial coefficient of x^j in the expansion of $(1+x)^l$. Then v_m generates V_m'/V_{m-1}', $m > 0$.

Theorem 3 is proved in Section 10.2.

Corollary 4. *For any number n there are pairwise different path components $U_1, \dots, U_n$ of $\mathcal{F}_k - \Sigma_k$ such that $v(U_1) = v(U_2) = \cdots = v(U_n)$, for all $v \in \bigcup_{m=0}^{\infty} V_m$.*

Corollary 4 is proved in Section 10.3. The special case $n = 2$ of it shows that Vassiliev invariants do not separate generic immersions.

2 The Discriminant

2.1 Generic immersions. An immersion $f\colon S^k \to \mathbb{R}^{2k-1}$ is a *generic immersion* if,

G1. For any $z \in \mathbb{R}^{2k-1}$, $f^{-1}(z)$ contains at most two points, and

G2. If $f(x) = f(y) = z$, for $x \neq y$ in S^k, then

$$T_z\mathbb{R}^{2k-1} = dfT_xS^k + dfT_yS^k.$$

If $k > 2$ then the set of generic immersions is an open dense subset of $\mathcal{F}_k$.

2.2 Self tangency points and versal deformation.

Notation. $\mathbb{R}^k = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{k-2}$ with coordinates $x = (x_1, x_2, x_3)$ and $\mathbb{R}^{2k-1} = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{k-2} \times \mathbb{R}^{k-2}$ with coordinates $x = (x_1, x_2, x_3, x_4, x_5)$.

2.2.1 Proposition. *If $k > 3$ then the codimension one stratum Σ_k^1 of Σ_k in $\mathcal{F}_k$ contains the immersions f satisfying* **G1**, *and* **G2**, *except at one pair of points p, q in S^k, with $f(p) = f(q)$ and $\dim(dfT_pS^k + dfT_qS^k) = 2k - 2$. Moreover, there are disjoint charts U containing p and V containing q, with coordinates x centered at p and y centered at q respectively, such that up to left action of a diffeomorphism, diffeotopic to identity*

$$\begin{aligned} f(x_1, x_2, x_3) &= (x_1, x_2, 0, x_3, 0), \\ f(y_1, y_2, y_3) &= (y_1, y_2, y_1^2 + \epsilon y_2^2, 0, y_3), \end{aligned}$$

where $\epsilon = \pm 1$.

Proposition 2.2.1 is proved in the Appendix. Let z be a double point of f as described above.

2.2.2 Definition. *If $\epsilon = +1$, we say that z is an* elliptic self tangency point *of f. If $\epsilon = -1$, we say that z is a* hyperbolic self tangency point *of f.*

A deformation F of a map f is called *versal* if every deformation is equivalent (up to left-right action of diffeomorphisms) to one induced from F.

2.2.3 Proposition. *The versal deformation of a self tangency point is the following one parameter deformation*

$$\begin{aligned} F_t(x_1, x_2, x_3) &= (x_1, x_2, 0, x_3, 0), \\ F_t(y_1, y_2, y_3) &= (y_1, y_2, y_1^2 + \epsilon y_2^2 + t, 0, y_3). \end{aligned}$$

Proposition 2.2.3 is proved in the Appendix.

2.3 m-immersions. An immersion $f\colon S^k \to \mathbb{R}^{2k-1}$ is a *m-immersion* if it satisfies **G1**, and **G2**, except at m self tangency points. We let $\Sigma_k^m \subset \Sigma_k \subset \mathcal{F}_k$ denote the space of m-immersions.

We say that f_0 belongs to a *smooth codimension one branch* of Σ_k if there is a path $\lambda\colon [-1,1] \to \Sigma_k$ such that $\lambda(0) = f_0$, $\dot\lambda(0) \neq 0$ and $\lambda(t) \in \Sigma_k^1$, if $t \neq 0$. In this terminology, Σ_k^m is the transverse intersection of m smooth codimension one branches of Σ_k.

3 Topology of the Self Intersection and Subdivision of the Discriminant

3.1 Topological transformation of the self intersection. Let f_t be a smooth path in $\mathcal{F}_k$ intersecting Σ_k^1 transversely in f_0. Let $\delta \geq 0$ be a number so small that for $|t| \leq \delta$, $t \neq 0$ the immersion f_t is generic. Let $f\colon S^k \times [-\delta,\delta] \to \mathbb{R}^{2k-1} \times [-\delta,\delta]$ be the map

$$f(x,t) = (f_t(x), t).$$

Then f is smooth and Proposition 2.2.3 implies that its self intersection M_f contains only transverse double points. Hence, M_f is a surface with boundary

$$M_f \cap \mathbb{R}^{2k-1} \times \{-\delta,\delta\} = M_{f_{-\delta}} \cup M_{f_\delta}.$$

Let $p\colon \mathbb{R}^{2k-1} \times [-\delta,\delta] \to [-\delta,\delta]$ be the projection. Proposition 2.2.3 shows that $p|M_f$ has exactly one critical point, which is nondegenerate.

Let F be the component of M_f that contains this critical point. Then F is a surface with boundary, admitting a Morse function with one critical point and we conclude:

(i) If the index of the critical point is 0 or 2 then F is a disk.

(ii) If the index of the critical point is 1 then F is a pair of pants or a Moebius strip with an open disk removed.

In case (i) f_0 has an elliptic self tangency point. In case (ii) f_0 has a hyperbolic self tangency point.

If k is odd then the codimension of $f^{-1}(M_f)$ in $S^k \times I$ is even. Hence, M_f is orientable ([4], Proposition 2.3) and F has to be a pair of pants in case (ii) if k is odd.

3.2 Subdivision of the discriminant. Let $\widetilde{F} = f^{-1}(F)$. Then $f\colon \widetilde{F} \to F$ is a (not necessarily connected) double cover and if k is even then this cover is the orientation double cover ([4], Proposition 2.5). Hence, we have the following Propositions:

3.2.1 Proposition. *If F is a disk then $\widetilde{F}$ is a union of two disks.* □

3.2.2 Proposition. *If k is even and f_0 has a hyperbolic self tangency then F is a pair of pants and $\widetilde{F}$ is a union of two pairs of pants or F is a Moebius strip with a disk removed and $\widetilde{F}$ is a sphere with four disks removed.* □

3.2.3 Proposition. *If k is odd and f_0 has a hyperbolic self tangency then F is a pair of pants and $\widetilde{F}$ is either a union of two pairs of pants or a sphere with four disks removed.* □

These propositions give a subdivision of the codimension one part of Σ_k:

3.2.4 Definition. *Let $E_k \subset \Sigma_k$ consist of immersions with one elliptic self tangency point.*

3.2.5 Definition. *Let $HD_k \subset \Sigma_k$ consist of immersions with one hyperbolic self tangency point such that the corresponding $\widetilde{F}$ is disconnected (i.e. $\widetilde{F}$ is a union of two pairs of pants).*

3.2.6 Definition. *Let HC_k consist of immersions with a hyperbolic self tangency point such that $\widetilde{F}$ is connected (i.e. $\widetilde{F}$ is a sphere with four disks removed).*

Using this terminology we can write $\Sigma_k^1 = E_k \cup HD_k \cup HC_k$.

4 The Structures Induced on the Self Intersection Manifold

In this section we introduce notation and state results that will be used in later sections.

4.1 Preliminaries.

Let $\mathbf{I}$ denote the identity matrix.

4.1.1 Definition. *If $n = 4s+1$, define $\mathbf{A}(n) \in SO(n)$ by*

$$\begin{bmatrix} 1 & & \\ & 0 & \mathbf{I}_{2s} \\ & \mathbf{I}_{2s} & 0 \end{bmatrix}.$$

If $n = 4s+2$, define $\mathbf{A}(n) \in SO(n)$ by

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \mathbf{I}_{2s} \\ & & \mathbf{I}_{2s} & 0 \end{bmatrix}.$$

if $n = 4s$, define $\mathbf{A}(n) \in SO(n)$ by

$$\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \mathbf{I}_{2s-1} \\ & & \mathbf{I}_{2s-1} & 0 \end{bmatrix}.$$

The group $\{\mathbf{I}, \mathbf{A}(n)\}$ acts on $SO(n)$ from the right. Denote the corresponding quotient space $SO(n)/\mathbf{A}(n)$ and let p be the projection.

4.1.2 Proposition. *Let $n = 4s+d$, $k \geq 1$ and $d = 0, 1, 2$. Then*

$$\pi_1(SO(n)/\mathbf{A}(n)) = \begin{cases} \mathbb{Z}_4 & \textit{if } s \textit{ is odd,} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \textit{if } s \textit{ is even.} \end{cases}$$

Proof. See [4], Proposition 5.2. □

Let $u \in \pi_1(SO(n))$ denote the nontrivial element.

4.2 Twist framing. Let k be odd and $f\colon S^k \to \mathbb{R}^{2k-1}$ be a generic immersion. Let c be a component of M_f and $c_S = f^{-1}(c)$. Note that c_S is either a union of two disjoint circles or one circle.

Choose a parameterization r of c, agreeing with the orientation induced on c if k is odd. Then r lifts to two parameterizations r_1 and r_2. If c_S is connected then the path product $r_1 * r_2$ is a parameterization of c_S.

Choose orthonormal framings Y_1 and Y_2 of $N(c_S \subset S^k)$ (the normal bundle of c_S in S^k) along r_1 and r_2 respectively such that they give a continuous framing of c_S. Then (dfY_1, dfY_2) is a framing or twist framing X of $N(c \subset \mathbb{R}^{2k-1})$.

Let $\vec{c}_S$ denote the loop (or loops) in $SO(TS^k)$ (the principal $SO(k)$-bundle of the tangent bundle of S^k) represented by c_S with the framing $(\dot{r}_i, Y_i)$ and $[\vec{c}_S]$ the corresponding element in $H_1(SO(TS^k);\mathbb{Z}_2)$. Let $[c,r,X] \in \pi_1(SO(n)/\mathbf{A}(n))$ denote the homotopy class of the loop induced by c with the (twist) framing $(\dot{r}, X)$ (which we may have to orthonormalize). Then define

4.2.1 Definition.

$$\omega_f(c) = \langle \xi, [\vec{c}_S]\rangle \cdot p_*(u) + [c,r,X] + p_*(u) \in \pi_1(SO(n)/\mathbf{A}(n)),$$

where $\xi \in H^1(SO(TS^k);\mathbb{Z}_2)$ denotes the unique spin structure on S^k.

This definition is independent of all choices ([4], Lemma 6.5).

Fix, once and for all, isomorphisms

$$\phi\colon \pi_1(SO(2k-1)/\mathbf{A}(2k-1)) \to \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } k \equiv 1 \mod 4, \\ \mathbb{Z}_4 & \text{if } k \equiv 3 \mod 4, \end{cases}$$

with $\phi(p_*(u)) = (1,1)$ or $\phi(p_*(u)) = 2$.

4.2.2 Definition. *The component c of the self intersection is* right twist framed *if $\phi(\omega_f(c)) = (1,0)$ or $\phi(\omega_f(c)) = 1$. It is* left twist framed *if $\phi(\omega_f(c)) = (0,1)$ or $\phi(\omega_f(c)) = 3$.*

If c_S is disconnected then the induced spin structure on c is trivial if $\omega_f(c) = 0$ and nontrivial if $\omega_f(c) = p_*(u)$.

4.3 A functional on curves in self intersection surfaces. Consider a generic immersion $g\colon S^k \times I \to \mathbb{R}^{2k-1} \times I$. Let F be a component of M_g and let c be a generic closed curve in F.

As above, we can associate to c with a parameterization r an element $\omega_g(c) \in \pi_1(SO(2k)/\mathbf{A}(2k))$. This gives a functional on oriented curves in F ([4], Definition 7.5).

Assume that g is level preserving and c is a component of $\partial F = F \cap \mathbb{R}^{2k-1} \times \partial I$. Let $g_i\colon S^k \times \{i\} \to \mathbb{R}^{2k-1} \times \{i\}$, $i = 0, 1$ be the restrictions of g.

If k is odd then

$$\omega_g(c) = \omega_{g_i}(c), \tag{$\dagger$}$$

where we use the isomorphism on π_1 induced by forgetting the second vector of the frame.

If k is even, $g^{-1}(c)$ is disconnected and g_i induces a spin structure on c. We write $c = t$ if this is the trivial spin structure and $c = n$ if it is the nontrivial (Lie group) one. Then (see [4], proof of Theorem 7.30),

$$\omega_g(c) = \begin{cases} 0 & \text{if and only if } c = t, \\ p_*(u) & \text{if and only if } c = n. \end{cases} \tag{††}$$

If c is a closed curve in F then there are two nonhomotopic parameterizations of c. Denote them c and $-c$. The following proposition describes how ω_g depends on the chosen parameterization.

4.3.1 Proposition. (i) *If $k \equiv 0 \mod 4$ or $k \equiv 1 \mod 4$ and c_S is connected then*

$$\omega_g(-c) = \omega_g(c) + p_*(u).$$

(ii) *If $k \equiv 2 \mod 4$ or $k \equiv 3 \mod 4$ or c_S is disconnected then*

$$\omega_g(-c) = \omega_g(c).$$

Proof. See [4], Lemma 7.9. □

We extend the functional ω_g to collections $L = \{c_1, \ldots, c_m\}$ of oriented closed curves in the natural way:

$$\omega_g(L) = \sum_{i=1}^{m} \omega_g(c_i).$$

Let L be a collection of oriented closed curves with m transverse intersections and L' be the collection obtained from smoothing each intersection, respecting orientation. Then (see [4], Lemma 7.14)

$$\omega_g(L') = \omega_g(L) + m \cdot p_*(u). \tag{‡}$$

Also, if K and L are isotopic collections then

$$\omega_g(K) = \omega_g(L).$$

5 Change in Structures on Passing the Discriminant

Notation. The notation follows that used in Section 3.

We write $\partial F = \partial F_{-\delta} \cup \partial F_\delta$, where $\partial F_{\pm\delta} = F \cap \mathbb{R}^{2k-1} \times \{\pm\delta\}$.

Let t, n, r, l denote a self intersection circle with trivial spin structure, nontrivial spin structure, right twist framing and left twist framing, respectively.

5.1 Transformation of self intersection structures.

If the number of boundary components of F is odd assume that the parameterization of f_t is chosen so that M_{f_δ} has more components than $M_{f_{-\delta}}$.

5.1.1 Proposition. *If we pass transversely through E_k then $\partial F_{-\delta} = \emptyset$ and $\partial F_\delta = t$.*

Proof. The spin structure induced by f_δ on ∂F_δ agrees with the spin structure induced from F on ∂F_δ. Hence, it must be the trivial spin structure. $\square$

5.1.2 Proposition. *If we pass transversely through HD_k then $\partial F_{-\delta} = t$ if and only if $\partial F_{-\delta} = t \cup t$ or $\partial F_{-\delta} = n \cup n$, and $\partial F_{-\delta} = n$ if and only if $\partial F_{-\delta} = t \cup n$.*

Proof. Note that the spin cobordism group of 1-manifolds is $\mathbb{Z}_2$, generated by a circle with the nontrivial spin structure and argue as in the proof of Proposition 5.1.1. $\square$

5.1.3 Proposition. *Let k be even and assume that we pass transversely through HC_k.*

(i) *If $k \equiv 0 \mod 4$ then $\partial F_{-\delta} = t$ if and only if $\partial F_\delta = n$.*

(ii) *If $k \equiv 2 \mod 4$ then $\partial F_{-\delta} = t$ and $\partial F_\delta = t$.*

Proof. Let c be the core of the Moebius strip. Orient it. Let c' be a closed curve in F cutting it transversely in one point.

Then $\partial F_{-\delta}$ is isotopic to the curve obtained from smoothing c and c' if c' is given one of its orientations, and ∂F_δ isotopic to that obtained by smoothing the intersection if c' is given the opposite orientation.

Let b_i, $i = 1, 2$ denote the curves obtained from smoothing. Then

$$\omega_f(b_i) = \omega_f(c) + \omega_f(\pm c') + p_*(u),$$

by equation (‡) of the previous section.

If $k \equiv 0 \mod 4$ then

$$\pi_1(SO(2k)/\mathbf{A}(2k)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Thus, with a suitable choice of b_1 and b_2, we get

$$\begin{cases} \omega_f(b_1) & = \omega_f(c) + \omega_f(c) + p_*(u) = p_*(u), \\ \omega_f(b_2) & = \omega_f(c) + \omega_f(-c) + p_*(u) \\ & = \omega_f(c) + \omega_f(c) + p_*(u) + p_*(u) = 0, \end{cases}$$

where we use Proposition 4.3.1. By (††) of the previous section, this proves (i).

If $k \equiv 1 \mod 4$ then

$$\pi_1(SO(2k)/\mathbf{A}(2k)) \cong \mathbb{Z}_4.$$

Thus, we get

$$\begin{cases} \omega_f(b_1) & = \omega_f(c) + \omega_f(c) + p_*(u) \\ & = p_*(u) + p_*(u) = 0, \\ \omega_f(b_2) & = \omega_f(c) + \omega_f(-c) + p_*(u) \\ & = \omega_f(c) + \omega_f(c) + p_*(u) = 0. \end{cases}$$

where we again use Proposition 4.3.1. By (††) of the previous section, this proves (ii). $\square$

5.1.4 Proposition. *Let k be odd and assume that we pass transversely through HC_k.*

(i) *If* $k \equiv 1 \mod 4$ *then*

(a) $\partial F_{-\delta} = t$ *if and only if* $\partial F_\delta = r \cup r$ *or* $\partial F_\delta = l \cup l$.

(b) $\partial F_{-\delta} = n$ *if and only if* $\partial F_\delta = r \cup l$.

(c) $\partial F_{-\delta} = r$ *if and only if* $\partial F_\delta = r \cup t$ *or* $\partial F_\delta = l \cup n$.

(d) $\partial F_{-\delta} = l$ *if and only if* $\partial F_\delta = l \cup t$ *or* $\partial F_\delta = r \cup n$.

(ii) *If* $k \equiv 3 \mod 4$ *then*

(a) $\partial F_{-\delta} = t$ *if and only if* $\partial F_\delta = r \cup l$.

(b) $\partial F_{-\delta} = n$ *if and only if* $\partial F_\delta = r \cup r$ *or* $\partial F_\delta = l \cup l$

(c) $\partial F_{-\delta} = r$ *if and only if* $\partial F_\delta = r \cup t$ *or* $\partial F_\delta = l \cup n$.

(d) $\partial F_{-\delta} = l$ *if and only if* $\partial F_\delta = l \cup t$ *or* $\partial F_\delta = r \cup n$.

Proof. Since the codimensions of $\widetilde{M}_f$ and $\widetilde{M}_{f\pm\delta}$ are even, there are induced orientations of F and ∂F.

Consider two closed curves a and b such that a is oriented isotopic to one of the components of ∂F_δ and b to the other one, and such that a and b meet transversely in two points. Smoothing $\{a, b\}$ at these two points we obtain two new curves c and d.

One of these, c say, is oriented isotopic to $\partial F_{-\delta}$. The other one d bounds a disk with induced spin structure. Hence, $\omega_f(d) = 0$ and using equation (‡) of the previous section we find

$$\omega_f(a) + \omega_f(b) = \omega_f(c).$$

By Proposition 4.1.2 and formula (†) of the previous section, this completes the proof. □

6 Formulas for Regular Homotopy

6.1 Homotopy groups. After the work of Smale mentioned in the Introduction there is a 1-1 correspondence between $\pi_0(\mathcal{F}_k)$ and $\pi_k(V_{2k-1,k})$. The latter groups have been computed. See [5]. If $k \geq 4$ they depend only on $k \mod 4$, in the following way:

Theorem (Paechter).

$$\pi_k(V_{2k-1,k}) = \begin{cases} 0 & \text{if } k \equiv 0 \mod 4, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } k \equiv 1 \mod 4, \\ \mathbb{Z}_2 & \text{if } k \equiv 2 \mod 4, \\ \mathbb{Z}_4 & \text{if } k \equiv 3 \mod 4. \end{cases}$$

□

6.2 Proof of Theorem 1. Assume that two generic immersions f and g are regularly homotopic. Then there is a generic path f_t in $\mathcal{F}_k$ with $f_0 = f$ and $f_1 = g$. That the path is generic means that it intersects Σ_k transversely in Σ_k^1 at finitely many parameter values.

It follows from Propositions 5.1.1, 5.1.2, 5.1.3 and 5.1.4 that r in the statement of Theorem 1 does not change when we pass Σ_k^1. This proves that r is an invariant of regular homotopy.

Hence, it induces a map, also denoted r, from $\pi_0(\mathcal{F}_k)$ into I_k, where I_k is as in the statement of the theorem. We must show that this map is a bijection.

Now, $\pi_0(\mathcal{F}_k)$ and I_k have the same finite number of elements. Hence, it is enough to prove surjectivity.

Using the methods of Example 8.2 of [4] we see that we can construct generic immersions f such that $r(f)$ is any element in I_k. Thus r is surjective. This finishes the proof. □

6.3 Path components of $\mathcal{F}_k - \Sigma_k$, $k \geq 6$. Let $k \geq 6$ and $f, g \in \mathcal{F}_k - \Sigma_k$.

6.3.1 Proposition. *f and g belongs to the same path component of $\mathcal{F}_k - \Sigma_k$ if and only if*

$$T(f) = T(g), \quad N(f) = N(g), \quad R(f) = R(g), \quad L(f) = L(g).$$

□

Proposition 6.3.1 follows from Theorem 1.4 in [4]. We shall use it below and in the proof of Corollary 4.

6.4 Another proof of Theorem 1, $k \geq 6$. Using Proposition 6.3.1 we can give a proof of Theorem 1 which is independent of Smales work, if $k \geq 6$. We have to show that if two generic immersions f and g satisfies $r(f) = r(g)$ then they are regularly homotopic.

Using Proposition 6.3.1 we can move a given immersion h along a path in $\mathcal{F}_k$ that avoids Σ_k to another one with the same self intersection lying close to Σ_k^1. We can then cross Σ_k^1 in such a way that the number of self intersection components decreases.

Repeating this procedure we can minimize the number of self intersection components. That is, we can find an immersion h_1 connected to h with a path in $\mathcal{F}_k$ such that the number of self intersection components of h_1 cannot be decreased. Since r does not change when we pass Σ_k^1, $r(h_1) = r(h)$.

If the invariants agree for two generic immersions f and g then $r(f_1) = r(g_1)$ and both f_1 and g_1 have a minimal number of self intersection circles (one or zero). It follows that T, N, R and L takes equal values on f_1 and g_1. By Proposition 6.3.1 we can connect f_1 to g_1 with a path in $\mathcal{F}_k - \Sigma_k$. Thus, f and g are regularly homotopic. □

7 Coorientations in Functional Spaces

In this section we formalize the concepts used in the Introduction to discuss jumps of invariants of generic immersions. These concepts are intuitively rather clear and maybe well known. However, the author does not know of any reference and includes this section to provide workable definitions which will be used later on, when we study Vassiliev invariants.

7.1 Hypersurfaces in functional spaces. The notion *functional space* will be used to denote an open subset of a space of r times differentiable maps, from one smooth manifold into another with the C^r topology, which has the subspace topology. Thus, a functional space is in most cases an infinite dimensional smooth manifold.

The spaces $\mathcal{F}_k$ are our main examples of functional spaces.

A *hypersurface* of a functional space $\mathcal{F}$ is a union of smooth submanifolds $M^1 \cup M^2 \cup \cdots \subset \mathcal{F}$. Where M^j has codimension j in the functional space and M^j is contained in the closure of M^1. The M^j are called the *strata* of the hypersurface.

7.2 Coorientation. Let $\mathcal{F}$ be a functional space. Let Δ be a hypersurface in $\mathcal{F}$ with topdimensional stratum Δ^1.

Since Δ^1 has codimension one, it locally separates $\mathcal{F}$. That is, for each point $f \in \Delta^1$ there is a path connected neighborhood $U(f)$ in $\mathcal{F}$ of f such that $U(f) - \Delta^1$ has exactly two components.

Since Δ^1 is smooth, the neighborhood $U(f)$ can be chosen so that $U(f) \cap \Delta \subset \Delta^1$.

For each point $f \in \Delta^1$, choose a neighborhood $U(f)$ that is separated by Δ^1 and meets Δ in Δ^1. Choose a *positive* component $U^+(f)$ of $U(f) - \Delta$. Denote the other one by $U^-(f)$ and call it *negative.*

In this way we get an open cover $\mathcal{U} = \bigcup_f U(f)$ of Δ^1 with positive and negative components of each $U(f) - \Delta$. We call such a cover a *signed cover.*

Let $\mathcal{U}$ be a signed cover of Δ^1. We say that $\mathcal{U}$ is a *fine signed cover* if for any two points f and g in Δ^1, $U(f) \cap U(g) \neq \emptyset$ implies $U(f) \cap U(g) \cap \Delta^1 \neq \emptyset$.

7.2.1 Definition. *A fine signed cover $\mathcal{U}$ is* coherent *if for any two elements $f, g \in \Delta^1$, $U(f) \cap U(g) \neq \emptyset$ implies that $U^+(f) \cap U^+(g) \neq \emptyset$.*

7.2.2 Definition. Δ *is* coorientable *if there exist a coherent fine signed cover of Δ^1. A choice of a specific fine signed cover of Δ^1 is a* coorientation *of Δ and Δ is* cooriented *by such a choice.*

7.2.3 Proposition. *If $k = 4s + d$, $d = 0, 1, 3$ then $\Sigma^1_k \subset \mathcal{F}_k$ is coorientable. If $k = 4s + 2$ then $\Sigma_k - HC_k$ is coorientable.*

Proof. If $k = 4s + r$, $r = 1, 2, 3$ we say that we pass Σ^1_k in the positive direction if the number of self intersection components increases. If $k = 4s$ we say that we pass in the positive direction if the number of self intersection circles with the trivial spin structure increases. (See Section 5.1.) $\square$

7.2.4 Remark. We shall use coorientation to study the jumps of invariants of generic immersions. Since HC_{4s+2} is nonseparating (see Section 7.3), no invariant jumps when we pass it. Thus, we do not need to consider any coorientation of HC_{4s+2}.

7.3 Separation. Let C be a connected component of Δ^1.

7.3.1 Definition. *The component C is* separating *if for any point $f \in C$ and any neighborhood W in $\mathcal{F}$ of f, W meets at least two different components of $\mathcal{F} - \Delta$.*

It follows from the propositions in Section 5.1 that if $k = 4s + r$, $r = 0, 1, 3$ then all components of Σ^1_k separates $\mathcal{F}_k$ and if $k = 4s + 2$, then all components except those in HC_k separates $\mathcal{F}_k$.

We define a function $\rho \colon \Delta^1 \to \mathbb{Z}$ as follows.

7.3.2 Definition.

$$\rho(f) = \begin{cases} 0 & \textit{if } f \textit{ belongs to a nonseparating component,} \\ 1 & \textit{if } f \textit{ belongs to a separating component.} \end{cases}$$

7.4 Intersection numbers of generic paths. Let $\Delta \subset \mathcal{F}$ be cooriented. Let $\lambda \colon I \to \mathcal{F}$ be a path that intersects Δ^1 transversely at $f = \lambda(t)$, $t \in I$.

7.4.1 Definition. The local intersection number *of λ and Δ^1 at f, $(\lambda \bullet \Delta^1)_f$ is $+1$ if $\lambda(t+\delta) \in U^+(f)$ for all small $\delta > 0$ and -1 if $\lambda(t+\delta) \in U^-(f)$ for all small $\delta > 0$.*

A *generic path* $I \to \mathcal{F}$ intersects Δ in Δ^1 transversely at a finite number of points. Any path $I \to \mathcal{F}$ is homotopic, by an arbitrarily small homotopy, to a generic path.

Let $\lambda \colon I \to \mathcal{F}$ be a generic path. Assume that it intersects Δ^1 at the points $f_1, \dots, f_n$.

7.4.2 Definition. The intersection number $(\lambda \bullet \Delta^1)$ *of λ and Δ^1 is*

$$(\lambda \bullet \Delta^1) = \sum_{i=1}^{n} \rho(f_i)(\lambda \bullet \Delta^1)_{f_i}.$$

Note that the coorientation of $\rho^{-1}(0)$ (i.e. the union of the codimension one components that does not separate) does not contribute to $(\lambda \bullet \Delta)$.

7.5 Consistent coorientation. Let Δ^2 denote the codimension two stratum of Δ. Let $g \in \Delta^2$. The versal deformation of g is a two parameter deformation. Let $\Lambda_g \colon D^2 \to \mathcal{F}$, $\Lambda_g(0) = g$ be a versal deformation. Then $\lambda_g = \Lambda_g | \partial D^2$ is a generic loop.

7.5.1 Definition. Δ *is* consistently coorientable *if Δ is coorientable and for any $g \in \Delta^2$, $(\lambda_g \bullet \Delta^1) = 0$.*

Let $\sigma \colon I \to \mathcal{F}$ be any path with endpoints in $\mathcal{F} - \Delta$. Then σ is homotopic, with endpoints fixed, to a generic path. Let λ_1 and λ_2 be two such a paths.

Assume that Δ is consistently cooriented. Then $(\lambda_1 \bullet \Delta^1) = (\lambda_2 \bullet \Delta^1)$ and in this case we define intersection numbers as follows.

7.5.2 Definition. *The* intersection number *of σ and Δ is*

$$(\sigma \bullet \Delta) = (\lambda \bullet \Delta^1),$$

where λ is any generic path homotopic to σ with endpoints fixed.

7.6 Continuous coorientation. Let g be any point Δ^2 and let Λ_g be as in Section 7.5. Then $\Lambda_g^{-1}(\Delta)$ is a collection of smooth arcs $\{a_i\}$ and $0 \in a_i$, for each i. (Some of these arcs may have 0 as an endpoint.)

If Δ is cooriented there is an induced coorientation on $a_i - \{0\}$, for each i.

7.6.1 Definition. *A coorientation of Δ is* continuous *if it is consistent and for any $g \in \Delta^2$ and arc a_i, as above, the induced coorientation extends continuously to all of a_i.*

7.6.2 Proposition. *If $k = 4s + d$, $d = 1, 3$ then Σ_k has an continuous coorientation. If $k = 4s + 2$ then $\Sigma_k - HC_k$ has an continuous coorientation.*

Proof. The coorientation in the proof of Proposition 7.2.3 is consistent since the number of self intersection circles can not change after moving around a loop in $\mathcal{F}_k$. Also, this consistent coorientation clearly extends continuously across the codimension two strata. $\square$

Notation. Let $(\emptyset|t)$ denote an elliptic self tangency point, $(t|tt)$ a self intersection component with a hyperbolic self tangency point such that under versal deformation there appears one or two self intersection circles, all with the trivial spin structure. The symbols $(t|nn)$, $(n|tn)$ and $(t|n)$ should be understood in a similar way. (See Section 5.)

7.6.3 Proposition. *If $k = 4s$ then Σ_k does not have any continuous coorientation.*

Proof. We shall use Lemma 9.3.2 on page 17, for $m = 1$, and the proof of Theorem 1. Combining these we see that any 1-immersion can be connected by a smooth path in Σ_k, intersecting Σ_k^2 transversely, to a 1-immersion with self intersection either $(\emptyset|t)$, or $(t|tt)$.

Therefore, if there exists an continuous coorientation of Σ_k then it is determined by the coorientations chosen on the two parts of Σ_k^1 containing immersions with self intersections as described above.

Figure 1 shows the result of transporting a coorientation from the part of Σ_k^1 consisting of immersions with self intersection $(t|tt)$ to the part consisting of immersions with self intersection $(t|nt)$. In Figure 2, we see other regions of these two parts of Σ_k^1 with the coorientations from Figure 1. We conclude that there is no coorientation on the part of Σ_k^1 containing immersions with self intersection of the type $(t|n)$ that extends continuously. This proves the proposition. $\square$

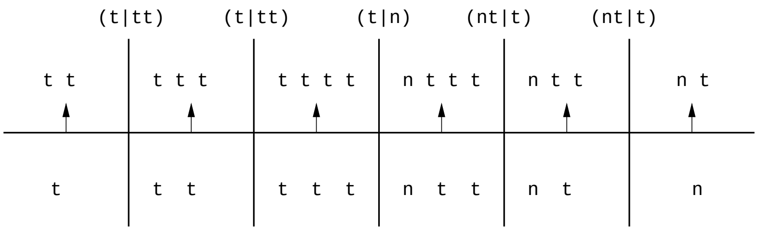

Fig. 1: Transporting the coorientation.

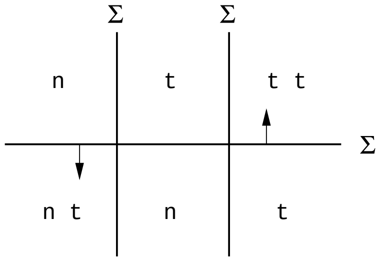

Fig. 2: Obstruction to continuous coorientation.

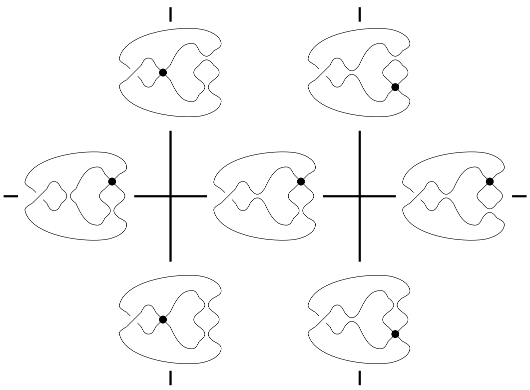

Fig. 3: The self intersections of the immersions in the walls of Figure 2.

8 Invariants and Jumps

8.1 The jump of an invariant. Let Δ be a cooriented hypersurface in a functional space $\mathcal{F}$. For reasons mentioned in the Introduction, elements in $H^0(\mathcal{F}-\Delta; G)$, where G is an Abelian group, will be called *G-invariants.* Let a be a G-invariant.

Let $f_0 \in \Delta^1$ and let $\lambda\colon [-1,1] \to \mathcal{F}$, $\lambda(0) = f_0$ be a versal deformation of f_0. Let $f_t = \lambda(t)$. Choose the versal deformation so that for small $\delta > 0$, $f_\delta \in U^+(f_0)$. Then $f_{-\delta} \in U^-(f_0)$.

8.1.1 Definition. *The* jump *of a, $\nabla a\colon \Delta^1 \to G$ is the function*

$$\nabla a(f_0) = a(f_\delta) - a(f_{-\delta}).$$

8.1.2 Remark. Note that ∇a is a locally constant G-valued function on Δ^1. Therefore, $\nabla a \in H^0(\Delta^1; G)$.

8.1.3 Remark. If $G = \mathbb{Z}_2$ then the above goes through without any coorientation of Δ.

8.2 The invariant of a jump. We describe the elements of $H^0(\Delta^1; G)$ which are jumps. Assume that Δ is cooriented or that $G = \mathbb{Z}_2$.

For a generic path $\lambda \colon I \to \mathcal{F}$, let $\lambda(t_i)$ denote the value of λ at its i^{th} intersection with Δ^1.

8.2.1 Proposition. *If $\alpha \in H^0(\Delta^1; G)$ then exists $a \in H^0(\mathcal{F} - \Delta; G)$ such that $\alpha = \nabla a$ if and only if for any generic loop σ in $\mathcal{F}$*

$$\sum_{i=1}^{n} (\sigma \bullet \Delta^1)_{\sigma(t_i)}\, \alpha(\sigma(t_i)) = 0,$$

where n is the number of intersections. Moreover, if such an invariant a exists it is unique up to addition of constants (one constant for each path component of $\mathcal{F}$).

Proof. The only if part is clear. Choose one base point in each path component of $\mathcal{F}$ and define a to be zero at these points. Given any point f in $\mathcal{F} - \Delta$, pick a generic path λ from the appropriate base point to f. Define

$$a(f) = \sum_{i=1}^{n} (\lambda \bullet \Delta^1)_{\lambda(t_i)}\, \alpha(\lambda(t_i)),$$

where n is the number of intersection points. Then a is locally constant and $\nabla a = \alpha$. The last statement is immediate. $\square$

9 Vassiliev Invariants of Generic Immersions

9.1 Jumps of higher order. Let G be an Abelian group. Let a be a G-invariant of a generic immersion, i.e. $a \in H^0(\mathcal{F}_k - \Sigma_k; G)$.

Coorient Σ_k ($\Sigma_k - HC_k$ if $k = 4s+2$) as in the proof of Proposition 7.2.3. Then there is a jump $\nabla a \in H^0(\Sigma_k^1; G)$ associated to a.

If $k = 4s + d$, $d = 1, 2, 3$ then Σ_k has an continuous coorientation and we can iterate this construction: Let $f_{0,\dots,0}$ (m zeroes) denote an immersion in Σ_k^m. Then $f_{0,\dots,0}$ has m distinct self tangency points.

Let $\Lambda \colon [-1,1]^m \to \mathcal{F}$ be a versal deformation of $f_{0,\dots,0}$ such that $\Lambda(t_1, \dots, t_m) \in \Sigma_k^l$ if l of the $t_i = 0$ and the other ones are different from 0. Let $f_{t_1,t_2,\dots,t_m} = \Lambda(t_1, \dots, t_m)$.

Fix nonzero values of t_i, $i \neq j$ then the intersection number of the path $f_{t_1,\dots,t_{j-1},t_j,t_{j+1},\dots,t_m}$ and Σ_k is independent of these fixed values. This is a consequence of the extendibility of the coorientation.

Assume that the versal deformation is chosen so that this intersection number is $+1$ for all t_j.

9.1.1 Definition. *Assume that $\nabla^{m-1}a \in H^0(\Sigma_k^{m-1}; G)$, the $(m-1)^{th}$ jump of a, has been defined. Define* the m^{th} jump *of a, $\nabla^m a \in H^0(\Sigma_k^m; G)$ by*

$$\nabla^m a(f_{0,\dots,0}) = \nabla^{m-1} a(f_{\delta,0\dots,0}) - \nabla^{m-1} a(f_{-\delta,0\dots,0}),$$

for a small $\delta > 0$.

9.1.2 Remark. It is easy to see that this function is locally constant on Σ_k^m and hence gives an element in $H^0(\Sigma_k^m; G)$.

Since the coorientation is continuous, we have

$$\nabla^{m-1}a(f_{\delta,0\dots,0}) - \nabla^{m-1}a(f_{-\delta,0\dots,0}) = \nabla^{m-1}a(f_{0,\dots,\delta,\dots,0}) - \nabla^{m-1}a(f_{0\dots,-\delta,\dots,0}),$$

where δ is at any of the other $m-1$ positions. I.e. it does not matter which self tangency we resolve.

9.1.3 Remark. If $k = 4s$ then Σ_k does not have any continuous coorientation. However, if $G = \mathbb{Z}_2$ the construction and Definition 9.1.1 makes sense without any reference to coorientation. That is, $a \in H^0(\mathcal{F}_{4s} - \Sigma_{4s}; \mathbb{Z}_2)$ gives $\nabla^m a \in H^0(\Sigma^m_{4s}; \mathbb{Z}_2)$.

For the remainder of this paper we let $G = \mathbb{Z}_2$ if $k = 4s$.

9.2 Vassiliev invariants.

We define Vassiliev invariants as follows:

9.2.1 Definition. *A G-invariant is a Vassiliev invariant of order m if $\nabla^{m+1}a \equiv 0$.*

9.2.2 Remark. Using this definition of finite order invariants, the only parts of the strata of the discriminant of higher codimension that gives any input to our theory are those consisting of immersions with several distinct self tangency points. Taking also other parts into consideration would add conditions on finite order invariants and could possibly lead to (even) fewer Vassiliev invariants. However, it is likely that all the invariants presented below satisfies also these stronger conditions.

9.3 Simple m-immersions.

9.3.1 Definition. *An m-immersion is* simple *if all of its self tangency points are either elliptic, or hyperbolic such that under a versal deformation all self intersection circles appearing get the trivial spin structure.*

Let f be an m-immersion.

9.3.2 Lemma. *There is a path $\lambda \colon I \to \Sigma^m_k \cup \Sigma^{m+1}_k$ intersecting Σ^{m+1}_k transversely in finitely many points such that $\lambda(0) = f$ and $\lambda(1)$ is simple.*

Proof. Introduce two hyperbolic self tangencies close to the hyperbolic self tangency points of f, as in Figure 4. Then all small circles appearing after a versal deformation can be cancelled by introducing an elliptic self tangency. Therefore, they have the trivial spin structure. $\square$

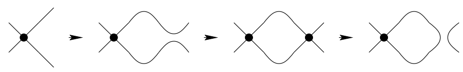

Fig. 4: Introducing a hyperbolic self tangency.

Notation. We write the self intersection of a simple m-immersion as $\{x_1, \dots, x_j\}$, where $x_i \in \{t, n, r, l, (t|tt), (\emptyset|t)\}$ for $i = 1, \dots, j$. (See Section 7.6.)

9.4 Properties of Vassiliev invariants.

9.4.1 Lemma. *A Vassiliev invariant v of order m is uniquely determined, up to addition of lower order Vassiliev invariants, by its m^{th} jump $\nabla^m v$.* □

9.4.2 Lemma. *If v_m is a Vassiliev invariant of order m then $\nabla^m v_m$ is constant (in each path component of $\mathcal{F}_k$).*

Proof. Lemma 9.3.2 shows that it is enough to prove this for simple m-immersions. Since the smooth part of the self intersection can be reduced to standard form by passing Σ_k^{m+1} transversely and $\nabla^m v_m$ does not jump on such hits we can assume that the smooth part of the self intersection is fixed. Call it X. Then, if we label a simple m-immersion by its self intersection, we have:

$$\begin{aligned}
&\nabla^m(\{(t|tt),\ldots,(t|tt),X\}) = \\
&\nabla^{m-1}(\{t,t,(t|tt),\ldots,(t|tt),X\}) - \nabla^{m-1}(\{t,(t|tt),\ldots,(t|tt),X\}) = \\
&\nabla^{m-1}(\{t,(t|tt),\ldots,(t|tt),X\}) + \nabla^m(\{t,(\emptyset|t),(t|tt),\ldots,(t|tt),X\}) \\
&-\nabla^{m-1}(\{t,(t|tt),\ldots,(t|tt),X\}) = \\
&\nabla^m(\{t,(\emptyset|t),(t|tt),\ldots,(t|tt),X\}) = \nabla^m(\{(\emptyset|t),(t|tt),\ldots,(t|tt),X\}).
\end{aligned}$$

Similarly it follows that

$$\begin{aligned}
&\nabla^m(\{(\emptyset|t),(t|tt),\ldots,(t|tt),X\}) = \nabla^m(\{(\emptyset|t),(\emptyset|t),\ldots,(t|tt),X\}) = \ldots \\
&= \nabla^m(\{(\emptyset|t),(\emptyset|t),\ldots,(\emptyset|t),X\}).
\end{aligned}$$

□

9.4.3 Corollary. $V'_m/V'_{m-1} = G$, *or* $V'_m/V'_{m-1} = 0$. □

9.4.4 Lemma. *The intersection number of any generic loop in $\mathcal{F}_k$ and Σ_k^1 endowed with the coorientation given in the proof of Proposition 7.2.3 is zero.*

Proof. Since this intersection number is equal to the increase in the number of self intersection components (with the trivial spin structure if $k = 4s$) along the path, this number must be zero for a loop. □

9.4.5 Proposition. $V'_1/V'_0 = G$ *and is generated*

(i) by the invariant which compute the number of self intersection components if $k = 4s + r$, $r = 1, 2, 3$ and multiplies this number by an element in G (considering G as a $\mathbb{Z}$-module).

(ii) by the invariant which compute the number of self intersection components with the trivial spin structure modulo 2 if $k = 4s$, (in this case we consider only $G = \mathbb{Z}_2$).

□

9.5 Proof of Theorem 2. Assume inductively that, for $p < m$, p^{th} order Vassiliev invariants v_p has been constructed and that they have the following properties (which clearly holds for v_0 and v_1):

P1. $\nabla^1 v_p(f_0) = v_{p-1}(f_{-\delta})$.

P2. If $\lambda\colon I \to \mathcal{F}_k$ is any generic path such that $(\lambda \bullet \Sigma^1_k) = 0$ then $v_p(\lambda(0)) = v_p(\lambda(1))$.

Define $\alpha \in H^0(\Sigma^1_k; G)$ by

$$\alpha(f_0) = v_{m-1}(f_{-\delta}).$$

Let $\lambda\colon I \to \mathcal{F}_k$ be a generic path with $(\lambda \bullet \Sigma^1_k) = 0$ intersecting Σ^1_k at $t_1, \ldots, t_n$. We claim that

$$\sum_{i=1}^{n} (\lambda \bullet \Sigma^1_k)_{\lambda(t_i)} \alpha(\lambda(t_i)) = 0.$$

To see this let t_j be the first point where $(\lambda \bullet \Sigma^1_k)_{\lambda(t_j)} \neq (\lambda \bullet \Sigma^1_k)_{\lambda(t_1)}$. Then

$$\begin{aligned} \alpha(\lambda(t_{j-1})) &= v_{m-1}(\lambda(t_{j-1} + \epsilon\delta)), \\ \alpha(\lambda(t_j)) &= v_{m-1}(\lambda(t_j - \epsilon\delta)), \end{aligned}$$

where $\epsilon = \pm 1$. The intersection number of $\lambda|[t_{j-1} + \epsilon\delta, t_j - \epsilon\delta]$ and Σ^1_k is zero. Hence, $\alpha(\lambda(t_{j-1})) = \alpha(\lambda(t_j))$. We proceed in a similar way.

If $j = 2$ then repeat the argument starting the path from $t_2 + \delta$. Otherwise, if $(\lambda \bullet \Sigma^1_k)_{\lambda(t_{j+1})} \neq (\lambda \bullet \Sigma)_{\lambda(t_1)}$ then the same argument shows that $\alpha(\lambda(t_{j+1})) = \alpha(\lambda(t_{j-1}))$ and if $(\lambda \bullet \Sigma^1_k)_{\lambda(t_{j+1})} = (\lambda \bullet \Sigma)_{\lambda(t_1)}$ then find the first point t_r with $(\lambda \bullet \Sigma^1_k)_{\lambda(t_r)} \neq (\lambda \bullet \Sigma)_{\lambda(t_1)}$. As above we see that $\alpha(\lambda(t_r)) = \alpha(\lambda(t_{r-1}))$.

Thus, the even number of intersections can be collected in pairs (t_i, t_j) with $(\lambda \bullet \Sigma^1_k)_{\lambda(t_i)} \neq (\lambda \bullet \Sigma^1_k)_{\lambda(t_j)}$ and $\alpha(\lambda(t_i)) = \alpha(\lambda(t_j))$. This proves the claim.

Since the intersection number of any generic loop and Σ^1_k is zero, Proposition 8.2.1 implies that we can find an invariant v_m such that $\nabla v_m = \alpha$. Then v_m has properties **P1** and **P2**.

We must compute the jumps of v_m. By definition $\nabla^1 v_m(f_0) = v_{m-1}(f_{-\delta})$. Hence,

$$\begin{aligned} \nabla^2 v_m(f_{0,0}) &= \nabla^1 v_m(f_{\delta,0}) - \nabla^1 v_m(f_{-\delta,0}) \\ &= v_{m-1}(f_{\delta,-\delta}) - v_{m-1}(f_{-\delta,-\delta}) \\ &= \nabla^1 v_{m-1}(f_{0,-\delta}) = v_{m-2}(f_{-\delta,-\delta}). \end{aligned}$$

Iterating this argument we get

$$\nabla^m v_m(f_{0,\ldots,0}) = v_0(f_{-\delta,\ldots,-\delta}) = 1, \quad \nabla^{m+1} v_m \equiv 0.$$

Thus, v_m is a m^{th} order Vassiliev invariant satisfying **P1** and **P2**.

The theorem follows for $k = 4s$ (in this case we only care about $\mathbb{Z}_2$ invariants) and for other k if we consider $\mathbb{Z}$-invariants.

If G is any Abelian group and $g \in G$, let $v^g_m = v_m \cdot g$. Then v^g_m is an m^{th} order G-invariant such that $\nabla^m v^g_m \equiv g$. This proves the general case. $\square$

10 Formulas and Nonseparation Properties

10.1 Vassiliev invariants and the number of self intersection components. The results in this section are formulated for $\mathbb{Z}$-invariants ($\mathbb{Z}_2$ if $k = 4s$). The end of the proof of Theorem 2 shows how to derive similar results for the other G.

Let f be a generic immersion in $\mathcal{F}_k$. Let v_m be the m^{th} order integer Vassiliev invariant constructed above. Recall that M_f denotes the self intersection manifold of the generic immersion f .

10.1.1 Lemma. (i) *If $k = 4s + d$, $d = 1, 2, 3$ $v_m(f)$ depends only on the number of components of M_f.*

(ii) *If $k = 4s$, $v_m(f)$ depends only on $T(f)$.*

Proof. This is true for v_1. Assume that it is true for v_{m-1}. We use induction on the path components of M_f, i.e., on $v_1(f)$ in case (i) and $T(f)$ in case (ii).

Connect f to g with a path intersecting Σ_k^1 once in the negative direction. Then $v_1(g) = v_1(f) - 1$ ($T(g) = T(f) - 1$ in case (i)) and by definition of v_m

$$v_m(f) = v_m(g) + v_{m-1}(g).$$

This finishes the proof. □

Let U be a path component of $\mathcal{F}_k$. Normalize v_m so that it takes the value zero on the generic immersions in U with a minimal number of self intersection components.

10.2 Proof of Theorem 3. Using Lemma 10.1.1 we see that $v_m(f)$ is determined by $v_1(f)$ in case (i) and $T(f)$ in case (ii). We therefore write $v_m(f) = v_m(l)$, where $l = v_1(f)$ in case (i) and $l = T(f)$ in case (ii). In this notation, the recursion formula for v_m is given by

$$v_m(l) = v_m(l-1) + v_{m-1}(l-1), \quad v_0 \equiv 1.$$

Hence,

$$v_m(l) = \sum_{j=1}^{l-1} v_{m-1}(j).$$

Assuming inductively that $v_{m-1}(l) = \binom{l}{m-1}$ (modulo 2 in case (ii)). Then the right hand side is the coefficinet of x^l in the polynominal

$$x\left((1+x) + (1+x)^2 + \cdots + (1+x)^{l-1}\right) = (1+x)^l - (1+x).$$

□

10.3 Proof of Corollary 4. Using the method of Example 8.2 in [4] combined with connected sum we can construct generic immersions f with any values of $T(f)$, $N(f)$, $R(f)$ and $L(f)$. The Corollary now follows from Proposition 6.3.1 and Theorem 3. □

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Appendix

Proof of Proposition 2.2.1. We shall show that a path in $\mathcal{F}_k$ generically intersects Σ_k transversely in points corresponding to immersions with the stated properties.

Notation. Let I denote the unit interval. Let $J^2(S^k \times I, \mathbb{R}^{2k-1})$ denote the space of 2-jets of maps $S^k \times I \to \mathbb{R}^{2k-1}$. We write $[f,(x,t),U \times J]$ for elements in this space. Here f is a germ at the point $(x,t) \in U \times J \subset S^k \times I$, where $U \times J$ is open. (Germs with equal derivatives up to second order are identified.)

The subset in $J^1(S^k \times I, \mathbb{R}^{2k-1})$ containing points $[f,(x,t),U \times I]$, such that $f_t(\cdot) = f(\cdot, t)$ is the germ of an immersion, is an open subset.

Consider the product space $J^2(S^k \times I, \mathbb{R}^{2k-1}) \times J^2(S^k \times I, \mathbb{R}^{2k-1})$. Let N_r denote the subset containing points $([f,(x,t),U \times J],[g,(y,s),V \times K])$ satisfying

$$s = t, \quad f(x,t) = g(y,s), \quad \dim(df_x T_x U + dg_y T_y V) = 2k-1-r.$$

Then N_r has codimension

$$1 + (2k-1) + ((2k-1) - (2k-1-r))(2k - (2k-1-r)) = 2k + r(r+1).$$

Using the jet-transversality theorem we see that any map $f\colon S^k \times I \to \mathbb{R}^{2k-1}$ is homotopic, by an arbitrarily small homotopy, to a map g whose 2-jet extension $j^2 g$ has the property that $j^2 g \times j^2 g$ is transverse to N_r outside a neighborhood of the diagonal of $(S^k \times I) \times (S^k \times I)$. Moreover, such a homotopy can be taken so that it is fixed on subsets where $j^2 f \times j^2 f$ is already transverse to N_r.

Counting dimension and using the compactness of the source space this means that $j^2 g \times j^2 g$ meets N_1 in isolated points and does not meet N_r, $r \geq 2$ at all. Using this and the openness of immersions, we see that any path in $\mathcal{F}_k$ is homotopic, by an arbitrarily small homotopy, to a path f_t in $\mathcal{F}_k$ that meets Σ_k at immersions $f_1, \dots f_n$, each with finitely many double points $z_j^i = f_i(p_j^i) = f_i(q_j^i)$, $p_k^i \neq q_k^i$, where

$$\dim(df_i T_{p_j^i} S^k + df_i T_{q_j^i} S^k) = 2k-2.$$

Adjusting the speed of the homotopy locally, we can see to that f_t meets Σ_k at immersions $f_1, \dots, f_m$ with only one self tangency each. This proves the first part of the proposition. To prove the second we have to involve one more derivative:

If $([f,(x,t),U\times J],[g,(y,t),V\times K])$ is a point in N_1 with $f(x,t)=f(y,t)=z$ and $\dim(df_xT_xU+dg_yT_yV)=2k-2$ then let w be a vector in $T_z\mathbb{R}^{2k-1}$ othogonal to $df_xT_xU+dg_yT_yV\subset T_z\mathbb{R}^{2k-1}$ and let $\{u,v\}$ be a basis in $df_xT_xU\cap dg_yT_yV$.

Let $\{u_x,v_x\}$, $\{u_y,v_y\}$ denote the preimages under df_t and dg_t respectively of $\{u,v\}$. (Here, df_t denotes the derivative of f with respect to x, keeping t fixed.) Let $\pi_w\colon\mathbb{R}^{2k-1}\to\mathbb{R}$ be the projection along the hyperplane with normal w and $\pi_w\circ f=f_w$.

Let N_1' denote the subset of N_1 containing maps with

$$\det\begin{pmatrix}\frac{\partial^2 g_w}{\partial y^2}(u_y,u_y)-\frac{\partial^2 f_w}{\partial x^2}(u_x,u_x) & \frac{\partial^2 g_w}{\partial y^2}(u_y,v_y)-\frac{\partial^2 f_w}{\partial x^2}(u_x,v_x)\\ \\ \frac{\partial^2 g_w}{\partial y^2}(v_y,u_y)-\frac{\partial^2 f_w}{\partial x^2}(v_x,u_x) & \frac{\partial^2 g_w}{\partial y^2}(v_y,v_y)-\frac{\partial^2 f_w}{\partial x^2}(v_x,v_x)\end{pmatrix}=0.$$

Then N_1' has codimension ≥ 1 in N_1. By the same argument as above, we can therefore avoid such points choosing a generic path in $\mathcal{F}_k$.

Consider an immersion f with self tangency at $z=f(p)=f(q)$ and the above determinant nonzero. Using a translation of $\mathbb{R}^{2k-1}$ we can assume $z=0$. Let x denote coordinates centered at p in U, where U is a neighborhood of p. We can choose the coordinates x so that after possibly renumbering the coordinates z in $\mathbb{R}^{2k-1}$, $f(x_1,x_2,x_3)=(x_1,x_2,0,x_3,0)$, $df(\frac{\partial}{\partial x_1}), df(\frac{\partial}{\partial x_2})$ spans $dfT_xU\cap dfT_xV$ and z_3 is a coordinate in the direction w described above.

Let V' be a small neighborhood of q. Then $f(V')$ is an embedded k-dimensional manifold intersecting the k-plane $z_5=0$ transversely and hence in a 2-dimensional manifold containing 0. The inverse image of this 2-manifold contains a 2-disk D around $q\in V'$. The image of the normal bundle of D at p spans a $(k-2)$-dimensional subspace of $T_0\mathbb{R}^{2k-1}$, transverse to the plane $z_5=0$.

Let $(d,y_3)\in D\times\mathbb{R}^{2k-1}$ be coordinates on this normal bundle. As in the proof of the tubular neighborhood theorem, we can find a diffetopy H_s of $\mathbb{R}^{2k-1}$ which is fixed on $z_5=0$ and such that $\pi_{4,5}H_1f(d,y_3)=(0,y_3)$, where $\pi_{4,5}\colon\mathbb{R}^{2k-1}\to\mathbb{R}^{k-2}\times\mathbb{R}^{k-2}$ is the projection along the plane $z_4=z_5=0$. For simplicity, we keep the notation f (instead of writing H_1f everywhere in what follows). We can choose coordinates (y_1,y_2) in D so that

$$\begin{pmatrix}\frac{\partial^2 f_3}{\partial y_1\partial y_1} & \frac{\partial^2 f_3}{\partial y_1\partial y_2}\\ \\ \frac{\partial^2 f_3}{\partial y_2\partial y_1} & \frac{\partial^2 f_3}{\partial y_2\partial y_2}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & \epsilon\end{pmatrix},$$

where $\epsilon=\pm1$. The disk $f(D)$ is then parameterized by

$$(y_1,y_2,\phi(y_1,y_2),0,0),$$

where

$$\phi(y_1,y_2)=y_1^2+\epsilon y_2^2+\psi(y_1,y_2),$$

and $\psi(y_1,y_2)$ vanishes up to order 3.

Let G denote the local diffeomorphism of $\mathbb{R}^{2k-1}$

$$G(z_1, z_2, z_3, z_4, z_5) = (z_1, z_2, z_3 - \psi(y_1, y_2), z_4, z_5).$$

Then G extends to all of $\mathbb{R}^{2k-1}$ and is diffeotopic to identity, keeping $z_3 = 0$ fixed. Let V be a tubular neighborhood of D parameterized by the normal bundle of D. Then $Gf|V$ is given by

$$Gf(y_1, y_2) = (y_1, y_2, y_1^2 + \epsilon y_2^2, 0, y_3).$$

This proves the second part of the proposition. □

Proof of Proposition 2.2.3. By a standard result in singularity theory, it is enough to prove that the deformation is infinitesimally versal.

Notation. If $x = (x_1, \ldots, x_d)$ are coordinates on $\mathbb{R}^d$ and $h\colon Y \to \mathbb{R}^d$, $h(y) = (h_1(y), \ldots, h_d(y))$ is some map from a space Y to $\mathbb{R}^d$ then

$$h(y) \cdot \frac{\partial}{\partial x} = \sum_{i=1}^{d} h_i(y) \frac{\partial}{\partial x_i}.$$

Also, if $f\colon \mathbb{R}^d \to \mathbb{R}^m$ then

$$h(y) \cdot \frac{\partial f}{\partial x} = \sum_{i=1}^{d} h_i(y) \frac{\partial f}{\partial x_i}.$$

Proving infinitesimal versality is the following: Given any variations

$$\alpha(x) = \sum_{i=1}^{5} \alpha_i(x) \cdot \frac{\partial}{\partial z_i}, \quad \beta(y) = \sum_{i=1}^{5} \beta_i(y) \cdot \frac{\partial}{\partial z_i},$$

we must find functions $\{a_i\}_{i=1}^3$ on U, $\{b_i\}_{i=1}^3$ on V (where U and V are as in Proposition 2.2.1) and a section $k : \mathbb{R}^{2k-1} \to T\mathbb{R}^{2k-1}$ such that

$$\begin{aligned}
\alpha(x) &= \sum_{i=1}^{3} a_i(x) \cdot \frac{\partial f}{\partial x_i} + k(f(x)) + c\dot{F},\\
\beta(y) &= \sum_{i=1}^{3} b_i(y) \cdot \frac{\partial f}{\partial y_i} + k(f(y)) + c\dot{F},
\end{aligned}$$

where $f = F_0$, $\dot{F} = \frac{\partial F_t}{\partial t}\big|_{t=0}$ and c is a constant.

Let $a_3 = \alpha_3$ and $b_3 = \beta_3$. Then infinitesimal versality will follow if we can prove that the following system of equations has a solution.

$$\begin{aligned}
\beta_1(y) &= b_1(y) + k_1(f(y)), & \alpha_1(x) &= a_1(x) + k_1(f(x)),\\
\beta_2(y) &= b_2(y) + k_2(f(y)), & \alpha_2(x) &= a_2(x) + k_2(f(x)),\\
\beta_3(y) &= y_1 b_1(y) + y_2 b_2(y) + k_3(f(y)) + c, & \alpha_3(x) &= k_3(f(x)),
\end{aligned}$$

where k_i is the i^{th} component function of the vector field k, $i = 1, 2, 3$. Let $k_3(z) = \alpha_3(z_1, z_2, z_4)$. Then the last equation holds. Let $c = \beta_3(0) - k_3(z)$. Then, by the below claim, we can find $b_1(y)$ and $b_2(y)$ so that the third equation holds. Choose $k_i(z) = \beta_i(z_1, z_2, z_5) - b_i(z_1, z_2, z_5)$, $i = 1, 2$ then the first two equations hold. Finally, let $a_i(x) = \alpha_i(x) - k_i(f(x))$. This proves that we have a solution and the proposition is fully proved once we demonstrate the following claim:

If $f(x, y)$ is smooth and $f(0, 0) = 0$ then there exist smooth functions $g(x, y)$ and $h(x, y)$ such that

$$f(x, y) = xh(x, y) + yg(x, y).$$

To see this write

$$\begin{aligned} f(x, y) &= f(0, y) + x\frac{\partial f}{\partial x}(0, y) + R(x, y), \\ f(x, y) &= f(x, 0) + y\frac{\partial f}{\partial y}(x, 0) + K(x, y), \end{aligned}$$

then

$$h(x, y) = \frac{1}{2}\begin{cases} \frac{f(x,0)+R(x,y)}{x} + \frac{\partial f}{\partial x}(0, y) & \text{if } x \neq 0, \\ \frac{\partial f}{\partial x}(0, 0) + \frac{\partial f}{\partial x}(0, y) & \text{if } x = 0, \end{cases}$$

is smooth. Similarly we can construct $g(x, y)$. □