ON THE TRIPLE POINTS OF SINGULAR MAPS

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Abstract. The number of triple points (mod 2) of a self-transverse immersion of a closed $2n$-manifold M into $3n$-space are known to equal one of the Stiefel-Whitney numbers of M. This result is generalized to the case of generic (i.e. stable) maps with singularities. Besides triple points and Stiefel-Whitney numbers, a certain linking number of the manifold of singular values with the rest of the image is involved in the generalized equation which corrects an erroneous formula in [9].

If n is even and the closed manifold is oriented then the equations mentioned above make sense over the integers. Together, the integer- and mod 2 generalized equations imply that a certain Stiefel-Whitney number of closed oriented $4k$-manifolds vanishes. This Stiefel-Whitney number is in fact the first in a family which vanish on such manifolds.

1. Introduction

In his classical paper [10] of 1946, Whitney showed that the number of double points of a self-transverse immersion of an n-manifold into $2n$-space is related to the Euler number of its normal bundle. Since then many results of a similar nature have been found. This paper deals with a generalization of one of these results, the Herbert–Ronga formula [5] which expresses the number of triple points of a self-transverse immersion of a closed $2n$-manifold into $3n$-space in terms of one of its characteristic numbers. More precisely, the Herbert–Ronga formula is extended to singular generic (i.e. stable) maps of $2n$-manifolds into $3n$-space. (In this paper all manifolds and maps are assumed to be C^∞-smooth, unless otherwise explicitly stated.) To state the formula, some notation is needed:

Let M be a closed $2n$-manifold and let $f\colon M \to \mathbb{R}^{3n}$ be a generic map. If $\Delta(f) \subset \mathbb{R}^{3n}$ denotes the set of double points of f then $\Delta(f)$ is an immersed n-dimensional submanifold with boundary. The self-intersection points of $\Delta(f)$ are the triple points of f. The boundary of $\Delta(f)$ is $\Sigma(f)$, the set of singular values of f.

Define $t_2(f) \in \mathbb{Z}_2$ as the mod 2-number of triple points of f. Let $\Sigma'(f)$ denote the $(n-1)$-dimensional submanifold of $\mathbb{R}^{3n}$ which is obtained by shifting $\Sigma(f)$ slightly along its outward normal vector field in $\Delta(f)$. Then $\Sigma'(f) \cap f(M) = \emptyset$. Define $l_2(f) \in \mathbb{Z}_2$ as the mod 2-linking number of the cycles $f(M)$ and $\Sigma'(f)$ in $\mathbb{R}^{3n}$. If $i_1 + \cdots + i_m = 2n$ then let $\bar{w}_{i_1} \ldots \bar{w}_{i_m}[M] \in \mathbb{Z}_2$ denote the product of the normal Stiefel-Whitney classes of M in dimensions $i_1, \ldots, i_m$ evaluated on the fundamental homology class F_M of M.

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Theorem 1. *Let M be a closed manifold of dimension $2n$ and let $f\colon M \to \mathbb{R}^{3n}$ be a generic map. Then*

$$t_2(f) + l_2(f) = \bar{w}_n^2[M] + \bar{w}_{n+1}\bar{w}_{n-1}[M] \tag{1}$$

Theorem 1 is proved in Section 2. It corrects the erroneous theorem on the second page of [9], in which the second term in the right hand side of Equation (1) is missing.

For closed oriented $4k$-manifolds Equation (1) can be lifted to an integer equation: If $n = 2k$ is even and M is oriented then there is an induced orientation on $\Delta(f)$ as well as on the triple points of f. Define $t(f) \in \mathbb{Z}$ as the algebraic number of triple points of f. The orientation of $\Delta(f)$ induces an orientation of its boundary $\Sigma(f)$ which in turn induces an orientation of $\Sigma'(f)$. Define $l(f) \in \mathbb{Z}$ as the linking number of the oriented cycles $f(M)$ and $\Sigma'(f)$ in $\mathbb{R}^{6k}$. Let $\bar{p}_k[M^{4k}]$ denote the k^{th} normal Pontryagin number of M. The following theorem is Lemma 4 in [1].

Theorem 2. *Let M be a closed oriented manifold of dimension $4k$ and let $f\colon M \to \mathbb{R}^{6k}$ be a generic map. Then*

$$3t(f) - 3l(f) = \bar{p}_k[M]. \tag{2}$$

Equation (2) turned out to be very useful: It is used in the derivation of a geometric formula for Smale invariants of immersions of spheres, see [1] and [2], and in the study of geometric features of the regular homotopy classification of immersions of 3-manifolds in 5-space, see [7].

If M is a closed oriented $4k$-manifold then the mod 2-reduction of $\bar{p}_k[M]$ equals $\bar{w}_{2k}^2[M]$. Hence Theorems 1 and 2 together imply that

$$\bar{w}_{2k+1}\bar{w}_{2k-1}[M] = 0 \tag{3}$$

for any closed oriented $4k$-manifold M. In fact, $\bar{w}_{2k+1}\bar{w}_{2k-1}$ is the first in a sequence of Stiefel-Whitney numbers which vanish on closed oriented $4k$-manifolds. More precisely,

Theorem 3. (Stong) *If M is an oriented $4k$-manifold and $(2k_1+1)+\dots+(2k_r+1) = 4k$ then*

$$\bar{w}_{2k_1+1}\dots\bar{w}_{2k_r+1}[M] = 0$$

This theorem was communicated by R. Stong to the second author together with a proof of the first case (3). A proof of Theorem 3 is presented in Section 3.

2. Proof of Theorem 1

Fix a generic map $f\colon M \to \mathbb{R}^{3n}$ of a closed $2n$-manifold. Let $\tilde{\Sigma} \subset M$ denote the $(n-1)$-dimensional submanifold of singular points of f and let $\Sigma = f(\tilde{\Sigma})$. Then f maps $\tilde{\Sigma}$ diffeomorphically to Σ.

Let $\tilde{\Delta} \subset M$ denote the closure of the preimages of multiple points of f. Then $\tilde{\Delta}$ is an immersed closed n-dimensional manifold with transverse double points at the preimages of triple points of f. Let $\tilde{\Delta}_{\text{res}}$ denote the resolution of $\tilde{\Delta}$ and let $\tilde{\iota}\colon \tilde{\Delta}_{\text{res}} \to M$ denote the natural immersion with image $\tilde{\Delta} \subset M$.

There is a natural involution $T\colon \tilde{\Delta}_{\text{res}} \to \tilde{\Delta}_{\text{res}}$ such that $f \circ \tilde{\iota} \circ T = f \circ \tilde{\iota}$. Since no triple point of f is singular we have a natural embedding $\tilde{\Sigma} \subset \tilde{\Delta}_{\text{res}}$ and $\tilde{\Sigma}$ is the fix point set of T.

Let $\nu(\tilde{\imath})$ denote the normal bundle of the immersion $\tilde{\imath}$ and let ν denote its restriction to $\tilde{\Sigma}$. Since ν is an n-dimensional vector bundle over an $(n-1)$-manifold there exists a non-zero section. Let $\tilde{s}$ be such a section.

A standard transversality argument allows us to extend $\tilde{s}$ to a section $\tilde{S}$ of $\nu(\tilde{\imath})$ which is transverse to the 0-section and which satisfies the two following conditions:

- If x is a double point of $\tilde{\imath}$ then $\tilde{S}(x) \neq 0$.
- If $\tilde{S}(x) = 0$ then $\tilde{S}(T(x)) \neq 0$.

Let $\Delta \subset \mathbb{R}^{3n}$ denote the closure of the double points of f. Then Δ is an immersed submanifold with boundary Σ and Δ has triple points at the triple points of f. Let Δ_{res} denote the resolution of Δ and let $\iota\colon \Delta_{\text{res}} \to \mathbb{R}^{3n}$ denote the natural immersion with image Δ. Let $\nu(\iota)$ denote the normal bundle of the immersion ι. Note that there is a natural map $\Pi\colon \tilde{\Delta}_{\text{res}} \to \Delta_{\text{res}}$ which is a double cover of $\Delta_{\text{res}} - \Sigma$ when restricted to $\tilde{\Delta}_{\text{res}} - \tilde{\Sigma}$, and which maps $\tilde{\Sigma}$ diffeomorphically onto Σ.

Define the section S of $\nu(\iota)$ as follows:

$$S(y) = \begin{cases} df(\tilde{S}(y_1)) + df(\tilde{S}(y_2)) & \text{if } y \in \Delta_{\text{res}} - \Sigma, \text{ where } y_1 \neq y_2,\ \Pi(y_1) = \Pi(y_2) = y, \\ 2df(\tilde{S}(y_1)) & \text{if } y \in \Sigma, \text{ where } \Pi(y_1) = y. \end{cases}$$

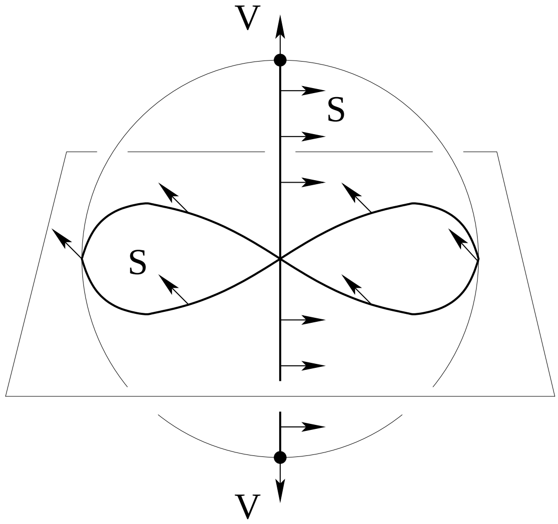

FIGURE 1. A piece of $f(M)$ (represented by a 2-sphere and a piece of a plane) with the double point set Δ (fat lines), its normal field S, and singularity set Σ (dots) with its outward normal field V in Δ.

Let $C(\Sigma) \subset \Delta_{\text{res}}$ be a small open collar on the boundary Σ of Δ_{res}. Let Δ'' denote the image of the immersion $y \mapsto \iota(y) + \epsilon S(y)$, $y \in \Delta_{\text{res}} - C(\Sigma)$ for some small $\epsilon > 0$. Then, if ϵ and the collar $C(\Sigma)$ are small enough, Δ'' is a chain with boundary $\partial\Delta'' = \Sigma''$ satisfying $\Sigma'' \cap f(M) = \emptyset$. If lk_2 denotes the mod 2-linking number, $\bullet$ denotes the mod 2-intersection number, and $\sharp(F)$ denotes the mod 2-number of elements in the finite set F, then

$$\mathrm{lk}_2(\Sigma'', f(M)) = \Delta'' \bullet f(M) = \sharp(\tilde{S}^{-1}(0)) + t_2(f), \tag{4}$$

since near each zero z of $\tilde{S}$ there is a unique intersection point of Δ'' and $f(M)$ near $f(z)$, and near each triple point of f there are exactly three such intersection points.

The homology class of the cycle $\tilde{\Delta}$ in M is Poincaré dual to n^{th} normal Stiefel-Whitney class $\bar{w}_n$ of M, see [6]. Thus

$$\bar{w}_n^2[M] = \tilde{\Delta} \bullet \tilde{\Delta} = \sharp(\tilde{S}^{-1}(0)), \tag{5}$$

since the image of a slight shift of the immersion $\tilde{\iota}$ along $\tilde{S}$ intersects $\tilde{\Delta}$ near each zero of $\tilde{S}$ and in *two* points near each double point of $\tilde{\iota}$.

Equations (4) and (5) imply

$$\mathrm{lk}_2(\Sigma'', f(M)) = \bar{w}_n^2[M] + t_2(f). \tag{6}$$

Recall that $\Sigma' \subset \mathbb{R}^{3n}$ is the submanifold which results when Σ is shifted slightly along its unit outward normal vector field V in Δ, and that $\Sigma' \cap f(M) = \emptyset$. We compare the linking numbers $\mathrm{lk}_2(\Sigma'', f(M))$ and $\mathrm{lk}_2(\Sigma', f(M))$:

Let $\tilde{\Sigma}_0 \subset M$ be the submanifold which results when $\tilde{\Sigma}$ is shifted a small distance along $\tilde{S}$. Let $\Sigma_0 = f(\tilde{\Sigma}_0)$ and for $p \in \Sigma$, let $p_0 = f(\tilde{p}_0)$ where $\tilde{p}_0$ is the point in $\tilde{\Sigma}_0$ corresponding to $\tilde{p} \in \tilde{\Sigma}$ with $f(\tilde{p}) = p$.

For small $\epsilon > 0$ and $p \in \Sigma$ let $l_p(\epsilon)$ be the segment of the straight line through $p + \epsilon V(p)$ and p_0 of length 2ϵ and centered at p_0. For $\epsilon > 0$ and the shifting of $\tilde{\Sigma}$ in M small enough,

$$\Gamma = \bigcup_{p \in \Sigma} l_p(\epsilon)$$

is a submanifold of $\mathbb{R}^{3n}$. If the collar $C(\Sigma)$ is chosen small enough and if the shifting distance along S is small enough then the boundary $\partial\Gamma$ of Γ is isotopic to $\Sigma' \cup \Sigma''$ in $\mathbb{R}^{3n} - f(M)$. Thus

$$\mathrm{lk}_2(\Sigma', f(M)) = \mathrm{lk}_2(\Sigma_0, f(M)) + \Gamma \bullet f(M) = \mathrm{lk}_2(\Sigma'', f(M)) + \Gamma \bullet f(M). \tag{7}$$

We compute $\Gamma \bullet f(M)$: The intersection $\Gamma \cap f(M)$ is a clean intersection. That is, $\Gamma \cap f(M) = \Sigma_0$ is a manifold and the tangent bundle

$$T\Sigma_0 = Tf(M) \cap T\Gamma \subset T\mathbb{R}^{3n}, \tag{8}$$

where all bundles in the left hand side are restricted to Σ_0.

As in [4], we find

$$\Gamma \bullet f(M) = w_{n-1}(\xi),$$

where ξ is the so called excess bundle over Σ_0:

$$\xi = T\mathbb{R}^{3n}/(T\Gamma + Tf(M)),$$

where all bundles are restricted to Σ_0.

To finish the proof it remains to calculate $w_{n-1}(\xi)$. Note that

$$T\Gamma|\Sigma_0 = T\Sigma_0 \oplus \epsilon^1,$$

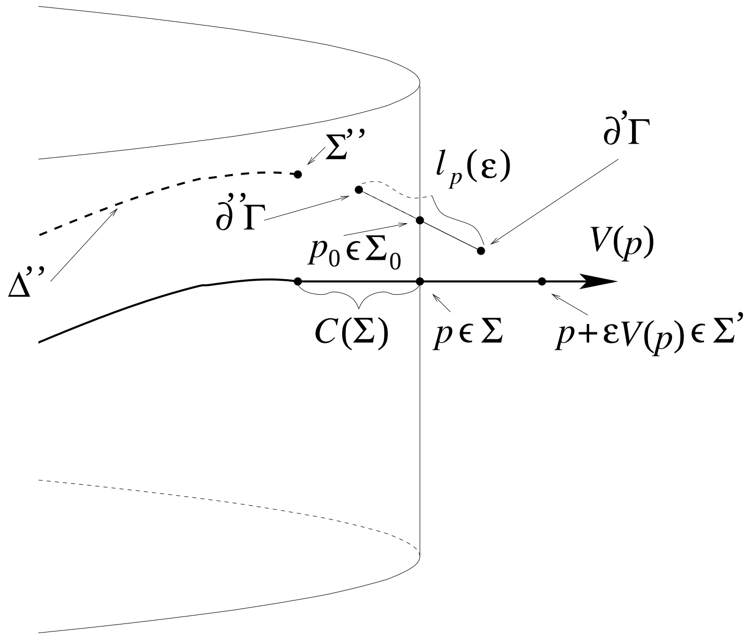

FIGURE 2. The normal space of Σ in $\mathbb{R}^{3n}$ at $p \in \Sigma$. In the figure the boundary of Γ is the union of $\partial'\Gamma$, isotopic to Σ' in $\mathbb{R}^{3n} - f(M)$, and $\partial''\Gamma$ isotopic to Σ'' in $\mathbb{R}^{3n} - f(M)$.

where ϵ^1 is the trivial line bundle directed along the intervals $l_p(\epsilon)$. Thus, by (8),

$$\xi \oplus Tf(M)|\Sigma_0 \oplus \epsilon^1 = T\mathbb{R}^{3n}|\Sigma_0. \tag{9}$$

The bundle $Tf(M)|\Sigma_0$ is identified with $TM|\tilde{\Sigma}_0$ by the differential of f. Hence if $i_0\colon \tilde{\Sigma}_0 \to M$ denotes the inclusion then $w(\xi) = i_0^*\bar{w}(M)$. Therefore, if F_V denotes the fundamental homology class of the manifold V and PD denotes the Poincaré duality operator,

$$\begin{aligned}\langle w_{n-1}(\xi), F_{\Sigma_0}\rangle &= \left\langle i_0^*\bar{w}(M), F_{\tilde{\Sigma}_0}\right\rangle = \left\langle \bar{w}(M), i_{0*}(F_{\tilde{\Sigma}_0})\right\rangle = \langle \bar{w}(M), \mathrm{PD}\,\bar{w}_{n+1}(M)\rangle \\ &= \langle \bar{w}(M) \cup \bar{w}_{n+1}(M), F_M\rangle = \bar{w}_{n-1}\bar{w}_{n+1}[M]. \end{aligned} \tag{10}$$

Here, the third equation follows from the well-known formula $\mathrm{PD}\,\bar{w}_{n+1}(M) = i_*F_{\tilde{\Sigma}}$, where $i\colon \tilde{\Sigma} \to M$ denotes the inclusion, together with $i_*F_{\tilde{\Sigma}} = i_{0*}F_{\tilde{\Sigma}_0}$. Equations (6), (7), and (10) prove the theorem. □

3. PROOF OF THEOREM 3

Let $\mathfrak{N}_*$, Ω_*, and Ω_*^U denote the cobordism ring, the oriented cobordism ring, and the complex cobordism ring, respectively. Note that there are natural forgetting homomorphisms

$$\Omega_*^U \longrightarrow \Omega_* \longrightarrow \mathfrak{N}_*.$$

For a manifold M, let $[M]$ denote its cobordism class.

Using some facts from cobordism theory which can all be found in Chapter 4 of Stong's book [8], we show that it is enough to prove the theorem for oriented $4k$-manifolds M such that either

(a) $[M]\in\Omega_{4k}$ maps to a square $[N\times N]\in\mathfrak{N}_{4k}$, or
(b) $[M]$ is a torsion element of Ω_{4k} (in fact, $[M]$ torsion implies $2\cdot[M]=0$):

Let $\mathrm{Tors}(\Omega_*)$ denote the torsion subgroup of Ω_*. The homomorphism $\Omega^U_*\to\Omega_*$ induces an epimorphism

$$\Omega^U_* \longrightarrow \Omega_*/\,\mathrm{Tors}(\Omega_*).$$

and the image $\Omega^U_*\to\mathfrak{N}_*$ consists of squares of elements in $\mathfrak{N}_*$.

Hence, if M is any oriented $4k$-manifold then there exists some oriented $4k$-manifold V such that $[V]$ is torsion in Ω_{4k} and $[M]+[V]=[N\times N]$ in $\mathfrak{N}_{4k}$. This implies that the theorem follows once it is proved for manifolds satisfying (a) or (b) above.

First consider (a): let $M=N\times N$. Then $\bar w(M)=\bar w(N)\times\bar w(N)$ and hence

$$\bar w_{2k+1}(M)=\sum_{i+j=2k+1}\bar w_i(N)\times\bar w_j(N).$$

Thus

$$\begin{aligned}&\langle\bar w_{2k_1+1}(M)\dots\bar w_{2k_r+1}(M),F_M\rangle=\\&=\sum\langle\bar w_{i_1}(N)\dots\bar w_{i_r}(N),F_N\rangle\cdot\langle\bar w_{j_1}(N)\dots\bar w_{j_r}(N),F_N\rangle\,.\end{aligned}\tag{11}$$

Since i_s+j_s is odd for all i_s,j_s there is a fixed point free involution T acting on the set of the terms in the sum in (11) such that $i_1+\dots+i_r=2k=j_1+\dots+j_r$:

$$\begin{aligned}T\colon\ &\langle\bar w_{i_1}(N)\dots\bar w_{i_r}(N),F_N\rangle\cdot\langle\bar w_{j_1}(N)\dots\bar w_{j_r}(N),F_N\rangle\\&\mapsto\langle\bar w_{j_1}(N)\dots\bar w_{j_r}(N),F_N\rangle\cdot\langle\bar w_{i_1}(N)\dots\bar w_{i_r}(N),F_N\rangle\,.\end{aligned}$$

Thus the terms in the left hand side of (11) which does not vanish for dimensional reasons cancel in pairs and hence $\bar w_{2k_1+1}\dots\bar w_{2k_r+1}[M]=0$.

Next consider (b): let $u\colon\mathfrak{N}_{4k}\to\mathbb{Z}_2$ denote the homomorphism induced by the product of odd-dimensional normal Stiefel-Whitney classes $\bar w_{2k_1+1}\dots\bar w_{2k_r+1}$, $\sum 2k_j+1=4k$. Odd-dimensional Stiefel-Whitney classes are mod 2-reductions of twisted integer classes, see [3], p. 140. Hence, a product of an even number of such classes is an integer class so the map

$$\Omega_{4k}\xrightarrow{\ \pi\ }\mathfrak{N}_{4k}\xrightarrow{\ u\ }\mathbb{Z}_2$$

lifts to a homomorphism

$$\Omega_{4k}\xrightarrow{\ U\ }\mathbb{Z}.$$

Thus U and therefore $u\circ\pi$ is zero on any torsion element of Ω_{4k}. □

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