KNOTS AND BERNOULLI NUMBERS

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ABSTRACT. Regular homotopy classes of immersions $S^{4j-1} \to \mathbb{R}^{4j+1}$ constitute an infinite cyclic group. The classes containing embeddings form a subgroup. We prove that the index of this subgroup equals the denominator of $B_j/4j$, where B_j is the j^{th} Bernoulli number, and that the signature of a Seifert-surface of an embedding determines its regular homotopy class.

1. INTRODUCTION

Immersions and embeddings form open subspaces of the space of C^r-maps $S^n \to \mathbb{R}^{n+k}$, $r \geq 1$. Smale [9] showed that the path components of the space of immersions (or, which is the same, the regular homotopy classes of immersions) $S^n \to \mathbb{R}^{n+k}$ are in one to one correspondence with the elements of $\pi_n(V_{n+k,n})$, the n^{th} homotopy group of the Steifel manifold of n-frames in $(n+k)$-space. He also suggested the following question ([9], p.329): *"What regular homotopy classes have an embedding for representative?"*.

In [2] Smale's question was answered in the case of immersions $S^3 \to \mathbb{R}^5$. This is the first in the sequence cases $S^{4j-1} \to \mathbb{R}^{4j+1}$, which is the subject of this paper. Partial answers to Smale's question will be given: Theorem 1 below tells us *how many* regular homotopy classes have an embedding for representative.

An answer to Smale's question gives information about how the inclusion of the space of embeddings into the space of immersions is organized. The sequence of cases studied in this paper contains the most interesting ones from the point of view of knot theory.

Knot theory is the study of path components of the space of embeddings. In codimension two (i.e. for $S^n \to \mathbb{R}^{n+2}$), where embeddings share many features of classical knots ($S^1 \to \mathbb{R}^3$), this subject is richer than in other dimensions.

For $n = 1$ (i.e. in the case $S^1 \to \mathbb{R}^3$), the group $\pi_1(V_{3,1})$ enumerating regular homotopy classes is trivial and all immersions are regularly homotopic. For $n = 2$ (i.e. in the case $S^2 \to \mathbb{R}^4$), $\pi_2(V_{4,2}) = \mathbb{Z}$, the regular homotopy class of an immersion is determined by the algebraic number of self intersection points and thus, only one class contains embeddings. For $k \geq 3$, $\pi_n(V_{n+2,n}) \cong \pi_n(SO(n+2)) \cong \pi_n(SO(\infty))$, where $SO(m)$ is the group of orthonormal $m \times m$-matrices of determinant one. By Bott-periodicity, $\pi_n(SO(\infty))$ is isomorphic to $\mathbb{Z}_2$ if $n = 8i$ or $n = 8i + 1$, it is isomorphic to $\mathbb{Z}$ if $n = 4j - 1$, and is 0 in the other cases. Thus, unless $n = 4j - 1$, the space of immersions $S^n \to \mathbb{R}^{n+2}$ has either one or two components (in the latter case Smale's problem is open). We restrict attention to the cases $S^{4j-1} \to \mathbb{R}^{4j+1}$, where the spaces of immersions have infinitely many components.

Theorem 1. *Let m_k be the denominator of $\frac{B_k}{4k}$, where B_k is the k^{th} Bernoulli number (see Section 3). Exactly every $m_j{}^{\text{th}}$ regular homotopy class of immersions $S^{4j-1} \to \mathbb{R}^{4j+1}$ has an embedding for representative.*

Theorem 1 is proved in Section 5.

The key step in proving Theorem 1 is to relate the Smale invariant of embeddings to another familiar invariant which turns out to classify embeddings up to regular homotopy:

Recall that for any embedding $f\colon S^n \to \mathbb{R}^{n+2}$ there exists a compact oriented $(n+1)$-manifold V such that $\partial V = f(S^n)$. We call such a manifold a *Seifert-surface* of f. Let W_f denote the closed manifold obtained from V by gluing a $(n+1)$-disk to V along ∂V.

If M^{4k} is a closed oriented $4k$-manifold then by Poincaré duality the symmetric bilinear form

$$\beta\colon H^{2k}(M;\mathbb{Q}) \otimes H^{2k}(M;\mathbb{Q}) \to \mathbb{Q}, \quad \beta(a,b) = \langle a \cup b, [M]\rangle,$$

is non-degenerate. (Here $\cup$ denotes the cup product and $[M] \in H_{4k}(M;\mathbb{Q})$ denotes the orientation class.) The *signature of* M $\sigma(M)$ is the number of positive eigenvalues of β minus the number of negative eigenvalues of β.

Theorem 2. *Let $f, g\colon S^{4j-1} \to \mathbb{R}^{4j+1}$ be two embeddings. Then f is regularly homotopic to g if and only if $\sigma(W_f^{4j}) = \sigma(W_g^{4j})$.*

Theorem 2 is proved in Section 4.

2. The Smale invariant and the immersion group

In this section we study the set **Imm** of regular homotopy classes of immersions. First we recall the Smale invariant which allows us to define a group structure in **Imm**. Then we identify the group operations geometrically. This identification allows us to regard the regular homotopy classes which has an embedding for representative as a subgroup **Emb** of **Imm**.

2.1. Definition of the Smale invariant.

We recall the definition of the Smale invariant of an immersion $f\colon S^n \to \mathbb{R}^{n+k}$:

Consider S^n as the unit sphere in $\mathbb{R}^{n+1}$ and let $s\colon S^n \to \mathbb{R}^{n+1} \subset \mathbb{R}^{n+2}$, denote the standard embedding. Fix a disk $D^n \subset S^n$ containing the south pole and a framing X of $S^n - D^n$. Using regular homotopy, deform f so that $f|D^n = s|D^n$.

Choose a diffeomorphism $h\colon H^n_+ \to S^n - D^n$ of degree $+1$, where H^n_+ is the hemisphere $\{x_0 \geq 0\}$ in the unit sphere $\{x_0^2 + \dots + x_n^2 = 1\}$ in $\mathbb{R}^{n+1}$. Let $r\colon S^n \to S^n$ be the map $(x_0, \dots, x_n) \mapsto (-x_0, \dots, x_n)$ of the unit sphere in $\mathbb{R}^{n+1}$. Define $\phi^f_s\colon S^n \to V_{n,n+k}$,

$$\phi^f_s(x) = \begin{cases} df(X(h(x))) & \text{for } x \in H^n_+, \\ ds(X(h(r(x)))) & \text{for } x \in H^n_-, \end{cases}$$

where H^n_- is the hemisphere $\{x_0 \leq 0\}$.

Definition 2.1. *The Smale invariant $\Omega(f)$ of f is*

$$\Omega(f) = [\phi^f_s] \in \pi_n(V_{n+k,n}).$$

Smale, [9] showed that Ω gives a bijection between the regular homotopy classes of immersions $S^n \to \mathbb{R}^{n+k}$ and the elements of $\pi_n(V_{n+k,n})$.

2.2. **The Smale invariant as an obstruction.** As in [9], Theorem E, we can interpret the Smale invariant as an obstruction to extending an immersion $f\colon S^n \to \mathbb{R}^{n+2}$ to the disk B^{n+1}:

By the Smale-Hirsch h-principle (see [4]), the space of maps of the principal bundle of TB^{n+1} into the bundle with fiber $V_{n+2,n+1}$ associated to the tangent bundle of $T\mathbb{R}^{n+2}$ is homotopy equivalent to the space of immersions of B^{n+1} into $\mathbb{R}^{n+2}$.

Let $f\colon S^n \to \mathbb{R}^{n+2}$, $n \geq 3$ be an immersion. By standard obstruction theory, there exists a normal vector field ν of f which is unique up to homotopy. Using this vector field we extend f to an immersion F of $S^n \times I$, where I is a small interval.

Consider $S^n \times I \subset B^{n+1}$ as a collar on $S^n \subset B^{n+1}$. The obstruction to extend F is then the homotopy class $[\Phi^F]$ of the map $\Phi^F : S^n \to V_{n+2,n+1}$, where

$$\Phi^F(x) = dF(Y(x)).$$

Lemma 2.2. *The homotopy class $[\Phi^F]$ is independent of the framing of TB^{n+1} and the normal vector field chosen. Under the natural isomorphism $\pi_n(V_{n+2,n+1}) \to \pi_n(V_{n+2,n})$ induced from the fibration $SO(2) \to V_{n+2,n+1} \to V_{n+2,n}$, $[\Phi^F]$ corresponds to the Smale invariant $\Omega(f)$ of f.*

Proof. All normal vector fields are homotopic and an homotopy between normal vector fields induces an homotopy of the corresponding Φ^F's. Hence, the choice of normal vector field does not affect $[\Phi^F]$.

Let Y and Z be two framings of TB^{n+1}. Interpreting the vectors in the frames as column vectors of a matrix we get a map $Y^{-1}Z\colon B^{n+1} \to SO(n+1)$. Its restriction to the boundary is null-homotopic. If Φ^F_Y and Φ^F_Z denote the maps obtained by using the framings Y and Z, respectively then

$$\Phi^F_Y(x) = \Phi^F_Z(x) \cdot (Y^{-1}Z)(x),$$

where $\cdot$ denotes the matrix acting on the framing. This proves that the choice of framing does not affect $[\Phi^F]$.

To relate $[\Phi^F]$ to the Smale invariant we use a special framing. Let D^n in the definition of the Smale invariant be the lower hemisphere H^n_- and let $X(x)$ denote the framing of the upper hemisphere H^n_+ used in the same definition.

We construct a framing Y of TB^{n+1} which we use to compute Φ^F: Let $Y(x) = (X(x), x)$ for $x \in S^n - H^n_+$, where we interpret x in the frame as the unit outer normal to the sphere. Extend this farming to all of B^{n+1} along vertical lines.

After regular homotopy, we may assume that the immersion f agrees with the standard embedding s on a disk slightly larger than H^n_-. Choose a normal field to f which agrees with the inward normal of s along the disk containing H^n_-. Using s we can extend the immersion F to an immersion of a disk E^n containing the set $x_0 \leq 0$. The map Φ^F is then homotopic to a map which is $d\Phi(X(x), x)$ on H^n_+ and $ds(r(x), X(r(x)))$ on H^n_-. That is, Φ^F is homotopic to a map which is a lifting of ϕ^s_f, used in the definition of the Smale invariant, into $V_{n+2,n+1}$. This proves the last statement. □

2.3. **Geometric group operations.** Let **Imm** denote the set of regular homotopy classes of immersions $S^n \to \mathbb{R}^{n+k}$, $k \geq 2$. The Smale invariant gives a bijection of **Imm** onto $\pi_n(V_{n+k,n})$ and therefore endows **Imm** Abelian group. We shall give a topological description of the group operations.

Let $f, g\colon S^n \to \mathbb{R}^{n+k}$ be immersions. The connected sum $f \star g$ of f and g is defined as follows: Using regular homotopies, make f agree with the standard immersion in a neighborhood of the north pole a_+ and g agree with a translate of the standard immersion close to the south pole a_-. Then connect f and g by a standard tube from $f(a_+)$ to $g(a_-)$. The connected sum is well-defined up to regular homotopy and has the following property:

Lemma 2.3.
$$\Omega(f \star g) = \Omega(f) + \Omega(g)$$

Proof. This is Theorem in Kervaire [5]. □

Let $r\colon S^n \to S^n$ be a reflection as in Section 2.1. For an immersion $f\colon S^n \to \mathbb{R}^{n+2}$ define $\widehat{f} = f \circ r$.

Lemma 2.4. *Let $f\colon S^n \to \mathbb{R}^{n+2}$, $n > 2$ be an immersion then*
$$\Omega(\widehat{f}) = -\Omega(f).$$

Proof. Assume that r is a reflection in the hyperplane $x_1 = 0$. Let $R\colon \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ be the map
$$R(y_0, y_1, \dots, y_n, y_{n+1}) = (y_0, -y_1, \dots, y_n, -y_{n+1}).$$
Then R is diffeotopic to the identity and $R \circ \widehat{s} = s$. Hence,
$$\Omega(\widehat{f}) = [\phi_{\widehat{s}}^{\widehat{f}}] = [\phi_{R\circ s\circ r}^{R\circ f\circ r}] = [\phi_{s\circ r}^{f\circ r}].$$
With notation as in the definition of the Smale invariant (Section 2.1) we take D^n to be slightly larger than H_-^n, $D^n = \{x_0 \le \epsilon\}$, where ϵ is a small positive number. The remaining part of the sphere $S^n - D^n$ is diffeomorphic to the n-disk $\{x_1^2 + \dots + x_n^2 \ge 1 - \epsilon^2\}$ by projecting onto the plane $x_0 = 0$. Choose the framing $X = (\partial_1, \dots \partial_n)$ of $S^n - D^n$.

Let $\phi_s^f(x) = [v_1(x), v_2(x), \dots, v_n(x)]$. Then, since
$$dr(X) = (-\partial_1, \partial_2, \dots, \partial_n),$$
$\phi_{s\circ r}^{f\circ r} = [-v_1(r(x)), v_2(r(x)), \dots, v_n(r(x))]$.

Consider $K\colon V_{n+2,n} \to V_{n+2,n}$,
$$K([v_1, v_2, \dots, v_n]) = [-v_1, v_2, \dots, v_n],$$
and the fibration $SO(2) \to SO(n+2) \to V_{n+2,n}$. The map K lifts to $\widetilde{K}\colon SO(n+2) \to SO(n+2)$,
$$\widetilde{K}([v_1, v_2, \dots, v_n, v_{n+1}, v_{n+2}]) = [-v_1, v_2, \dots, v_n, v_{n+1}, -v_{n+2}],$$
and $\widetilde{K}$ is clearly homotopic to identity therefore $\widetilde{K}_* = 1$. From the commutative diagram with exact rows
$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \pi_n(SO(n+2)) \arrow[r] \arrow[d, "\widetilde{K}_*"'] & \pi_n(V_{n+2,n}) \arrow[r] \arrow[d, "K_*"'] & 0 \arrow[d] \\
0 \arrow[r] & \pi_n(SO(n+2)) \arrow[r] & \pi_n(V_{n+2,n}) \arrow[r] & 0
\end{tikzcd},$$
it follows that $K_* = 1$. Hence,
$$\Omega(\widehat{f}) = [\phi_{s\circ r}^{f\circ r}] = [\phi_s^f \circ r] = -\Omega(f).$$
□

Remark 2.5. For $n=2$ the above Lemma is false. In this case $\Omega(\widehat{f})=\Omega(f)$. This follows immediately from the characterization of the Smale invariant of a generic immersion $S^2\to\mathbb{R}^4$ as the algebraic number of self intersection points. This is the general behavior of immersions in double dimension: For $S^k\to\mathbb{R}^{2k}$, $k\geq 1$, $\Omega(\widehat{f})=\Omega(f)$. On the other hand, if R is a reflection in a hyperplane in $\mathbb{R}^{2k}$ then $\Omega(R\circ f)=-\Omega(f)$, for immersions $f\colon S^k\to\mathbb{R}^{2k}$.

Remark 2.6. If $f,g\colon S^n\to\mathbb{R}^{n+2}$, $n>2$ are embeddings then $f\star g$ is regularly homotopic to an embedding and $\widehat{f}$ is an embedding. Thus, the set **Emb** of regular homotopy classes which has embeddings for representatives is invariant under the group operations in **Imm**. Hence, **Emb** $\subset$ **Imm** is a subgroup. We call it *the embedding subgroup.*

3. Bernoulli numbers, the J-homomorphism, and almost parallelizable $4n$-manifolds

In this section we list some well-known results that will be needed in the proofs of Theorems 1 and 2.

The Bernoulli numbers B_k are the rational numbers defined by

$$\frac{x}{e^x-1}=1-\frac{x}{2}+\sum_{k\geq 1}(-1)^{k-1}B_k\frac{x^k}{(2k)!}.$$

Let m_k be the integer defined by $\frac{B_k}{4k}=\frac{n_k}{m_k}$, where m_k and n_k are relatively prime.

The Hopf-Whitehead J-homomorphism,

$$J\colon \pi_k(SO(m))\to\pi_{k+m}(S^m),$$

maps $\xi\in\pi_k(SO(m))$ to the framed cobordism class of the standard sphere $S^k\subset\mathbb{R}^{k+1}\subset\mathbb{R}^{k+m}$ with the normal framing $(x,\partial_{k+2},\dots,\partial_{k+m})\cdot\mathbf{A}(x)$, where the map $\mathbf{A}\colon S^k\to SO(m)$ is a smooth representative of ξ. (Recall that the Pontryagin construction gives a one to one correspondence between framed cobordism classes of framed k-dimensional submanifolds of $\mathbb{R}^{m+k}$ and $\pi_{k+m}(S^m)$.)

The Adams conjecture (formulated in [1] and proved in [8]) implies that the order j_n of the image $J\pi_{4n-1}(SO(m))$ of the stable group $\pi_{4n-1}(SO(m))\cong\mathbb{Z}$ satisfies $j_n=m_n$.

There are connections between the numerical characteristics of an almost parallelizable $4n$-manifold and j_n (and therefore m_n) which we shall utilize:

Lemma 3.1. *Let ξ be a stable $SO(m)$-bundle over a CW-complex K ($\dim K\leq n$) and let f be a cross-section of ξ restricted to the skeleton K^{4n-1}. Then the obstruction class $\nu(\xi,f)\in H^{4n}(K;\pi_{4n-1}(SO(m)))$ is related to the Pontryagin class $p_n(\xi)$ by $p_n(\xi)=\pm a_n(2n-1)!\,\nu(\xi,f)$, where a_n is 2 for n odd and 1 for n even.*

Proof. This is Lemma 2 in [6]. □

Lemma 3.2. *The Pontryagin number $\langle p_n(M^{4n}),[M^{4n}]\rangle$ of any almost parallelizable $4n$-manifold is divisible by $m_na_n(2n-1)!$.*

Proof. This is Theorem 1 in [6]. □

Lemma 3.3. *There exists an almost parallelizable manifold M^{4n} with*

$$\langle p_n(M^{4n}),[M^{4n}]\rangle=m_na_n(2n-1)!\,.$$

Moreover, this manifold admits a handle decomposition with one 0-handle, 2n-handles, one 4n-handle, and no other handles.

Proof. The first part is Theorem 2 in [6] and follows immediately from Lemma 3.1 and the definition of J. The second part follows in the case $n=1$ (i.e. $\dim(M)=4$) from Harer [3] where a handle decomposition of a K-3 surface without 1-handles and 3-handles is proved to exist. In case $n>1$ ($\dim(M)>7$), using surgery we can kill all homotopy groups of M below the middle dimension, keeping it almost parallelizable. Then M is simply connected and $H_q(M^{4n};\mathbb{Z})=0$ for $0<q<2n$. This implies the existence of a handle decomposition as required. □

4. Regular homotopy and signature

This section constitutes a proof of Theorem 2. We introduce some notation:

Let $\mathbf{Imm}_j$ denote the group of regular homotopy classes of immersions $S^{4j-1}\to\mathbb{R}^{4j+1}$ and $\mathbf{Emb}_j$ its embedding subgroup (see Remark 2.6). As mentioned in the Introduction $\mathbf{Imm}_j$ is an infinite cyclic group and hence $\mathbf{Emb}_j$ is either infinite cyclic or trivial. In Proposition 4.1 below we prove that the former is the case.

Let $\xi\in\mathbf{Emb}_j$. Represent ξ by an embedding $f\colon S^{4j-1}\to\mathbb{R}^{4j+1}$ and define $\sigma(\xi)=\sigma(W_f^{4j})\in\mathbb{Z}$ (see Theorem 2 for notation).

Proposition 4.1.

$$\sigma\colon \mathbf{Emb}\to\mathbb{Z}$$

is a nontrivial homomorphism.

Remark 4.2. Note that Theorem 2 follows immediately from Lemma 4.1.

Proof. First we must check that σ is well-defined: Let f_0 and f_1 be regularly homotopic embeddings. Choose a regular homotopy f_t connecting them and let $F\colon S^{4j-1}\times I\to\mathbb{R}^{4j+1}\times\mathbb{R}$ be given by

$$F(x,t)=f_t(x).$$

Then F is an immersion.

Let V_0^{4j} and V_1^{4j} be the Seifert surfaces of f_0 and f_1, respectively. Gluing them in at the ends of F we get an immersion G of the connected sum of W_{f_0} and $-W_{f_1}$ into $\mathbb{R}^{4j+2}$. The normal bundle of G is 2-dimensional orientable and there are trivializations of it over $V_1\cup V_2$. The obstruction to extend these trivializations over $S^{4j-1}\times I$ lives in

$$H^{q+1}(S^{4j-1}\times I,S^{4j-1}\times\partial I;\pi_q(SO(2)))=0.$$

Hence, the normal bundle of G is trivial and thus the tangent bundle of the connected sum of W_{f_0} and W_{f_1} is stably trivial. Therefore, all the Pontryagin classes of this tangent bundle are zero and by Hirzebruch signature theorem (see [7]) the signature also vanishes. Since signature is additive under connected sum and changes sign when orientation is reversed it follows that $\sigma(W_{f_0})=\sigma(W_{f_1})$ and the map σ is well-defined.

Moreover, from this last remark it follows that σ is a homomorphism: If $\xi,\eta\in\mathbf{Emb}$ are represented by embeddings f and g, respectively. Then $\xi+\eta$ is represented by $f\star g$ and $W_{f\star g}$ is the connected sum of W_f and W_g.

Finally, we must show that σ is non-trivial: Let W^{4j} be a manifold having properties as in Lemma 3.3. Let $V^{4j}=W^{4j}-D^{4j}$. Then the tangent bundle of V^{4j} is trivial. By the Smale-Hirsch h-principle we can find an immersion of V^{4j} into

$\mathbb{R}^{4j+1}$. By general position, this immersion is regularly homotopic to an immersion which embedds the $2j$-skeleton of V^{4j}. Now, since W^{4j} has only $2j$-handles, V^{4j} is a regular neighborhood of this $2j$-skeleton. Changing the immersion along the fibers we obtain an embedding of V^{4j} into $\mathbb{R}^{4j+1}$.

Let M^{4n} be an almost parallelizable manifold. Then there is a cellular decomposition of M^{4n} such that TM is trivial when restricted to the $(4n-1)$-skeleton. It follows from this that all Pontryagin classes $p_k(M^{4n})$, $0<k<n$, of M^{4n} vanish. The signature theorem then implies that the signature $\sigma(M^{4n})$ is proportional to the Pontryagin number $\langle p_n(M^{4n}), [M^{4n}]\rangle$ of M^{4n}.

Thus, restricting the embedding of V^{4j} to the boundary, we obtain an embedding with non-trivial signature. □

5. The index of the embedding subgroup

This section constitutes a proof of Theorem 1.

Proposition 5.1. *The subgroup* $\mathbf{Emb}_j \subset \mathbf{Imm}_j$ *has index* m_j.

Remark 5.2. Note that Theorem 1 follows immediately from Proposition 5.1.

Proof. The generator of $\mathbf{Emb}_j$ is an embedding with Seifert-surface of as low signature as possible. To calculate the index of $\mathbf{Emb}_j$ in $\mathbf{Imm}_j$, we shall calculate the Smale invariant of an embedding representing the generator of $\mathbf{Emb}_j$.

Let V^{4j} be a Seifert-surface of an embedding $f\colon S^{4j-1} \to \mathbb{R}^{4j+1}$ and let W^{4j} denote the closed manifold obtained from V^{4j} by adding a disk to it along $f(S^{4j})$. Then W^{4j} is almost parallelizable since V^{4j} embedds in $\mathbb{R}^{4j+r}$, r large, with trivial normal bundle. By Lemma 3.2, $m_j a_j(2j-1)!$ divides $\langle p_j(W^{4j}), [W^{4n}]\rangle$. As mentioned at the end of the proof of Proposition 4.1 the signature of an almost parallelizable $4j$-manifold is proportional to its j^{th} Pontryagin class. Therefore, an embedding $f\colon S^{4j-1}\to\mathbb{R}^{4j+1}$ such that $\langle p_j(W_f^{4j}), [W_f^{4j}]\rangle = \pm m_j a_j(2j-1)!$, represents a generator of $\mathbf{Emb}_j$.

Let W^{4j} be as in Lemma 3.3 with a handle decomposition as described there. Let V^{4j} be $W^{4j}-D^{4j}$, as in the proof of Proposition 4.1 we can construct an embedding $F\colon V^{4j}\to\mathbb{R}^{4j+1}$.

Let $h\colon S^{4j-1}\to \partial V^{4j}$ be a diffeomorphism. Then $f' = F\circ h$ represents a generator of $\mathbf{Emb}_j$ since $\langle p_j(W_{f'}^{4j}), [W_{f'}^{4j}]\rangle = \pm m_j a_j(2j-1)!$.

Let $n(x)\in T_{h(x)}V$, $x\in S^{4j-1}$ be an outward normal vector field of ∂V^{4j} in TV^{4j} and let $(x,t)\in S^{4j-1}\times(0,\delta)$ be coordinates in a collar C on ∂V^{4j} in V^{4j}. Straightening the embedding F close to the boundary ∂V^{4j}, we may assume that it has the form $F(x,t) = f'(x) - t dF(n(x))$ for $(x,t)\in C$.

We use the coordinates (x,t) also to parameterize a collar B on ∂D^{4j} in D^{4j}. The embedding F allows us to think about V^{4j} as a subset of $\mathbb{R}^{4j+1}$. Then

$$W^{4j} = \operatorname{int}(D^{4j}) \cup_\Psi \operatorname{int}(V^{4j}), \tag{†}$$

where $\Psi\colon B\to C\subset\mathbb{R}^{4j+1}$ is the map $\Psi(x,t) = f'(x) + (t-\delta)dF(n(x))$. Let $\overline{V}$ and $\overline{D}$ be the compact manifolds obtained by deleting half of the collar C and half of the collar B, respectively. Then

$$W^{4j} = \overline{V}\cup_f \overline{D},$$

where $f(x,\frac{\delta}{2}) = f'(x) + \frac{\delta}{2}dF(n(x))$. Clearly, f is diffeotopic to f' and hence represents a generator of $\mathbf{Emb}_j$.

Let ϵ be a trivial line bundle over W^{4j}. Let ξ denote the principal $SO(4j+1)$ bundle over W^{4j} associated to $TW^{4j} \oplus \epsilon$. Since W^{4j} is almost parallelizable, we have the decomposition

$$\xi = \overline{V} \times SO(4j+1) \cup_\psi \overline{D} \times SO(4j+1),$$

where $\psi \colon \partial\overline{D} \to SO(4j+1)$ is a gluing map.

By Lemma 3.1, the obstruction to trivialize ξ is $\frac{1}{a_j(2j-1)!}\langle p_j(\xi), [W^{4j}]\rangle$ times a generator of $\pi_3(SO(4j+1))$. On the other hand, this obstruction is just the homotopy class $[\psi] \in \pi_{4j-1}(SO(4j+1))$.

The bundle $T\overline{V} \oplus \epsilon$ is naturally identified with $T\mathbb{R}^{4j+1}|\overline{V}$: Let ν be a normal vector field of V^{4j} in $\mathbb{R}^{4j+1}$. For $p \in \overline{V}$, identify the vector $v_p + \lambda_p$ in $T_p\overline{V} \oplus \epsilon_p$ with $v_p + \nu(p) \in T_p\mathbb{R}^{4j+1}$. We can now identify the homotopy class of the gluing map ψ:

Pick a trivialization (Y, w) of $T\overline{D} \oplus \epsilon$. We need to compare it with a trivialization of $T\mathbb{R}^{4j+1}|\overline{V}$ along $\partial\overline{V}$. We use the standard trivialization of $T\mathbb{R}^5$ restricted to $\overline{V}$.

Using the decomposition (†) we see that (Y, w) induces the trivialization $(d\Psi(Y(x)), \nu(h(x)))$ of $T\mathbb{R}^{4j+1}$ along $\partial\overline{V}$. But $d\Psi(Y(x)) = dF(Y(x)) = \Phi^F(x)$ (notation as in Lemma 2.2). Hence, the induced trivialization of $T\mathbb{R}^{4j+1}$, considered as a $(4j+1)$-tuple of vectors gives (after orthonormalization) a lifting of Φ^F to $SO(4j+1)$ and thus $[\psi] = [\Phi^F] \in \pi_3(SO(4j+1))$. Applying Lemma 2.2 and recalling that $\langle p_j(W^{4j}), [W^{4j}]\rangle = m_j a_j(2j-1)!$, we conclude

$$\begin{aligned}\Omega(f) = [\Phi^F] = [\psi] &= \frac{1}{a_j(2j-1)!}\langle p_j(\xi), [W^{4j}]\rangle \\ &= \pm m_j \in \pi_{4j-1}(SO(4j+1)) = \pi_{4j-1}(V_{4j+1,4j-1}).\end{aligned}$$

□

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