GEOMETRY OF SELF INTERSECTION SURFACES AND VASSILIEV INVARIANTS OF GENERIC IMMERSIONS $S^k \to \mathbb{R}^{2k-2}$.

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ABSTRACT. The regular homotopy class of a generic immersion $S^k \to \mathbb{R}^{2k-2}$ is described in terms of its self intersection manifold with natural additional structures.

There is a natural notion of Vassiliev invariants of generic immersions. These may take values in any Abelian group G. It is proved that, for any m, the group of m^{th} order G-valued invariants modulo invariants of lower order is isomorphic to G and that Vassiliev invariants are not sufficient to separate generic immersions, which cannot be obtained from each other by a regular homotopy through generic immersions.

1. INTRODUCTION

1.1. **Finding the regular homotopy class of a generic immersions.** The set of immersions $S^k \to \mathbb{R}^{k+n}$ is an open subset of the space of r times differentiable maps with the C^r-topology, $r \geq 1$. The *space of immersions* is this set with the subspace topology. If two immersions belong to the same path component of the space of immersions they are called *regularly homotopic*.

Whitney [9] classified immersed plane curves up to regular homotopy: Two curves are regularly homotopic if and only if they have the same tangential degree. He gave a formula for the tangential degree of a generic plane curve in terms of its double points.

Smale [8] classified immersions $S^k \to \mathbb{R}^{k+n}$ up to regular homotopy: Comparing any immersion f to the standard embedding $S^k \subset \mathbb{R}^{k+1} \subset \mathbb{R}^{k+n}$, we can associate to it its Smale invariant $\Omega(f) \in \pi_k(V_{k+n,k})$, the k^{th} homotopy group of the Stiefel manifold of k-frames in $(k+n)$-space. The invariant Ω is the higher dimensional analogue of the tangential degree of a plane curve. It gives a one to one correspondence between the regular homotopy classes of immersions $S^k \to \mathbb{R}^{k+n}$ and the elements of the group $\pi_k(V_{k+n,k})$.

Smale generalized the formula of Whitney: If $f\colon S^k \to \mathbb{R}^{2k}$, $k \geq 2$ is a generic immersion then $\Omega(f)$ can be computed geometrically: Count the double points of f (modulo 2 if k is odd, with signs if k is even).

The author [3] found a formula for $\Omega(f)$ of a generic immersion $f\colon S^k \to \mathbb{R}^{2k-1}$, $k \geq 4$ in terms of the geometry of its self intersection.

Theorem 1 below, presents a similar formula for generic immersions $S^k \to \mathbb{R}^{2k-2}$, $k \geq 4$. To state it, we give a brief description of self intersections:

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The self intersection M_f of a generic immersion $f\colon S^k \to \mathbb{R}^{2k-2}$ contains only transverse double points. Hence, M_f is a closed surface. If $f^{-1}(M_f) = \widetilde{M}_f$ then $f\colon \widetilde{M}_f \to M_f$ is a double cover. If k is odd then this is the orientation double cover. If k is even then orientations of S^k and $\mathbb{R}^{2k-2}$ induces orientation of M_f. It follows that there is always an induced orientation on $\widetilde{M}_f$.

Spin structures on S^k and $\mathbb{R}^{2k-1}$ induces a spin structure on $\widetilde{M}_f$, which is invariant under the covering involution. Corresponding to a spin structure on $\widetilde{M}_f$ there is a $\mathbb{Z}_2$-quadratic function on $H_1(\widetilde{M}_f;\mathbb{Z}_2)$.

To define the invariant $r(f)$, which determines f up to regular homotopy, we need to study four separate cases:

If $k = 4s+1$ then any component of M_f has even Euler characteristic (Theorem 7.30 in [2]). Thus, the unique torsion element in $H_1(M_f;\mathbb{Z})$ whose reduction modulo 2 is dual to the first Stiefel-Whitney class of the tangent bundle of M_f, is in the image $f_*H_1(\widetilde{M}_f;\mathbb{Z})$. Reducing any preimage of it modulo 2 we get a uniquely determined element in $H_1(\widetilde{M}_f;\mathbb{Z}_2)$. The invariant $r(f)$ is the quadratic function corresponding to the spin structure on $\widetilde{M}_f$ evaluated on this element. It is an element of $\mathbb{Z}_2$.

If $k = 4s+2$ then there is an induced $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-quadratic function on $H_1(M_f;\mathbb{Z}_2)$ (see Section 3.4). That is, there are two spin structures induced on M_f. The invariant $r(f)$ is the Arf invariant of this quadratic function. It is an element of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

If $k = 4s+3$ then there is an induced $\mathbb{Z}_4$-quadratic function on $H_1(M_f;\mathbb{Z}_2)$ (see Section 3.5). That is, there is a pin structure induced on M_f. The invariant $r(f)$ is the Brown invariant of this quadratic function. It is an element of $\mathbb{Z}_8$.

Finally, if $k = 4s$ then all immersions are known to be regularly homotopic and we define $r(f) = 0$ in this case.

Theorem 1. *Let $k \geq 4$ and let $f, g\colon S^k \to \mathbb{R}^{2k-2}$ be two generic immersions. Then f and g are regularly homotopic if and only if $r(f) = r(g)$, where r is as described above.*

Theorem 1 is proved in Section 6.2.

1.2. **Vassiliev invariants and their insufficiency.** Let $\mathcal{F}_k$ denote the space of immersions $f\colon S^k \to \mathbb{R}^{2k-2}$ and let $k \geq 6$. The set of generic immersions (see Definition 2.1.1) is an open dense subset of $\mathcal{F}_k$. For an Abelian group G, a G-invariant of generic immersions is a G-valued locally constant function on the space of generic immersions.

Let $\Sigma_k \subset \mathcal{F}_k$ denote the subset of all non-generic immersions. Then there is an isomorphism from the group of G-invariants onto $H^0(\mathcal{F}_k - \Sigma_k; G)$.

The subspace $\Sigma_k \subset \mathcal{F}_k$ will be called the *discriminant*. The subset of Σ_k consisting of all immersions with a single self tangency point (see Definition 2.2.3) will be denoted Σ_k^1. It is an dense open subset of Σ_k and a smooth codimension one submanifold of $\mathcal{F}_k$.

We can associate a jump to any G-invariant of generic immersions: Since Σ_k^1 is a smooth submanifold of codimension one, it locally separates $\mathcal{F}_k$. Given an immersion $f_0 \in \Sigma_k^1$ and an invariant of generic immersions, we define the jump of this invariant at f_0 to be the difference of the values of the invariant at two generic immersions which are close to f_0 and are on opposite sides of Σ_k^1. Such a difference is defined only up to sign.

In order to make the jump well defined we make a choice of a positive and a negative part of a small neighborhood of f_0, cut into two parts by Σ_k^1. A coherent choice like that for all $f_0 \in \Sigma_k^1$ is called a *coorientation* of Σ_k.

If we compute the jump using the signs coming from a coorientation then the jump will be a locally constant G-valued function on Σ_k^1. Thus, if Σ_k is cooriented then any invariant $a \in H^0(\mathcal{F}_k - \Sigma_k; G)$ has a jump $\nabla a \in H^0(\Sigma_k^1; G)$.

The discriminant admits a *continuous* coorientation (defined in Section 7.2). Therefore, the above construction can be iterated to give a m^{th} jump $\nabla^m a \in H^0(\Sigma_k^m; G)$ of a, where Σ_k^m consists of all immersions with m distinct self tangency points.

We define the m^{th} *order Vassiliev* G-invariants to be the G-invariants a that satisfies $\nabla^{m+1} a \equiv 0$. This definition is similar to the definition of Vassiliev invariants of knots in 3-space (see [1]).

The Vassiliev invariants form an increasing sequence of subgroups of the group of all invariants. If V_m denotes the group of m^{th} order Vassiliev G-invariants, we have

$$V_0 \subset V_1 \subset \cdots \subset V_m \subset \cdots \subset H^0(\mathcal{F}_k - \Sigma_k; G).$$

The cohomology of $\mathcal{F}_k - \Sigma_k$ splits naturally into a sum over the path components of $\mathcal{F}_k$, which are described by Theorem 1. So does V_m. Let V_m' denote a summand of V_m corresponding to a path component of $\mathcal{F}_k$.

The following theorem describes the structure of the filtered group of Vassiliev invariants.

Theorem 2. *If $k \geq 6$ then $V_m'/V_{m-1}' \cong G$, for all m.*

Theorem 2 is proved in Section 8.5. If $k < 6$ then generic one parameter families of immersions contains more complicated degeneracies than those considered here and the above theorem does not hold.

The Vassiliev invariants can be explicitly computed. The following theorem shows how to do that in case of $\mathbb{Z}$-invariants

Theorem 3. *Let $k \geq 6$. Let $\kappa\colon \mathbb{Z} \to \{0,1\}$ satisfy $\kappa(n) = 0$ if $n \geq 0$ and $\kappa(n) = 1$ if $n < 0$. Let*

$$v(f) = (-1)^{\kappa(\chi(M_f))+1} \left[\frac{|\chi(M_f)|}{2}\right],$$

where χ denotes the Euler characteristic and $[\alpha]$ denotes the integer part of α. If $m \geq 1$, let

$$v_m(f) = (-1)^{m\kappa(v(f))} \binom{|v(f)| + \kappa(v(f))(m-1)}{m},$$

where $\binom{l}{j}$ denotes the binomial coefficient of x^j in the expansion of $(1+x)^l$. Then v_m generates V_m'/V_{m-1}', $m > 0$.

Theorem 3 is proved in Section 9.2. We remark that, for $\mathbb{Q}$-valued invariants $v_m = v^m$, the m^{th} power of v generates V_m'/V_{m-1}'.

In the theory of classical knots in 3-space, Vassiliev conjectured that his invariants distinguish knots. This conjecture has neither been proved nor disproved. If $k \geq 8$ then there is an explicit description of the path components of the space of generic immersions $S^k \to \mathbb{R}^{2k-2}$, see [2]. This allows us to disprove the analogue of the Vassiliev conjecture in this case:

Corollary 4. *If $k \geq 8$ then, for any number n there are pairwise different path components $U_1, \ldots, U_n$ of $\mathcal{F}_k - \Sigma_k$ such that $v(U_1) = v(U_2) = \cdots = v(U_n)$, for all $v \in \bigcup_{m=0}^{\infty} V_m$.*

Corollary 4 is proved in Section 9.3. The special case $n = 2$ of it shows that Vassiliev invariants do not separate generic immersions.

2. The discriminant and versal deformations

2.1. **Generic immersions.** Let $f\colon S^k \to \mathbb{R}^{2k-2}$ be an immersion.

Definition 2.1.1. *f is a* generic immersion *if,*

G1. *For any $z \in \mathbb{R}^{2k-2}$, $f^{-1}(z)$ contains at most two points, and*
G2. *If $f(x) = f(y) = z$, for $x \neq y$ in S^k, then*

$$T_z\mathbb{R}^{2k-2} = dfT_xS^k + dfT_yS^k.$$

If $k > 4$ then the set of generic immersions is an open dense subset of $\mathcal{F}_k$.

2.2. **The codimension one stratum.** Let $x = (x_1, x_2, x_3, x_4)$, $x_i \in \mathbb{R}$, $i = 1,2,3$ and $x_4 \in \mathbb{R}^{k-3}$, be coordinates on $\mathbb{R}^k$ and let $y = (y_1, y_2, y_3, y_4, y_5, y_6)$, $y_i \in \mathbb{R}$, $i = 1, \ldots, 4$ and $y_j \in \mathbb{R}^{k-3}$, $j = 4,5$ be coordinates on $\mathbb{R}^{2k-2}$.

Proposition 2.2.1. *If $k > 5$ then the codimension one stratum Σ_k^1 of Σ_k in $\mathcal{F}_k$ consists of the immersions f satisfying* **G1**, *and* **G2**, *except at one pair of points p, q in S^k, with $f(p) = f(q)$ and $\dim(dfT_pS^k + dfT_qS^k) = 2k - 3$. Moreover, there are disjoint charts U containing p and V containing q, with coordinates x centered at p and y centered at q respectively, such that up to left action of a diffeomorphism, diffeotopic to identity*

$$\begin{aligned} f(x_1, x_2, x_3) &= (x_1, x_2, x_3, 0, x_4, 0), \\ f(y_1, y_2, y_3) &= (y_1, y_2, y_3, y_1^2 + \epsilon_1 y_2^2 + \epsilon_2 y_3^2, 0, y_4), \end{aligned}$$

where $\epsilon_i = \pm 1$, $i = 1, 2$. □

Proposition 2.2.1 is proved in the same way as Proposition 2.2.1 in [3].

Remark 2.2.2. If $k = 5$ then the codimension one stratum Σ_5^1 also contains immersions with triple points. The local model is simply 3 pairwise transversal 5-planes in 8-space intersecting at zero, each meeting the intersection of the other two transversally.

Let z be a double point of f as described above.

Definition 2.2.3. *If $\epsilon_i = +1$, $i = 1, 2$, we say that z is an* elliptic self tangency point *of f. If $\epsilon_i = -1$, some i, we say that z is a* hyperbolic self tangency point *of f.*

2.3. **Versal deformations.**

Proposition 2.3.1. *The versal deformation of a self tangency point is the following one parameter deformation*

$$\begin{aligned} F_t(x_1, x_2, x_3, x_4) &=, (x_1, x_2, x_3, t, x_4, 0) \\ F_t(y_1, y_2, y_3, y_4) &= (y_1, y_2, y_3, y_1^2 + \epsilon_1 y_2^2 + \epsilon_2 y_3^2, 0, y_4). \end{aligned}$$

□

Proposition 2.3.1 is proved in the same way as Proposition 2.3.1 in [3].

Remark 2.3.2. If $k = 5$ then (cf. Remark 2.2.2) we must also consider versal deformations of triple points. The model is simple: Just move one of the three planes along a direction transversal to all three 3-planes.

2.4. m**-immersions.** Let $f\colon S^k \to \mathbb{R}^{2k-2}$ be an immersion.

Definition 2.4.1. *f is an m-*immersion *if it satisfies* **G1**, *and* **G2**, *except at m self tangency points.*

Definition 2.4.2. $\Sigma_k^m \subset \Sigma_k \subset \mathcal{F}_k$ *is the space of m-immersions.*

We say that f_0 belongs to a *smooth codimension one branch* of Σ_k if there is a path $\lambda\colon [-1,1] \to \Sigma_k$ such that $\lambda(0) = f_0$, $\dot\lambda(0) \neq 0$ and $\lambda(t) \in \Sigma_k^1$, if $t \neq 0$. With this terminology Σ_k^m is the transverse intersection of m smooth codimension one branches of Σ_k.

3. Spin, Pin and a functional on curves

In this section we describe the tools used to define r in the formulation of Theorem 1.

3.1. **Spin structures.** If ξ, η and ζ are oriented vector bundles and $\zeta = \xi \oplus \eta$ then spin structures on any two of these bundles determine a spin structure on the third.

If V is a submanifold of a manifold W, we let $N(V \subset W)$ denote the normal bundle of V in W.

Let $f\colon S^k \to \mathbb{R}^{2k-2}$ be a generic immersion. Then we have the following splitting (see the Introduction for notation):

$$f^*(T\mathbb{R}^{2k-2}) = TS^k|\widetilde{M}_f \oplus i^*N(\widetilde{M}_f \subset S^k), \tag{$\dagger$}$$

where $i\colon \widetilde{M}_f \to \widetilde{M}_f$ is the involution ($f \circ i = f$). Since $T\mathbb{R}^{2k-2}$ and TS^k are spin bundles we get an induced spin structure on $N(\widetilde{M}_f \subset S^k)$. Moreover, we have a splitting

$$TS^k|\widetilde{M}_f = T\widetilde{M}_f \oplus N(\widetilde{M}_f \subset S^k). \tag{$\dagger\dagger$}$$

Therefore, we get an induced spin structure on $T\widetilde{M}_f$.

If F is any surface then there is a 1-1 correspondence between spin structures on F and $\mathbb{Z}_2$-quadratic functions on $H_1(F;\mathbb{Z}_2)$, see [4]. The Arf invariant of a $\mathbb{Z}_2$-quadratic function q on a symplectic vector space V over $\mathbb{Z}_2$ is defined to be

$$\mathrm{Arf}(q) = \sum_{(a,b)} q(a)q(b),$$

where (a,b) ranges over any symplectic basis. It determines q up to symplectomorphism of V.

The quadratic function associated to the spin structure on $\widetilde{M}_f$ will be denoted $\widetilde{q}_f$. As we shall see, it is equivariant and thus induces a spin structure on any component $F \subset M_f$, with $f^{-1}(F)$ disconnected.

3.2. **Pin structures on surfaces.** Let V be a vector space over $\mathbb{Z}_2$ with a non-singular symmetric bilinear form $(x,y) \mapsto x \cdot y$. A $\mathbb{Z}_4$-quadratic function on V is a function $q\colon V \to \mathbb{Z}_4$ such that $q(x+y) = q(x) + q(y) + 2(x \cdot y)$, for all $x, y \in V$.

There are four indecomposable $\mathbb{Z}_4$-quadratic spaces:

$$\begin{aligned}
\mathcal{P}_+ &= (\mathbb{Z}_2(a), \cdot, q), & a \cdot a &= 1, & q(a) &= 1,\\
\mathcal{P}_- &= (\mathbb{Z}_2(a), \cdot, q), & a \cdot a &= 1, & q(a) &= -1,\\
\mathcal{T}_0 &= (\mathbb{Z}_2(b) \oplus \mathbb{Z}_2(c), \cdot, q), & b \cdot b = c \cdot c &= 0,\ b \cdot c = 1, & q(b) = q(c) &= 0,\\
\mathcal{T}_4 &= (\mathbb{Z}_2(b) \oplus \mathbb{Z}_2(c), \cdot, q), & b \cdot b = c \cdot c &= 0,\ b \cdot c = 1, & q(b) = q(c) &= 2.
\end{aligned}$$

A $\mathbb{Z}_4$-quadratic space V is called *split* if it contains a subspace H such that $\dim H = \frac{1}{2}\dim V$, $q|H = 0$ and $H \cdot H = \{0\}$. Two $\mathbb{Z}_4$-quadratic spaces V and W belong to the same *Witt class* if there exists spilt spaces S_1 and S_2 such that $V \oplus S_1 \cong W \oplus S_2$. The Witt classes forms the *Witt group* which is generated by $[\mathcal{P}_+]$ (the Witt class of $\mathcal{P}_+$) with relations $8[\mathcal{P}_+] = 0, 4[\mathcal{P}_+] = [\mathcal{T}_4]$ and $[\mathcal{P}_+] + [\mathcal{P}_-] = 0$.

Given a $\mathbb{Z}_4$-quadratic space V with quadratic function q we define

$$\lambda(V,q) = \sum_{x \in V} e^{\frac{\pi i}{2} q(x)}.$$

Then

$$\lambda(V,q) = \sqrt{2}^{\dim V} \left(\frac{1+i}{\sqrt{2}}\right)^m.$$

Since $\frac{1+i}{\sqrt{2}}$ is an 8^{th} root of unity, m modulo 8 is well defined. This is *Brown's invariant*. It is denoted $\beta(V,q)$ and gives an isomorphism between the Witt group and $\mathbb{Z}_8$. We shall sometimes write $\beta(q)$, dropping V. More details about $\mathbb{Z}_4$-quadratic spaces can be found in [6].

Let F be a surface. By a pin structure on F we shall mean a Pin^--structure on the tangent bundle TF of F. There is a 1-1 correspondence between pin structures on F and $\mathbb{Z}_4$-quadratic functions on $H_1(F;\mathbb{Z}_2)$ (See [5], Theorem 3.2).

3.3. **A functional on curves in a self intersection surface.** The quadratic functions mentioned in the introduction all emanates from a functional on curves in self intersection surfaces. It is briefly described below. Details can be found in [2], Sections 5 and 7. Before we start the description we need some preliminaries:

Let $\mathbf{I}$ denote the identity matrix.

Definition 3.3.1. *If $n = 4s+2$, define $\mathbf{A}(n) \in SO(n)$ by*

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \mathbf{I}_{2s} \\ & & \mathbf{I}_{2s} & 0 \end{bmatrix}.$$

if $n = 4s$, define $\mathbf{A}(n) \in SO(n)$ by

$$\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \mathbf{I}_{2s-1} \\ & & \mathbf{I}_{2s-1} & 0 \end{bmatrix}.$$

The group $\{\mathbf{I}, \mathbf{A}(n)\}$ acts on $SO(n)$ from the right. Denote the corresponding quotient space $SO(n)/\mathbf{A}(n)$ and let p be the projection.

Proposition 3.3.2. *Let $n = 4s + r$, $k \geq 1$ and $r = 0, 1, 2$. Then*

$$\pi_1(SO(n)/\mathbf{A}(n)) = \begin{cases} \mathbb{Z}_4 & \text{if } s \text{ is odd,} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } s \text{ is even.} \end{cases}$$

□

Let $u \in \pi_1(SO(n))$ denote the nontrivial element.

The functional is constructed as follows: Let $f\colon S^k \to \mathbb{R}^{2k-2}$ be a generic immersion. Let c be a closed curve in M_f and $c_S = f^{-1}(c)$. Note that c_S is either a union of two disjoint circles or one circle.

Choose a parameterization $r\colon I \to c$ of c and a normal vector $n\colon I \to TM_f$ of c in TM_f. Then r lifts to two parameterizations r_1 and r_2. If c_S is connected then the path product $r_1 * r_2$ is a parameterization of c_S.

Choose orthonormal framings Y_1 and Y_2 of $N(\widetilde{M_f} \subset S^k)|c_S$ along r_1 and r_2 respectively. Let n_1 and n_2 be such that $df(n_i) = n$. Assume that Y_i is chosen so that $(\dot{r}_i, n_i, Y_i)$ is a positively oriented frame, $i = 1, 2$. Then $(\dot{r}, n, dfY_1, dfY_2)$ is a framing or twist framing X of $T\mathbb{R}^{2k-1}|c$.

Let $\vec{c}_S$ denote the loop (or loops) in $SO(TS^k)$ (the principal $SO(k)$-bundle of the tangent bundle of S^k) represented by c_S with the framing $(\dot{r}_i, n_i, Y_i)$ along r_i, $i = 1, 2$ and $[\vec{c}_S]$ the corresponding element in $H_1(SO(TS^k); \mathbb{Z}_2)$.

Let $[c, r, X] \in \pi_1(SO(n)/\mathbf{A}(n))$ denote the homotopy class of the loop induced by c with the (twist) framing $(\dot{r}, n, X)$ (which we may have to orthonormalize). Then define

Definition 3.3.3.

$$\omega_f(c) = \langle \xi_S, [\vec{c}_S] \rangle \cdot p_*(u) + [c, r, X] + p_*(u) \in \pi_1(SO(n)/\mathbf{A}(n)),$$

where $\xi_S \in H^1(SO(TS^k); \mathbb{Z}_2)$ denotes the unique spin structure on S^k.

This definition is independent of all choices.

If c is a closed curve in M_f then there are two non-homotopic parameterizations of c. Denote them c and $-c$. The following proposition describes how ω_f depends on the parameterization.

Proposition 3.3.4. (a) *If $k \equiv 0 \mod 4$ or $k \equiv 1 \mod 4$ and c_S is connected then*

$$\omega_f(-c) = \omega_f(c) + p_*(u).$$

(b) *If $k \equiv 2 \mod 4$ or $k \equiv 3 \mod 4$ or c_S is disconnected then*

$$\omega_f(-c) = \omega_f(c).$$

□

We extend the functional ω_f to collections $L = \{c_1, \ldots, c_m\}$ of oriented closed curves in M_f:

$$\omega_f(L) = \sum_{i=1}^{m} \omega_f(c_i).$$

Let L be a collection of oriented closed curves in M_f with m transverse intersections and L' be the collection obtained from smoothing each intersection, respecting orientation. Then

$$\omega_f(L') = \omega_f(L) + m \cdot p_*(u). \qquad (\ddagger)$$

Also, if K and L are isotopic collections then

$$\omega_f(K) = \omega_f(L).$$

3.4. **Spin structures and the functional on curves.**

Proposition 3.4.1. *Let c be an embedded curve in $\widetilde{M}_f$. Then*

$$\widetilde{q}_f([c]) = \begin{cases} 0 & \text{if and only if } \omega_f(f(c)) = 0, \\ 1 & \text{if and only if } \omega_f(f(c)) = p_*(u), \end{cases}$$

where $[c]$ is the $\mathbb{Z}_2$-homology class represented by c.

Proof. Let ξ denote the cohomology class corresponding to the spin structure on $\widetilde{M}_f$. From the decompositions (†) and (††) in Section 3.1 we see that

$$\langle \xi, [\vec{c}] \rangle = \langle \xi_S, [\vec{c}_S] \rangle + \langle \xi_S, [\vec{ic}_S] \rangle + \langle \xi_{\mathbb{R}^{2k-2}}, [\vec{c}_{\mathbb{R}^{2k-2}}] \rangle,$$

where $\vec{c}_S$ is an extension of the framing $\vec{c}$ of c in $T\widetilde{M}_f$ to a framing of TS^k, $\vec{ic}_S$ is an extension of the framing induced by i on ic in $T\widetilde{M}_f$ to TS^k and $\vec{c}_{\mathbb{R}^{2k-2}}$ is the framing of c induced from these two by df.

Recalling that $\widetilde{q}_f([c]) = \langle \xi, [\vec{c}] \rangle + 1$, see [4], and comparing with Definition 3.3.3 proves the proposition. □

Corollary 3.4.2. *The spin structure on $\widetilde{M}_f$ is equivariant and takes the same value on all $a \in H_1(\widetilde{M}_f; \mathbb{Z}_2)$ such that $f_*(a) = 0 \in H_1(M_f; \mathbb{Z}_2)$. This value is 0 if $k = 4s$ or $k = 4s + 3$ and it is 1 if $k = 4s + 1$ or $k = 4s + 2$.*

Proof. This follows from the definition of ω_f and Propositions 3.4.1, 3.3.2. □

Let $k = 4s + 2$. Choose isomorphisms

$$\phi\colon \pi_1(SO(2k-2)/\mathbf{A}(2k-2)) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2,$$

such that $\phi(p_*(u)) = (1, 1)$. The functional $\phi \circ \omega_f$ induces a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-quadratic function q_f on $H_1(M_f; \mathbb{Z}_2)$:

$$q_f(\alpha) = \phi(\omega_f(a)),$$

where a is a simple closed curve representing $\alpha \in H_1(M_f; \mathbb{Z}_2)$. Thus, in these dimensions, we get two spin structures on M_f. They correspond to the first and second component of the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-quadratic function q_f.

3.5. **Pin structures and and the functional on curves.**

Let $k = 4s+3$. Choose isomorphisms

$$\phi\colon \pi_1(SO(2k-2)/\mathbf{A}(2k-2)) \to \mathbb{Z}_4.$$

The functional $\phi \circ \omega_f$ induces a $\mathbb{Z}_4$-quadratic function q_f on $H_1(M_f; \mathbb{Z}_2)$:

$$q_f(\alpha) = \phi(\omega_f(a)),$$

where a is a simple closed curve representing $\alpha \in H_1(M_f; \mathbb{Z}_2)$. (See [5], Lemma 3.4.) Thus, in these dimensions, we get a pin structures on M_f. It corresponds to the $\mathbb{Z}_4$-quadratic function q_f.

4. The regular homotopy invariants

The regular homotopy classes of immersions $S^k \to \mathbb{R}^{2k-2}$ constitutes $\pi_0(\mathcal{F}_k)$ which depends on the residue class of k modulo 4.

4.1. $k = 4s$**, one regular homotopy class.** To conform with the other cases, we use the following definition to express the fact that all immersions f are regularly homotopic if $k = 4s$.

Definition 4.1.1. *For a generic immersion* $f : S^k \to \mathbb{R}^{2k-2}$, $k = 4s$ *we define* $r(f) = 0$.

4.2. $k = 4s + 1$**, spin structure and a curve dual to** w_1**.** Let $\widetilde{M}_f = f^{-1}(M_f)$. The map $f : \widetilde{M}_f \to M_f$ is the orientation double cover if k is odd ([2], Proposition 2.6). There is an induced spin structure on $\widetilde{M}_f$ with corresponding $\mathbb{Z}_2$-quadratic function $\widetilde{q}_f$.

Let $a \in H_1(M_f; \mathbb{Z})$ be the unique torsion class, whose reduction modulo 2 is dual to $w_1 \in H^1(M_f; \mathbb{Z}_2)$. Since any component of M_f has even genus (Theorem 7.30 in [2]), there is a class $b' \in H_1(\widetilde{M}_f; \mathbb{Z})$ such that $f_*(b') = a$. Reducing b' modulo 2, we get a uniquely determined class $b \in H_1(\widetilde{M}_f; \mathbb{Z}_2)$.

Definition 4.2.1. *For a generic immersion* $f : S^k \to \mathbb{R}^{2k-2}$, $k = 4s + 1$ *we define* $r(f) = \widetilde{q}_f(b) \in \mathbb{Z}_2$.

Remark 4.2.2. The invariant $r(f)$ described above is actually equal to $\mathrm{Arf}(\widetilde{q}_f)$. This can be seen as follows: First, let F be a component of M_f such that $\widetilde{F} = f^{-1}(F)$ has two components. Then the projection b_F of b onto $H_1(\widetilde{F}; \mathbb{Z}_2)$ is zero so $\widetilde{q}_f([b_F]) = 0$. On the other hand $\mathrm{Arf}(\widetilde{q}_f|\widetilde{F}) = 0$ since $\widetilde{F}$ has two components and the spin structure on $\widetilde{F}$ is equivariant. Second, let F be a component of M_f such that $\widetilde{F}$ is connected. Then F is a connected sum of $2g$ projective planes. Choose simple closed curves a_i, b_i, $i = 1, \dots, 2g - 1$ in $\widetilde{M}_f$ which gives a symplectic basis in $H_1(\widetilde{M}_f; \mathbb{Z}_2)$. Let $i : \widetilde{F} \to \widetilde{F}$ be the involution. We can choose the curves so that $i(a_i) = a_{i+g-1}$, $i(b_i) = b_{i+g-1}$ for $i = 1, \dots, g - 1$ and $b_{2g-1} = b_F$. Since the spin structure is equivariant, $q_f(a_i)q_f(b_i) = q_f(a_{i+g-1})q_f(b_{i+g-1})$. Hence, $\mathrm{Arf}(q_f|\widetilde{F}) = q_f(a_{2g-1})q_f(b_{2g-1})$. Now, $f_*(a_{2g-1}) = 0$ so $q_f(a_{2g-1}) = 1$ by Corollary 3.4.2. Thus, $\mathrm{Arf}(q_f|\widetilde{F}) = q_f(b_{2g-1})$. It follows that $r(f) = \mathrm{Arf}(\widetilde{q}_f)$.

4.3. $k = 4s + 2$**, two spin structures.** Since $k = 4s + 2$ is even, M_f is orientable and there is an induced $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-quadratic function $q_f : H_1(M_f; \mathbb{Z}_2) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Definition 4.3.1. *For a generic immersion* $f : S^k \to \mathbb{R}^{2k-2}$, $k = 4s + 2$ *we define* $r(f) = \mathrm{Arf}(q_f) = \mathrm{Arf}(q_f^1) \oplus \mathrm{Arf}(q_f^2) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

4.4. $k = 4s + 3$**, a pin structure.** If $k = 4s + 3$ then there is an induced pin structure on M_f with corresponding $\mathbb{Z}_4$-quadratic function q_f.

Definition 4.4.1. *For a generic immersion* $f : S^k \to \mathbb{R}^{2k-2}$, $k = 4s + 2$ *we define* $r(f) = \beta(q_f) \in \mathbb{Z}_8$*, where* β *is Brown's invariant.*

5. Transformation of the self intersection and invariance of r

The self intersections of the immersions in a generic one parameter family are transformed in different ways. We describe these and show that r remains constant under such transformations.

5.1. **Subdivision of the discriminant.** Let f_t be a smooth path in $\mathcal{F}_k$ intersecting Σ_k^1 transversely in f_0. Let $\delta \geq 0$ be a number so small that for $|t| \leq \delta$, $t \neq 0$ the immersion f_t is generic. Let $f(x,t) = f_t(x)$. Then Proposition 2.3.1 implies that

$$f\colon S^k \times [-\delta,\delta] \to \mathbb{R}^{2k-2} \times [-\delta,\delta],$$

is a generic immersion. Thus, its self intersection M_f is a 3-manifold with boundary $M_f \cap \mathbb{R}^{2k-2} \times \partial[-\delta,\delta]$. Let $p\colon \mathbb{R}^{2k-2} \times [-\delta,\delta]$ denote the projection onto $[delta,\delta]$. Then $p|M_f$ has exactly one critical point. Let K be the component of M_f containing this critical point.

We write (using obvious notation) $\partial K = \partial K_{-\delta} \cup \partial K_\delta$. Then ∂K_δ is obtained from $\partial K_{-\delta}$ by a single Morse modification. Assume that the parameterization of f_t is such that

$$\chi(\partial K_{-\delta}) > \chi(\partial K_\delta),$$

where χ denotes the Euler characteristic.

Let $\widetilde{K} = f^{-1}(K)$. Then $f\colon \widetilde{K} \to K$ is a (not necessarily connected) double cover and if k is even then this cover is the orientation double cover ([2] Proposition 2.6). By counting the number of boundary components of ∂K and $\partial\widetilde{K}$ we get a subdivision of the codimension one part of the discriminant.

Definition 5.1.1. *Let $E_k \subset \Sigma_k$ contain all immersions with one elliptic self tangency point.*

Definition 5.1.2. *Let $Y_k(i,j)$ contain all immersions with a hyperbolic self tangency point such that under versal deformation the corresponding ∂K has i components and $\partial\widetilde{K}$ has j components.*

Remark 5.1.3. The possible values of i in Definition 5.1.2 are 2, 3 and 4. If $i = 2$ then j equals 2, 3 or 4. If $i = 3$ then j equals 3, 4 or 6.

We get the decomposition

$$\Sigma_k^1 = E_k \cup Y_k(2,2) \cup Y_k(2,3) \cup Y_k(2,4) \cup Y_k(3,3) \cup Y(3,4) \cup Y_k(3,6).$$

5.2. **Transformations of self intersection surfaces.** If $g > 0$, let T_g denote a connected sum of g tori and P_g a connected sum of g if projective planes. Let T_0 and P_0 denote 2-spheres.

We shall in several lemmas below explain how the self intersection of a generic immersion changes upon crossing the discriminant. The results are collected in Table 1

To see that r defined in Section 4 is invariant under regular homotopy we make extensive use of the following: Let $f_0\colon S^k \to \mathbb{R}^{2k-2}$ be an immersion in Σ_k^1 with one hyperbolic self tangency point. Making a small versal deformation of f_0 in the direction of decreasing Euler characteristic, we get a generic immersion f_δ and there is a new-born circle c in M_{f_δ}. See Figure 1.

Lemma 5.2.1.

$$\omega_{f_\delta}(c) = 0.$$

Proof. Use the coordinates from Proposition 2.3.1. The preimages of c are $c_1 = x_1^2 + x_2^2 = t, x_3 = 0, x_4 = 0$ and $c_2 = y_1^2 + y_2^2 = t, y_3 = 0, y_4 = 0$. Choose the framing

$$\left(\frac{1}{t}\left(x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}\right), \frac{\partial}{\partial x_3}, \frac{1}{t}\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_1}\right), \frac{\partial}{\partial x_4}\right)$$

Part of Σ^1_k	k	$\partial K_{-\delta}$	∂K_δ
E_k	all	T_0	$\emptyset$
$Y_k(2,2)$	even	T_g	T_{g+1}
	odd	P_g	P_{g+2}
$Y_k(2,3)$	even	T_g	T_{g+1}
	odd	T_g	P_{2g+2}
$Y_k(2,4)$	all	T_g	T_{g+1}
$Y_k(3,3)$	even	$T_g \cup T_d$	T_{g+d}
	odd	$P_g \cup P_d$	P_{g+d}
$Y_k(3,4)$	even	$T_g \cup T_d$	T_{g+d}
	odd	$T_g \cup P_d$	P_{2g+d}
$Y_k(3,6)$	all	$T_g \cup T_d$	T_{g+d}

TABLE 1. Transformation of self intersection surfaces on passing the discriminant.

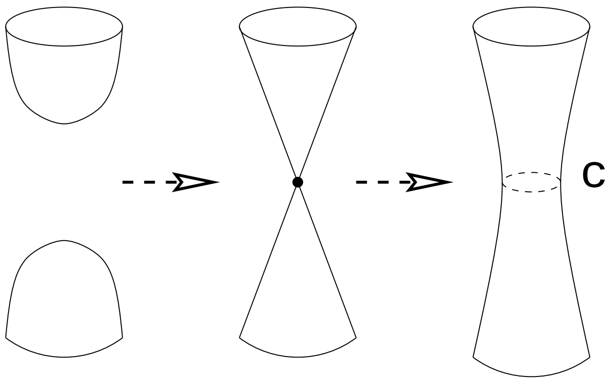

FIGURE 1. A new-born circle

of c_1 and the corresponding framing of c_2. Then

$$\langle \xi, [\vec{c}_1] \rangle = 1 = \langle \xi, [\vec{c}_2] \rangle,$$

so that $\langle \xi, [\vec{c}_S] \rangle = 0$. Moreover, since

$$\frac{\partial f_4}{\partial y} = [2y_1, 2y_2, -2y_3, 0],$$

the induced framing on c is, after orthonormalization,

$$\left(\frac{1}{t} \left(z_2 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_2} \right), \frac{\partial}{\partial z_3}, \frac{1}{t} \left(z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \right), \frac{\partial}{\partial z_4}, \frac{\partial}{\partial z_5}, \frac{\partial}{\partial z_6} \right).$$

Hence, $[c, r, X] = p_*(u)$. Therefore,

$$\omega_f(c) = \langle \xi, [\vec{c}_S] \rangle \cdot p_*(u) + [c, r, X] + p_*(u) = 0 + p_*(u) + p_*(u) = 0.$$

□

Lemma 5.2.2. *If we pass transversely through E_k then $\partial K_{-\delta} = T_0$ and $\partial K_\delta = \emptyset$. Moreover, $r(f_{-\delta}) = r(f_\delta)$.*

Proof. The statement about the topological type of ∂K is clear. For the second claim note that an S^2-component of M_f does not contribute to $r(f)$ in any of the cases. $\square$

We introduce some notation that will be used in all proofs below: The lemmas below treat versal deformations f_t of hyperbolic self tangency points. Such a versal deformation can be taken constant outside small neighborhoods of the preimages of the self tangency point. We will assume that this is the case in all proofs below.

With this assumption, there is a disk D in $K_{-\delta}$ such that $K_{-\delta} - D \times [-\delta, \delta] \subset K$. Such a disk D contains one point from each of the different sheets coming together at the self tangency. Also, $\partial D \times \delta$ bounds a new-born handle in ∂K_δ. It is either a torus with a hole or a Klein bottle with a hole. (The latter case appears only if k is odd.)

If $k = 4s+1$ and $f\colon S^k \to \mathbb{R}^{2k-2}$ is a generic immersion then $r(f) = \widetilde{q}_f(b)$ (see Section 4.2.1). We shall denote the classes corresponding to $f_{-\delta}$ and f_δ by b_δ and $b_{-\delta}$, respectively.

Lemma 5.2.3. *If we pass transversely through $Y_k(2,2)$ then*

(a) *if k is even, $\partial K_{-\delta} = T_g$ and $\partial K_\delta = T_{g+1}$, $g > 0$ or,*
(b) *if k is odd, $\partial K_{-\delta} = P_g$ and $\partial K_\delta = P_{g+2}$, $g > 0$.*

Moreover, $r(f_{-\delta}) = r(f_\delta)$.

Proof. Since both $\partial\widetilde{K}_{-\delta}$ and $\partial\widetilde{K}_\delta$ are connected the only possibilities are that ∂K_δ is obtained from $\partial K_{-\delta}$ by attaching a handle (which may be twisted if k is odd). If k is odd then $\partial K_{\pm\delta}$ must be non-orientable since they have connected preimages and the coverings in question are the orientation double coverings. If k is even there is an induced orientation on the self intersection. The statements about topological types follow. To prove that r does not change we have to treat each residue class of k separately:

Assume that $k = 4s+1$. Then there exists a collection L of embedded curves in $\widetilde{K}_{-\delta}$ such that it represents the class $b_{-\delta}$ and is disjoint from $f_{-\delta}^{-1}(D)$. Letting $L' = L \times \{\delta\}$. We have $\omega_{f_{-\delta}}(f_{-\delta}(L)) = \omega_{f_\delta}(f_\delta(L'))$.

If the added handle is a torus then L' represents b_δ and therefore $\widetilde{q}_{f_\delta}(b_\delta) = \widetilde{q}_{f_{-\delta}}(b_{-\delta})$.

If the added handle is a Klein bottle then $L' \cup c_1$ represents b_δ, where c_1 is one of the preimages of the newborn circle c. Since $f_\delta(L')$ and c does not intersect we can use formula (‡) in Section 3.3 and Lemma 5.2.1 to see that $\omega_{f_\delta}(L' \cup c) = \omega_{f_\delta}(L)$. Therefore, $\widetilde{q}_{f_\delta}(b_\delta) = \widetilde{q}_{f_{-\delta}}(b_{-\delta})$. Thus, in either case

$$r(f_{-\delta}) = q_{f_{-\delta}}(b_{-\delta}) = q_{f_\delta}(b_\delta) = r(f_\delta).$$

Assume that $k = 4s+2$. Then the new-born handle contains c with $q_f(c) = (0,0)$. The Arf invariant is

$$\operatorname{Arf}(q_f) = \sum_{a,b} q_f(a) q_f(b),$$

where $\{a, b\}$ ranges over a symplectic basis. Thus, the newborn handle does not contribute to the Arf invariant. Hence,

$$r(f_{-\delta}) = \operatorname{Arf}(q_{f_{-\delta}}) = \operatorname{Arf}(q_{f_\delta}) = r(f_\delta).$$

Assume that $k = 4s+3$. Since K_δ is obtained from $K_{-\delta}$ by adding a handle, $H_1(M_{f_\delta}; \mathbb{Z}_2) \cong H_1(M_{f_{-\delta}}; \mathbb{Z}_2) \oplus V$ as $\mathbb{Z}_4$-quadratic spaces, where V is 2-dimensional.

Let c be the newborn circle. Then $[c] \in V$. Since $\omega_f(c) = 0$, $q_f([c]) = 0$ and thus, $V = \mathcal{P}_+ \oplus \mathcal{P}_-$ or $V = \mathcal{T}_0$. Hence, the Witt class of $H_1(M_{f_\delta}; \mathbb{Z}_2)$ is the same as that of $H_1(M_{f_-\delta}; \mathbb{Z}_2)$. Thus,

$$r(f_{-\delta}) = \beta(q_{f_{-\delta}}) = \beta(q_{f_\delta}) = r(f_\delta).$$

□

Lemma 5.2.4. *If we pass transversely through $Y_k(2,3)$ then*

(a) *if k is even, $\partial K_{-\delta} = T_g$ and $\partial K_\delta = T_{g+1}$ or,*
(b) *if k is odd $\partial K_{-\delta} = T_g$ and $\partial K_\delta = P_{2g+2}$.*

Moreover, $r(f_{-\delta}) = r(f_\delta)$.

Proof. We note that $\partial \widetilde{K}_{-\delta}$ must be disconnected and $\partial \widetilde{K}_\delta$ connected. The surface ∂K_δ is obtained from $\partial K_{-\delta}$ by attaching a handle. The topological types follow if k is even. If k is odd then the coverings $\partial \widetilde{K}_{\pm\delta} \to \partial K_{\pm\delta}$ are the orientation double coverings. Hence, $\partial K_{-\delta}$ is orientable, ∂K_δ is non-orientable and the added handle is twisted. Thus, the statement about topological types follows also in this case.

The invariance of r: Assume that $k = 4s+1$. If L is a collection representing $b_{-\delta}$ then $L' \cup c_1$ represents b_δ. Assume that $k = 4s+2$. Then we add a handle with a $(0,0)$-curve in it. Assume that $k = 4s+3$. Then $H_1(\partial K_\delta; \mathbb{Z}_2) \cong H_1(\partial K_{-\delta}; \mathbb{Z}_2) \oplus \mathcal{P}_+ \oplus \mathcal{P}_-$, since we add a twisted handle with a newborn curve c such that $q_{f_\delta}([c]) = 0$.

As in the proof of Lemma 5.2.3 we conclude that $r(f_{-\delta}) = r(f_\delta)$. □

Lemma 5.2.5. *If we pass transversely through $Y_k(2,4)$ then $\partial K_{-\delta} = T_g$ and $\partial K_\delta = T_{g+1}$ Moreover, $r(f_{-\delta}) = r(f_\delta)$.*

Proof. The preimages $\partial \widetilde{K}_{\pm\delta}$ are disconnected. Hence, $\partial K_{\pm\delta}$ are orientable. The statement about topological types follows.

If $k = 4s+2$ or $k = 4s+3$ the invariance of r follows from the fact that $H_1(M_{f_{-\delta}}; \mathbb{Z}_2)$ and $H_1(M_{f_\delta}; \mathbb{Z}_2)$ are isomorphic as $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-quadratic spaces and $\mathbb{Z}_4$-quadratic spaces, respectively.

If $k = 4s+1$ then orientable components of M_f does not contribute to r. Hence, r does not change. □

Lemma 5.2.6. *If we pass transversely through $Y_k(3,3)$ then*

(a) *if k is even, $\partial K_{-\delta} = T_g \cup T_d$ and $\partial K_\delta = T_{g+d}$, $g, d > 0$, or,*
(b) *if k is odd $\partial K_{-\delta} = P_g \cup P_d$ and $\partial K_\delta = P_{g+d}$, $g, d > 0$, (g and d are even if $k = 4s+1$).*

Moreover, $r(f_{-\delta}) = r(f_\delta)$.

Proof. The preimages of each one of the components of $\partial K_{\pm\delta}$ are connected. The self intersection ∂K_δ is the connected sum of the two components of $\partial K_{-\delta}$. The statements about topological types follow.

Assume that $k = 4s+1$. If L represents $b_{-\delta}$ then L' represents b_δ. The invariance of r follows.

If $k = 4s+2$ or $k = 4s+3$ then the invariance of r follows exactly as in Lemma 5.2.3. □

Lemma 5.2.7. *If we pass transversely through $Y_k(3,4)$ then*

(a) *if k is even, $\partial K_{-\delta} = T_g \cup T_d$ and $\partial K_\delta = T_{g+d}$, $d > 0$, or,*

(b) *if k is odd $\partial K_{-\delta} = T_g \cup P_d$ and $\partial K_\delta = P_{2g+d}$, $d > 0$, (d is even if $k = 4s+1$). Moreover, $r(f_{-\delta}) = r(f_\delta)$.*

Proof. The preimage $\partial \widetilde{K}_\delta$ is connected. One component of $\partial K_{-\delta}$ has connected preimage, the other one disconnected preimage and ∂K_δ is the connected sum of these. The statements about topological types follow.

The invariance of r follows exactly as in the proof of Lemma 5.2.6. □

Lemma 5.2.8. *If we pass transversely through $Y_k(3,6)$ then $\partial K_{-\delta} = T_g \cup T_d$ and $\partial K_\delta = T_{g+d}$ Moreover, $R(f_{-\delta}) = R(f_\delta)$.*

Proof. The preimages $\partial \widetilde{K}_{\pm\delta}$ are disconnected and ∂K_δ is the connected sum of the two components of $\partial K_{-\delta}$. The statement about topological types follows.

If $k = 4s + 1$ then invariance of r follows as in the proof of Lemma 5.2.5.

If $k = 4s + 2$ or $k = 4s + 3$ then invariance of r follows as in the proof of Lemma 5.2.6. □

Remark 5.2.9. It follows from Lemmas 5.2.2- 5.2.8 that r is constant on path components of $\mathcal{F}_k$: If f and g belong to the same path component of $\mathcal{F}_k$ then there is a generic path connecting then. Such a path intersects Σ_k only transversally in Σ_k^1 and on such hits r does not change.

Remark 5.2.10. To show that r is invariant under regular homotopy when $k = 5$ we must also check that it is invariant when we pass the triple point stratum of the discriminant. This holds true since passing a triple point does not affect the functional ω_f at all.

6. Formulas for regular homotopy

6.1. **Homotopy groups.** After the work of Smale (see the Introduction) there is a 1-1 correspondence between the groups $\pi_0(\mathcal{F}_k)$ and $\pi_k(V_{2k-2,k})$. The latter groups have been computed. See [7]. If $k \geq 4$ then they depend only on $k \mod 4$, as follows:

Theorem (Paechter).

$$\pi_k(V_{2k-2,k}) = \begin{cases} 0 & \text{if } k \equiv 0 \mod 4, \\ \mathbb{Z}_2 & \text{if } k \equiv 1 \mod 4, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } k \equiv 2 \mod 4, \\ \mathbb{Z}_8 & \text{if } k \equiv 3 \mod 4. \end{cases}$$

□

6.2. **Proof of Theorem 1.** From Smale's classification of sphere immersions and the above theorem, the case $k = 4$ is trivial.

In the case $k \geq 5$ we have a function $r \colon \mathcal{F}_k \to I_k$. In Section 5.2 it was shown to be constant on the path components of $\mathcal{F}_k$. Therefore, it induces functions, also denoted r, from $\pi_0(\mathcal{F}_k)$ to I_k, where I_k is $\mathbb{Z}_2$ if $k = 4s + 1$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ if $k = 4s + 2$ and $\mathbb{Z}_8$ if $k = 4s + 3$. We must show that these are bijections.

Smale's classification of immersions and the above theorem implies that $\pi_0(\mathcal{F}_k)$ and I_k have the same finite number of elements. Therefore, it is enough to show that r is surjective.

If $k \geq 6$ and $k = 4s + 1$, $k = 4s + 2$ or $k = 4s + 3$ then, using the methods in Examples 8.4, 8.6 and 8.7 in [2], a generic immersion $f\colon S^k \to \mathbb{R}^{2k-2}$, with self intersection a Klein bottle, a torus or a projective plane, respectively and with $r(f)$ equal to a generator of I_k can be constructed. Taking connected sum of such immersions we see that for any $a \in I_k$ there is a generic immersion g in $\mathcal{F}_k$ such that $r(g) = a$.

Thus, r is bijective if $k \geq 6$.

For $k = 5$ we have to construct an immersion f with $r(f) = 1$. This can be done as follows: Consider $U = T^2 \times D^3 \subset S^5$. The standard embedding $s\colon S^5 \to \mathbb{R}^6 \subset \mathbb{R}^8$ gives after restriction an embedding $s\colon \partial U \to \mathbb{R}^8$.

Using a tubular neighborhood of an embedded Klein bottle in $\mathbb{R}^8$, we can construct an immersion $f\colon U \to \mathbb{R}^8$ with self intersection a Klein bottle K. If necessary, we can twist the map in the normal bundle along the circle in T^2 which maps to an orientation preserving circle in K to obtain that the Arf-invariant of the spin structure induced on $T^2 = \widetilde{K}$ is 1. This immersion f is an embedding on ∂U, which is a 4-manifold.

Composing the standard embedding with a translation in $\mathbb{R}^8$, we may assume that it is disjoint from $f(U)$. Using a result of Hudson, Theorem 3.1 in [3], we conclude that $f|\partial U$ and $s|\partial U$ are PL isotopic embeddings since the codimension is larger that 3 and all homotopy obstructions vanish.

By Haeffliger's theorem (Theorem 3.5 in [2]), they are also diffeotopic since $2 \cdot 8 > 3 \cdot 5$. We can thus map $U' = T^2 \times 2D^3$ into $\mathbb{R}^8$ in such a way that when restricted to $\partial U'$ the map agrees with s. Filling it in we get an immersion $f\colon S^5 \to \mathbb{R}^8$ with self intersections. However, the only self intersection component of f with connected preimage is K since all other preimages are separated by ∂U or $\partial U'$. It follows that $r(f) = 1$. Thus, r is bijective also if $k = 5$. $\square$

6.3. **Path components of $\mathcal{F}_k - \Sigma_k$, $k \geq 8$ and Theorem 1.** If $k \geq 8$ we know the path components of $\mathcal{F}_k - \Sigma_k$ and can therefore give a proof of Theorem 1 which is independent of Smale's work.

Let $f\colon S^k \to \mathbb{R}^{2k-2}$ be a generic immersion. Let F be a component of M_f. To describe the path components of $\mathcal{F}_k - \Sigma_k$ we shall associate an invariant σ to each such component F:

- (a) If $\widetilde{F} = f^{-1}(F)$ is disconnected then there is an induced spin structure on F. Let $\sigma(F) \in \mathbb{Z}_2$ be the corresponding Arf invariant.
- (b) If $\widetilde{F}$ is connected
  - (b0) and $k = 4s$. Let $\sigma(F) = 0$.
  - (b1) and $k = 4s+1$ then F is non-orientable and of even genus. Let $\sigma(F) \in \mathbb{Z}_2$ be $\widetilde{q}_f(b_F)$, where b_F is the projection of b on $H_1(\widetilde{F}; \mathbb{Z}_2)$.
  - (b2) and $k = 4s + 2$ then there are two spin structures induced on F. Let $\sigma(F) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$ be the corresponding Arf invariant.
  - (b3) and $k = 4s+3$ then there is an induced pin structure on F. Let $\sigma(F) \in \mathbb{Z}_8$ be the corresponding Brown invariant.

Let $k \geq 8$ and $f, g \in \mathcal{F}_k - \Sigma_k$.

Proposition 6.3.1. *f and g belongs to the same path component of $\mathcal{F}_k - \Sigma_k$ if and only if there is a 1-1 correspondence between the path components F of M_f and G of M_g such that if F corresponds to G then F and G are diffeomorphic and $\sigma(F) = \sigma(G)$.* $\square$

Proposition 6.3.1 follows from Theorem 1.4 in [2]. We shall use it below and in the proof of Corollary 4.

Remark 6.3.2. Using the above Proposition 6.3.1 and the moves described in the Lemmas in Section 5.2 we can reduce the self intersection manifold of any generic immersion to the following form:

(a) The empty set if $k = 4s$.
(b) The empty set or a Klein bottle F with $\sigma(F) = 1$ if $k = 4s + 1$.
(c) The empty set or a torus F (with $\sigma(F)$ any nonzero value in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$), if $k = 4s + 2$.
(d) The empty set or a collection of at most four projective planes with the same Brown invariant (± 1), if $k = 4s + 3$.

Applying Proposition 6.3.1 again we get Theorem 1.

Remark 6.3.3. The above reduction of the self intersection is not always straightforward. For example, if $k = 4s$ and $f\colon S^k \to \mathbb{R}^{2k-2}$ is a generic immersion with self intersection a torus F, with the spin structure that has Arf invariant equal to 1, then to connect f to an embedding with a generic path we must first add a handle to F in such a way that we get a non-trivially covered genus two surface. (We pass through $Y_k(2,3)$.) Then contract a trivially covered circle (pass through $Y_k(2,2)$ in the negative direction) and obtain a non-trivially covered torus. This torus can now be transformed into a sphere by passing $Y_k(2,3)$ in the negative direction. Finally we can kill this sphere by passing through E_k and we get an embedding.

Similar remarks applies in the other cases.

7. Coorientation

7.1. **Hypersurfaces in functional spaces.** The notion *functional space* will be used to denote an open subset of a space of r times differentiable maps, from one smooth manifold into another with the C^r topology, which has the subspace topology. Thus, a functional space is in most cases an infinite dimensional smooth manifold.

The spaces $\mathcal{F}_k$ are our main examples of functional spaces.

A *hypersurface* of a functional space $\mathcal{F}$ is a union of smooth submanifolds $M^1 \cup M^2 \cup \dots \subset \mathcal{F}$. Where M^j has codimension j in the functional space and M^j is contained in the closure of M^1. The M^j are called the *strata* of the stratified subspace.

7.2. **Consistent and continuous coorientation.** Let $\mathcal{F}$ be a functional space and Δ a hypersurface with top-dimensional stratum Δ^1 of codimension one. Since Δ^1 has codimension one it locally separates $\mathcal{F}$.

A *coorientation* of Δ is a coherent choice of a positive and a negative part of small neighborhoods of the points in Δ^1, which are cut into two parts by Δ^1.

A *consistent coorientation* is a coorientation such that the intersection number of any small generic loop and Δ^1 vanishes.

A *continuous coorientaion* is a consistent coorientation that extends continuously over the codimension two stratum of Δ.

For a more thorough description of these concepts we refer to [3], Section 7.

7.3. **Continuous coorientation of the discriminant.** Let $k \geq 6$. Let $f_0 \in \Sigma_k^1$ and let f_t, $-\delta \leq t \leq \delta$ be a versal deformation of f_0. We say that f_δ lies on the positive side of Σ_k^1 at f_0 if the Euler characteristic of the self intersection of f_δ is smaller than the Euler characteristic of the self intersection of $f_{-\delta}$.

Proposition 7.3.1. *The coorientation of Σ_k defined above is continuous.*

Proof. This coorientation is consistent since the intersection number of Σ_k^1 and a generic path is half of the difference of the Euler characteristics of the self intersections at the starting point and at the endpoint. This is zero if the path is a loop.

That this coorientation is extendible follows from the fact that the change of the Euler characteristic of the self intersection can be calculated from the local picture near a self tangency point. □

8. Vassiliev invariants of generic immersions

We assume throughout the rest of the paper that $\dim S^k = k \geq 6$, so that we do not have to consider triple points.

8.1. **Jumps of higher order.** Let G be an Abelian group. Let a be a G-invariant of a generic immersion, i.e. $a \in H^0(\mathcal{F}_k - \Sigma_k; G)$. Coorient Σ_k as in Proposition 7.3.1.

Let $f_0 \in \Sigma_k^1$ and let f_t be a versal deformation of f_0 intersecting Σ_k^1 in the positive direction.

Definition 8.1.1. *Define* the jump of a *as*

$$\nabla a(f_0) = a(f_\delta) - a(f_{-\delta}),$$

for a small $\delta > 0$.

Remark 8.1.2. The function ∇a is a locally constant G-valued function on Σ_k^1 and therefore gives an element in $H^0(\Sigma_k^1; G)$.

We can iterate this construction: Let $f_{0,\dots,0}$ (m zeroes) denote an immersion in Σ_k^m. Then $f_{0,\dots,0}$ has m distinct self tangency points.

Let $\Lambda\colon [-1,1]^m \to \mathcal{F}$ be a versal deformation of $f_{0,\dots,0}$ such that $\Lambda(t_1,\dots,t_m) \in \Sigma_k^l$ if l of the $t_i = 0$ and the other ones are different from 0. Let $f_{t_1,t_2,\dots,t_m} = \Lambda(t_1,\dots,t_m)$. Fix nonzero values of t_i, $i \neq j$ then the intersection number of the path $f_{t_1,\dots,t_m}$, traced out as t_j varies, and Σ_k is independent of these fixed values. This is a consequence of the continuity of the coorientation of Σ_k.

Assume that the versal deformation is chosen so that this intersection number is $+1$ for all t_j.

Definition 8.1.3. *Assume that $\nabla^{m-1} a \in H^0(\Sigma_k^{m-1}; G)$, the $(m-1)^{th}$ jump of a, has been defined. Define* the m^{th} jump *of a, $\nabla^m a \in H^0(\Sigma_k^m; G)$ by*

$$\nabla^m a(f_{0,\dots,0}) = \nabla^{m-1} a(f_{\delta,0\dots,0}) - \nabla^{m-1} a(f_{-\delta,0\dots,0}),$$

for a small $\delta > 0$.

Remark 8.1.4. It is easy to see that this function is locally constant on Σ_k^m and hence gives an element in $H^0(\Sigma_k^m; G)$.

Since the coorientation is continuous, we have

$$\nabla^{m-1} a(f_{\delta,0\dots,0}) - \nabla^{m-1} a(f_{-\delta,0\dots,0}) = \nabla^{m-1} a(f_{0,\dots,\delta,\dots,0}) - \nabla^{m-1} a(f_{0\dots,-\delta,\dots,0}),$$

where δ is at any of the other $m-1$ positions. That is, it does not matter which self tangency we resolve.

8.2. **Vassiliev invariants.** We define Vassiliev invariants as follows:

Definition 8.2.1. *A G-invariant is a Vassiliev invariant of order m if $\nabla^{m+1}a \equiv 0$.*

Remark 8.2.2. Using this definition of finite order invariants, the only parts of the strata of the discriminant of higher codimension that give any input to our theory are those consisting of immersions with several distinct self tangency points. Taking also other parts into consideration would add conditions on finite order invariants and could possibly lead to (even) fewer Vassiliev invariants. However, it is likely that all the invariants presented below satisfies also these stronger conditions.

We let V_m denote the group of Vassiliev invariants of order m. The cohomology group $H^0(\mathcal{F}_k - \Sigma_k)$ splits as sum over path components of $\mathcal{F}_k$. We let V'_m denote the projection of V_m onto a summand of $H^0(\mathcal{F}_k - \Sigma_k)$ corresponding to such a path component.

8.3. **Simple m-immersions.**

Definition 8.3.1. *An m-immersion is* simple *if all of its self tangency points are either elliptic, or hyperbolic such that under a versal deformation all self intersection components involved in the Morse modification are spheres.*

Let f be an m-immersion.

Lemma 8.3.2. *There is a path $\lambda\colon I \to \Sigma_k^m \cup \Sigma_k^{m+1}$ intersecting Σ_k^{m+1} transversally in finitely many points such that $\lambda(0) = f$ and $\lambda(1)$ is simple.*

Proof. Introduce two hyperbolic self tangencies close to the hyperbolic self tangency points of f, as in Figure 2. (That this is possible can be seen from the local model in Proposition 2.2.1). □

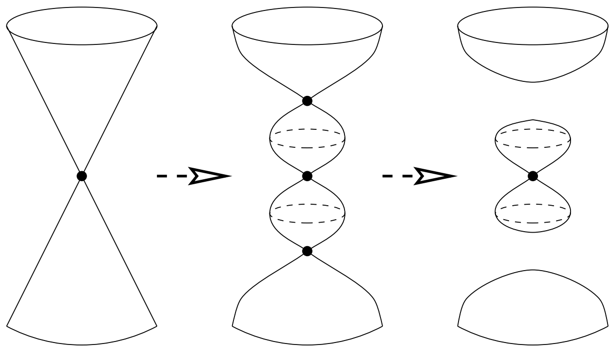

FIGURE 2. Introducing a hyperbolic self tangency

We write the non-smooth part of the self intersection of a simple m-immersion as $\{x_1, \dots, x_j\}$, where $x_i \in \{e, h\}$ for $i = 1, \dots, j$, where e denotes an elliptic self tangency point and h a hyperbolic one.

8.4. **Properties of Vassiliev invariants.**

Lemma 8.4.1. *A Vassiliev invariant v of order m is uniquely determined, up to addition of lower order Vassiliev invariants, by its m^{th} jump $\nabla^m v$.* $\square$

Lemma 8.4.2. *If v is a Vassiliev invariant of order m then $\nabla^m v$ is constant.*

Proof. By lemma 8.3.2, we need only prove this for simple m-immersions. Since the smooth part of the self intersection can be reduced to standard form by passing Σ_k^{m+1} transversally and $\nabla^m v$ does not jump on such hits we can assume that the smooth part of the self intersection is fixed. Call it X. Then, if we label a simple m-immersion by its self intersection, we have:

$$\begin{aligned}
&\nabla^m v(\{h,\dots,h,X\}) = \\
&\nabla^{m-1} v(\{S^2,h,\dots,h,X\}) - \nabla^{m-1} v(\{S^2,S^2,h,\dots,h,X\}) = \\
&\nabla^{m-1} v(\{S^2,h,\dots,h,X\}) + \nabla^m v(\{S^2,e,h,\dots,h,X\}) \\
&- \nabla^{m-1} v(\{S^2,h,\dots,h,X\}) = \\
&\nabla^m v(\{S^2,e,h,\dots,h,X\}) = \nabla^m v(\{e,h,\dots,h,X\}).
\end{aligned}$$

Similarly it follows that

$$\begin{aligned}
&\nabla^m v(\{e,h,\dots,h,X\}) = \nabla^m v(\{e,e,\dots,h,X\}) = \dots \\
&= \nabla^m v(\{e,e,\dots,e,X\}).
\end{aligned}$$

$\square$

Let χ denote the Euler characteristic.

Corollary 8.4.3. *$V_1'/V_0' = G$ and is generated by*

$$v_1(f) = (-1)^{\kappa(\chi(M_f))+1}\left[\frac{|\chi(M_f)|}{2}\right] \in \mathbb{Z},$$

if $G = \mathbb{Z}$. (See Theorem 3 for notation.)

Proof. Clearly, $\nabla v_1 \equiv 1$. So, v_1 is non-trivial and by Lemma 8.4.2 it generates V_1'/V_0'. If $G \neq \mathbb{Z}$, we consider $v_1^g = v_1 \cdot g$, for $g \in G$. $\square$

8.5. **Proof of Theorem 2.** Assume inductively that, for $p < m$, p^{th} order Vassiliev invariants v_p have been constructed and that they have the following properties (which clearly holds for v_0 and v_1):

P1. $\nabla^1 v_p(f_0) = v_{p-1}(f_{-\delta})$.

P2. If $\lambda\colon I \to \mathcal{F}_k$ is any generic path such that $(\lambda \cdot \Sigma_k^1) = 0$ then $v_p(\lambda(0)) = v_p(\lambda(1))$, where $\lambda \cdot \Sigma_k^1$ is the intersection number computed using the coorientation of Σ_k and the orientation of λ.

Define $\alpha \in H^0(\Sigma_k^1; G)$ by

$$\alpha(f_0) = v_{m-1}(f_{-\delta}).$$

Let $\lambda\colon I \to \mathcal{F}_k$ be a generic path with $(\lambda \cdot \Sigma_k^1) = 0$ intersecting Σ_k^1 at $t_1,\dots,t_n$. We claim that

$$\sum_{i=1}^{n} (\lambda \cdot \Sigma_k^1)_{\lambda(t_i)} \alpha(\lambda(t_i)) = 0.$$

To see this let t_j be the first point where $(\lambda\cdot\Sigma_k^1)_{\lambda(t_j)} \neq (\lambda\cdot\Sigma_k^1)_{\lambda(t_1)}$. Then

$$\begin{aligned}\alpha(\lambda(t_{j-1})) &= v_{m-1}(\lambda(t_{j-1}+\epsilon\delta)),\\ \alpha(\lambda(t_j)) &= v_{m-1}(\lambda(t_j-\epsilon\delta)),\end{aligned}$$

where $\epsilon = \pm 1$. The intersection number of $\lambda|[t_{j-1}+\epsilon\delta, t_j-\epsilon\delta]$ and Σ_k^1 is zero. Hence, $\alpha(\lambda(t_{j-1})) = \alpha(\lambda(t_j))$. We proceed in a similar way.

If $j=2$ then repeat the argument starting the path from $t_2+\delta$. Otherwise, if $(\lambda\cdot\Sigma_k^1)_{\lambda(t_{j+1})} \neq (\lambda\cdot\Sigma)_{\lambda(t_1)}$ then the same argument shows that $\alpha(\lambda(t_{j+1})) = \alpha(\lambda(t_{j-1}))$ and if $(\lambda\cdot\Sigma_k^1)_{\lambda(t_{j+1})} = (\lambda\cdot\Sigma)_{\lambda(t_1)}$ then find the first point t_r with $(\lambda\cdot\Sigma_k^1)_{\lambda(t_r)} \neq (\lambda\cdot\Sigma)_{\lambda(t_1)}$. As above we see that $\alpha(\lambda(t_r)) = \alpha(\lambda(t_{r-1}))$.

Thus, the even number of intersections can be collected in pairs (t_i,t_j) with $(\lambda\cdot\Sigma_k^1)_{\lambda(t_i)} \neq (\lambda\cdot\Sigma_k^1)_{\lambda(t_j)}$ and $\alpha(\lambda(t_i)) = \alpha(\lambda(t_j))$. This proves the claim.

Since the intersection number of any generic loop and Σ_k^1 is zero, Proposition 8.2.1 in [3] implies that we can find an invariant v_m such that $\nabla v_m = \alpha$. Then v_m has properties **P1** and **P2**.

We must compute the jumps of v_m. By definition, $\nabla^1 v_m(f_0) = v_{m-1}(f_{-\delta})$. Hence,

$$\begin{aligned}\nabla^2 v_m(f_{0,0}) &= \nabla^1 v_m(f_{\delta,0}) - \nabla^1 v_m(f_{-\delta,0})\\ &= v_{m-1}(f_{\delta,-\delta}) - v_{m-1}(f_{-\delta,-\delta})\\ &= \nabla^1 v_{m-1}(f_{0,-\delta}) = v_{m-2}(f_{-\delta,-\delta}).\end{aligned}$$

Iterating this argument we get

$$\nabla^m v_m(f_{0,\dots,0}) = v_0(f_{-\delta,\dots,-\delta}) = 1, \quad \nabla^{m+1} v_m \equiv 0.$$

Thus, v_m is a m^{th} order Vassiliev invariant satisfying **P1** and **P2**.

The theorem follows for $\mathbb{Z}$-invariants.

If G is any Abelian group and $g\in G$, let $v_m^g = v_m\cdot g$. Then v_m^g is an m^{th} order G-invariant such that $\nabla^m v_m^g \equiv g$. This proves the general case. □

9. Formulas and non-separation properties

9.1. **Vassiliev invariants and the Euler characteristic of the self intersection.** The results in this section are formulated for $\mathbb{Z}$-invariants. The end of the proof of Theorem 2 shows how to derive similar results for other Abelian coefficient groups.

Let f be a generic immersion in $\mathcal{F}_k$. Let v_m be the m^{th} order integer Vassiliev invariant constructed above.

Lemma 9.1.1. *v_m depends only on $\chi(M_f)$, the Euler characteristic of the self intersection of f.*

Proof. This is true for v_1. Assume that it is true for v_k, $k<m$. Consider the invariant $v^m(f) = (v_1(f))^m$. We compute

$$\begin{aligned}\nabla v^n(f_0) &= v^n(f_\delta) - v^n(f_{-\delta})\\ &= (v_1(f_{-\delta}) + \nabla v_1(f_0))^m - (v_1(f_{-\delta}))^m\\ &= mv^{m-1}(f_{-\delta}) + \tbinom{m}{2} v^{m-2}(f_{-\delta}) + \dots + nv^1(f_{-\delta}) + 1,\end{aligned}$$

and

$$\begin{aligned}
\nabla^2 v^n(f_{0,0}) &= \nabla v^n(f_{\delta,0}) - \nabla v^n(f_{-\delta,0}) \\
&= m v^{m-1}(f_{\delta,-\delta}) + \binom{m}{2} v^{m-2}(f_{\delta,-\delta}) + \cdots + n v^1(f_{\delta,-\delta}) + 1 \\
&- m v^{m-1}(f_{-\delta,-\delta}) - \binom{m}{2} v^{m-2}(f_{-\delta,-\delta}) - \cdots - n v^1(f_{-\delta,-\delta}) - 1 \\
&= m\nabla v^{m-1}(f_{0,-\delta}) + \binom{m}{2} \nabla v^{m-2}(f_{0,-\delta}) + \cdots + n\nabla v^1(f_{0,-\delta}) \\
&= m(m-1)v^{m-2}(f_{-\delta,-\delta}) + \binom{m}{2}(m-2)v^{m-3}(f_{-\delta,-\delta}) + \cdots + n.
\end{aligned}$$

After m steps we arrive at

$$\nabla^m v^m(f_{0,\ldots,0}) = m!\nabla v(f_{0,-\delta,\ldots,-\delta}) = m!\,.$$

Now, $\nabla^m v_m = 1$ so $m!v_m - v^m$ is a Vassiliev invariant of degree at most $m-1$. By our inductive assumption, it depends only on $\chi(M_f)$. This is also true for v^m, thus for $m!v_m$ and hence for v_m itself. □

Let U be a path component of $\mathcal{F}_k$. Normalize v_m so that it takes the value zero on the generic immersions in U with minimal absolute value of the Euler characteristic of the self intersection.

9.2. **Proof of Theorem 3.** Using Lemma 9.1.1 we see that $v_m(f)$ is determined by $v_1(f)$. We therefore write $v_m(f) = v_m(l)$, where $l = v_1(f)$. As in the formulation of the theorem we let $\kappa(n) = 0$ if $n \geq 0$ and $\kappa(n) = 1$ if $n < 0$ for $n \in \mathbb{Z}$. In this notation, the recursion formula for v_m is

$$v_m(l) = v_m(l + (-1)^{\kappa(l)+1}) + (-1)^{\kappa(l)} v_{m-1}(l - \kappa(l)), \quad v_m(0) = 0.$$

Hence,

$$v_m(l) = (-1)^{\kappa(l)} \sum_{j=1}^{|l|-1+\kappa(l)} v_{m-1}((-1)^{\kappa(l)} j).$$

Assume inductively that

$$v_{m-1}(j) = (-1)^{(m-1)\kappa(l)} \binom{|l|+\kappa(l)(m-2)}{m-1}.$$

Then the right hand side is the coefficient of x^m in the polynomial

$$\begin{aligned}
&x(-1)^{\kappa(l)}(-1)^{\kappa(l)(m-1)}\left((1+x)^{\kappa(l)(m-2)} + \cdots + (1+x)^{\kappa(l)(m-2)+|l|-1+\kappa(l)}\right) \\
&= (-1)^{\kappa(l)m}\left((1+x)^{\kappa(l)(m-1)+|l|} - (1+x)^{\kappa(l)(m-2)}\right),
\end{aligned}$$

which is $\binom{|l|+\kappa(l)(m-1)}{m}$ □

9.3. **Proof of Corollary 4.** Using the methods of Section 8 in [2] and connected sum we can construct generic immersions with for example n tori, m of which has the trivial spin structure and $n-m$ of which has the nontrivial one. The Corollary now follows from Proposition 6.3.1 and Theorem 3. □

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