Introduction

This paper is concerned with the connection between curvature and the lengths of simple closed geodesics on manifolds homeomorphic to the standard two-dimensional sphere. On a nonsimply connected, complete and compact Riemannian manifold the existence of geodesic loops follows from the fact that any conjugacy class in the fundamental group can be represented by a geodesic. This is proved by considering the universal covering manifold and deck-transormations as in Klingenberg [6], Theorem 6.7.1. If the manifold has a trivial fundamental group the existence theorem becomes more difficult. However, in the two dimensional case there is a straightforward approach, due to Poincaré. One considers the variational problem of finding the shortest smooth curve that divides the surface into two parts of equal total Gaussian curvature. There exists a solution-curve to this problem and it is not hard to see that it is a geodesic. A proof for arbitrary dimensional compact manifolds requires different techniques and can be found in Klingenberg [7]. There it is also shown that any simply connected, compact surface has at least three distinct geodesic loops and that on certain ellipsoids there are no more than three.

Section one of this paper is a survey of some standard results of Riemannian geometry that is used in, and have inspired, later sections. Section two is concerned with three examples showing that although a surface has a short geodesic loop it may, in some sense, have small L^2-norm of curvature and small area. However, the examples involve surfaces having points where the curvature is negative. For surfaces with non-negative curvature the situation is different. In section three we study this situation and prove two results. Firstly, if there is a short geodesic loop, then the L^2-norm of curvature has to be large. Secondly, the area is at most the product of the length of the shortest geodesic loop and the diameter of the surface.

1 Some Riemannian geometry

In the coming sections some general facts from Riemannian geometry are used. These results are presented here for arbitrary dimensional Riemannian manifolds. We let M be a complete manifold with a riemannian metric $\langle\, , \,\rangle$ and corresponding Levi-Civita connection $\nabla : \Upsilon(M) \times \Upsilon(M) \to \Upsilon(M)$ where $\Upsilon(M)$ is the vector space of smooth vector fields on M. That is, if $V, W, X \in \Upsilon(M)$ and f is a smooth function on M, the following holds

$$\nabla_{fV} W = f \nabla_V W, \tag{1.1}$$
$$\nabla_V fW = (Vf)W + f \nabla_V W, \tag{1.2}$$
$$X\langle V, W\rangle = \langle \nabla_X V, W\rangle + \langle V, \nabla_X W\rangle, \tag{1.3}$$
$$[V, W] = \nabla_V W - \nabla_W V, \tag{1.4}$$

where $[\, , \,]$ is the Lie-bracket. As usual, $\|X\| = \langle X, X\rangle^{1/2}$ and if $c : [a, b] \to M$ is a curve on M the length of c, $l(c)$ is defined by

$$l(c) = \int_a^b \|c'(t)\| \, dt. \tag{1.5}$$

The curvature tensor R of ∇ is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.6}$$

for $X, Y, Z \in \Upsilon(M)$. By a straightforward calculation one verifies that this tensor has the following four properties

$$R(X, Y)Z + R(Y, X)Z = 0, \tag{1.7}$$
$$R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0, \tag{1.8}$$
$$\langle R(X, Y)Z, W\rangle - \langle R(Z, W)X, Y\rangle = 0, \tag{1.9}$$
$$\langle R(X, Y)Z, W\rangle + \langle R(X, Y)W, Z\rangle = 0. \tag{1.10}$$

Given two linearly independent tangent vectors ξ, η in the tangent space M_p at the point $p \in M$ we define the *sectional curvature of the 2-section determined by* ξ, η to be

$$\mathbf{k}(\xi, \eta) = \frac{\langle R(\xi, \eta)\eta, \xi\rangle}{\|\xi \wedge \eta\|^2}. \tag{1.11}$$

By the bilinearity of the curvature tensor and the wedge-product, $\mathbf{k}$ only depends on the 2-section, not on the particular basis chosen and knowledge of $\mathbf{k}$ on every 2-section in M_p completely determines R. If M^2 is two-dimensional there is only one 2-section in each tangent space of M and we may look upon $\mathbf{k}$ as a function defined on M. This function is denoted by K and is called the *Gaussian curvature of* M.

1.1 The first variation of arclength. Let $c : [a,b] \to M$ be a piecewise smooth curve. A variation $\nu(t,s)$ of c is a mapping

$$\nu : [a,b] \times (-\epsilon, \epsilon), \quad \epsilon > 0, \quad \nu(t,0) = c(t), \tag{1.12}$$

such that there exists a subdivision $a = t_0 < \ldots < t_k = b$ of $[a,b]$ for which $\nu|_{[t_j,t_{j+1}]\times(-\epsilon,\epsilon)}$ is smooth for each j. Writing $\partial_t \nu$ and $\partial_s \nu$ for $\nu_* \partial_t$ and $\nu_* \partial_s$ respectively (ν_* is the differential of ν) and letting ∇_t and ∇_s denote $\nabla_{\partial_t \nu}$ and $\nabla_{\partial_s \nu}$, we get

$$\nabla_s \partial_t \nu - \nabla_t \partial_s \nu = 0, \tag{1.13}$$

by the fact that the Lie bracket commutes with differentials and (1.4).

THEOREM. *Assume $c : [a,b] \to M$ is differentiable and ν a differentiable variation of c. For each s in $(-\epsilon, \epsilon)$ let $L(s)$ be the length of c_s,*

$$L(s) = \int_a^b \|\partial_t \nu(t,s)\| \, dt.$$

Then L is differentiable and

$$\frac{dL}{ds} = \langle \partial_s \nu, \partial_t \nu / \|\partial_t \nu\| \rangle |_a^b - \int_a^b \langle \partial_s \nu, \nabla_t (\partial_t \nu / \|\partial_t \nu\|) \rangle \, dt. \tag{1.14}$$

If c is parameterized proportional to arc length, say $\|c'\| = l$, then

$$\frac{dL}{ds}(0) = l^{-1} \left(\langle V, c' \rangle |_a^b - \int_a^b \langle V, \nabla_t c' \rangle \, dt \right), \tag{1.15}$$

where $V(t) = (\partial_s \nu)(t, 0)$.

PROOF: We have

$$\begin{aligned}
\partial_s L &= \partial_s \int_a^b \|\partial_t \nu\| \, dt \\
&= \int_a^b \partial_s (\langle \partial_t \nu, \partial_t \nu \rangle^{1/2}) \, dt \\
&= \int_a^b \|\partial_t \nu\|^{-1} \langle \nabla_s \partial_t \nu, \partial_t \nu \rangle \, dt \qquad [\text{ by } (1.3)] \\
&= \int_a^b \langle \nabla_t \partial_s \nu, \partial_t \nu / \|\partial_t \nu\| \rangle \, dt \qquad [\text{ by } (1.13)] \\
&= \int_a^b \{ \partial_t \langle \partial_s \nu, \partial_t \nu / \|\partial_t \nu\| \rangle - \langle \partial_s \nu, \nabla_t \partial_t \nu / \|\partial_t \nu\| \rangle \} \, dt \qquad [\text{ by } (1.3)] \\
&= \langle \partial_s \nu, \partial_t \nu / \|\partial_t \nu\| \rangle |_a^b - \int_a^b \langle \partial_s \nu, \nabla_t \partial_t \nu / \|\partial_t \nu\| \rangle \, dt,
\end{aligned}$$

which is (1.14). Equation (1.15) follows immediately. $\square$

By defining a geodesic as a curve $\gamma : (a, b) \to M$ satisfying

$$\nabla_t \gamma' = 0, \tag{1.16}$$

one can show that a curve γ is a geodesic if and only if $L'(0) = 0$ for every variation ν of γ fixing endpoints. Thus a geodesic is stationary curve for the length function defined on all curves connecting two fixed points.

1.2 The Exponential map and the Gauss lemma. A fundamental property of geodesics is that, given a point $p \in M$ and a vector $\xi \in M_p$, there exists a unique geodesic γ_ξ through p with velocity vector ξ at p. This follows from the fact that the defining condition for geodesics $\nabla_{\gamma'} \gamma' = 0$ is a second order differential equation in the parameter t of γ. The point p and the vector ξ are exactly the required initial data for existence and uniqueness of γ_ξ (see Curtis and Miller [5], chapter 4, for details). If $\gamma_\xi : (-\epsilon, \epsilon) \to M$ is a geodesic parameterized by t, the curve $c : (-\epsilon/s, \epsilon/s) \to M$ defined by $c(t) = \gamma_\xi(st)$ (s fixed) is also a geodesic, since $\nabla_{c'} c' = s^2 \nabla_{\gamma'_\xi} \gamma'_\xi = 0$. Also $c'(0) = s\xi$ so $c = \gamma_{s\xi}$. The *exponential map* $\exp_p : M_p \to M$ is defined by $\exp_p \xi = \gamma_\xi(1)$ for all ξ such that 1 is in the domain of γ_ξ. From the above we know that for any fixed ξ there is an $s > 0$ such that $\gamma_{s\xi}(1)$ is defined and by the the existence theorems for second order differential equations we may let s vary continuously with ξ. It follows that $\exp_p$ is defined on a neighbourhood of the origin in M_p. However, by our assumption that M is complete, the Hopf-Rinow theorem (Cheeger and Ebin [4], theorem 1.8) tells us that $\exp_p$ is actually defined on all of M_p for each p in M. Furthermore, it is a smooth map and, by the implict function theorem, a local diffeomorphism in a neighbourhood of the origin. Actually since we have $\exp_p : M_p \to M$ for each p, we can define the union of these maps and denote it by $\exp : TM \to M$, that is $\exp(p, \xi) = \exp_p(\xi)$.

If we let $\rho(t) = t\xi$ be a ray from $0 \in M_p$ through ξ, it is easy to verify that $\exp_*(\rho'(t)) = \gamma'_\xi$ and that $\|\rho'(t)\| = \|\gamma'_\xi(t)\|$. Moreover, there is the following result known as the Gauss lemma.

PROPOSITION. *If $\rho(t) = t\xi$ is a ray through the origin in M_p and $\eta \in (M_p)_{\rho(t)}$ is perpendicular to $\rho'(t)$, then $\exp_{p_*}(\eta)$ is perpendicular to $\exp_{p_*}(\rho'(t))$.*

PROOF: Let $c(s)$ be a curve in M_p such that $c(0) = \xi$ and $c'(0) = \eta$ and such that every point on c is at the same distance from the origin of M_p. That is, c is a great circle on the sphere of radius $\|\xi\|$ in M_p. Let $\nu(t, s)$ be a rectangle i M defined by

$$\nu(t, s) = \exp_p\left(\rho_s(t)\right),$$

where $\rho_s : [0, 1] \to M_p$ is the ray from 0 to $c(s)$ in M_p. From the definitions of $\exp_p$ and ν we know that the lengths of the curves $t \mapsto \nu(t, s)$ are independent of s. Also $t \mapsto \nu(t, 0)$

is a geodesic, $\partial_s\nu(0,0) = 0$ and $\partial_s\nu(t,0) = \exp_{p_*}(t\eta)$. The first variation formula then gives

$$0 = \frac{d}{ds}L\left(\exp_p(\rho_s)\right)\big|_{s=0} = \langle \partial_s\nu(\tau,0), \gamma'_\xi(\tau)\rangle|_0^t = t\langle \exp_{p_*}(\eta), \gamma'_\xi(t)\rangle,$$

and the proposition follows. $\square$

1.3 Jacobi fields. We will now examine the relationship between curvature and the exponential map. Let ξ and η be orthonormal vectors in M_p. Using the natural identification of tangent spaces of M_p with M_p itself ξ and η induce vectorfields X and Y on all of M_p. Consider the family ρ_s of rays in M_p defined by

$$\rho_s(t) = (X + sY)\, t.$$

Then $\exp_p \circ \rho_s(t)$ is a geodesic through p with tangent vector $\xi + s\eta$. In order to measure the effect of $\exp_{p_*}$ on the length of vectors in M_p we shall compute the first few terms in the Taylor expansion of $\|\exp_{p_*}(tY)\|$.

First some preliminaries. It is clear from the definition that $\exp_{p_*}(tY)$ arises as a variation field of a one parameter family of geodesics $\exp_p \circ \rho_s$. Fields of this type are called *Jacobi fields* and are characterized by a certain second order differential equation. To find this equation let $\nu : [a,b] \times (-\epsilon, \epsilon) \to M$ be such that for fixed s, $\nu(t,s)$ is a geodesic. We now determine the differential equation satisfied by $\partial_s\nu|_{\nu(t,0)}$. By equation (1.4) we have

$$\nabla_t\partial_s\nu - \nabla_s\partial_t\nu - \nu_*[\partial_t, \partial_s] = 0.$$

However, the Lie-bracket above vanishes so $\nabla_t\partial_s\nu = \nabla_s\partial_t\nu$. Therefore $\nabla_t\nabla_t\partial_s\nu = \nabla_t\nabla_s\partial_t\nu$. Since $\nabla_t\partial_t\nu = 0$, we may write

$$\nabla_t\nabla_t\partial_s\nu = \nabla_t\nabla_s\partial_t\nu = \nabla_t\nabla_s\partial_t\nu - \nabla_s\nabla_t\partial_t\nu.$$

Using the definition of the curvature tensor and writing V for $\partial_s\nu(t,0)$ and γ' for $\partial_t\nu(t,0)$ this can be written as

$$\nabla_t\nabla_t V = R(\gamma', V)\gamma'. \tag{1.17}$$

Equation (1.17) is called the *Jacobi equation* and a vector field along a geodesic γ satisfying it is called a *Jacobi field*. Let $\{E_i(t)\}$ be orthonormal and parallel along $\nu(t,0)$ then the Jacobi equation may be written as the linear second order system of ordinary differental equations

$$\begin{pmatrix} \langle J, E_1\rangle \\ \vdots \\ \langle J, E_n\rangle \end{pmatrix}'' = \langle R(\gamma', E_i)\gamma', E_j\rangle \begin{pmatrix} \vdots \\ \langle J, E_j\rangle \\ \vdots \end{pmatrix}.$$

From the theory of ordinary differential equations we know that the solution space to this system is $2n$-dimensional and there exists a unique solution with prescribed initial value and first derivative. This is equivalent to prescribing $J(0)$ and $J'(0) = \nabla_t J|_{t=0}$. Noticing that $\nabla_t \gamma' = 0$, we have

$$\langle J, \gamma' \rangle'' = \langle J'', \gamma' \rangle = \langle R(\gamma', J)\gamma', \gamma' \rangle = 0.$$

Therefore any Jacobi field J along γ may be written uniquely as

$$J = J_0 + (at + b)\gamma',$$

where $\langle J_0, \gamma' \rangle = 0$. Finally we have to see that the two, apparently different, definitions of Jacobi fields actually is the same. That is, we need to show that if J satisfies (1.17), then J comes from a variation of geodesics. In fact let $c(s)$ be a curve such that $c'(0) = J(0)$ and let γ' and $J'(0)$ be extended to parallel fields along $c(s)$. Then the variation field of $\nu(t, s) = \exp_{c(s)}(t(\gamma' + sJ'(0)))$ is a Jacobi field by the fact that $\nu(t, s)$, for fixed s, is a geodesic. Moreover,

$$\begin{aligned} \partial_s \nu|_{t=0} &= \partial_s c(s) = c'(s), \\ \nabla_t \partial_s \nu|_{t=0} &= \nabla_s \partial_t \nu|_{t=0} = \nabla_s(\gamma' + sJ'(0)) = J'(0), \end{aligned}$$

by the fact that γ' and $J'(0)$ are parallel along $c(s)$. Hence $\partial_s \nu$ satisfies the same initial conditions as J. Therefore it equals J by the uniqueness property above.

We are now ready to calculate the Taylor series for $\|\exp_*(tY)\|^2$. Set $\exp_*(X) = T$ and $\exp_*(tY) = J$. We see that $J'(0) = \eta$. Then

$$\begin{aligned} \langle J, J \rangle|_{t=0} =& 0, \\ \langle J, J \rangle'|_{t=0} =& 2\langle J, J' \rangle|_{t=0} = 0, \\ \langle J, J \rangle''|_{t=0} =& 2\langle J', J' \rangle|_{t=0} + 2\langle J'', J \rangle|_{t=0} = 2\|\eta\| = 2. \end{aligned}$$

Note that in fact $J''|_{t=0} = R(T, J)T|_{t=0} = 0$, so that

$$\langle J, J \rangle'''|_{t=0} = 6\langle J'', J' \rangle|_{t=0} + 2\langle J''', J \rangle|_{t=0} = 0.$$

Also,

$$J''' = \nabla_T R(T, J)T|_{t=0} = (\nabla_T R)(T, J)T|_{t=0} + R(T, J')T|_{t=0},$$

so that

$$J'''|_{t=0} = R(T, J')T|_{t=0} = R(T, \eta)T|_{t=0}.$$

Then

$$\begin{aligned} J''''|_{t=0} &= 8\langle J''', J' \rangle|_{t=0} + 6\langle J'', J'' \rangle|_{t=0} + 2\langle J'''', J \rangle|_{t=0} \\ &= 8\langle R(T, \eta)T, \eta \rangle|_{t=0} = -8\langle R(\xi, \eta)\eta, \xi \rangle. \end{aligned}$$

Therefore

$$\|exp_*(tY)\|^2 = t^2 - \tfrac{1}{3}\langle R(\xi, \eta)\eta, \xi \rangle t^4 + O(t^5). \tag{1.18}$$

Equation (1.18) says that geodesics come together to the order of $\left(\frac{1}{3}\langle R(\xi, \eta)\eta, \xi \rangle\right)^{1/2} t^2$ when compared to the rays ρ_s. Thus, in spaces of positive sectional curvature geodesics locally converge, whereas in spaces of negative sectional curvature they locally diverge when compared with rays in the tangent space.

1.4 The Gauss-Bonnet theorem for surfaces. In this section we assume that the dimension of M is 2. The Gauss-Bonnet theorem is a classical result of differential geometry. It is proved in Chavel [3], section 4.6. Before stating it we need the following definition.

DEFINITION. *For $\gamma : (a,b) \to M$, an immersed C^2 curve, the geodesic curvature of γ is defined by*

$$\kappa_g = \frac{\langle \nabla_t \gamma', \iota\gamma' \rangle}{\|\gamma'\|^3} \tag{1.19}$$

where ι denotes the rotation of tangent spaces of M by $\pi/2$ radians.

THEOREM. *For M diffeomorphic to the sphere $\mathbf{S}^2$*

$$\iint_M K\, dA = 4\pi. \tag{1.20}$$

For $D \subset M$ with compact closure and diffeomorphic to the open unit disk and the boundary of D given by an orientation preserving map $\mathbf{S}^1 \to M$ with finitely many corners

$$\int_{\partial D} \kappa_g\, ds + \iint_D K\, dA = 2\pi - \sum_j \alpha(p_j), \tag{1.21}$$

where $\alpha(p_j)$ is the exterior angle at $p_j \in \partial D$.

1.5 The coarea formula. The coarea formula says that an integral over an n-dimensional manifold can be computed by first integrating over $(n-1)$-dimensional level surfaces of a function f and then over the range of f. Some notation used in the theorem below ought to be explained. As usual we have an n-dimensional manifold M with Riemannian measure dV, that of course also can be looked upon as an n-form on M. Hence we can look upon dV as measuring the volumes of parallelepipeds in $M_q, q \in M$. If N is a submanifold of M one may understand $dV_{|N}$ as the restriction of dV to parallelepipeds in $M_q, q \in N$. The $(n-1)$-dimensional measure on N induced by the metric $\imath^* g$ where g is the metric on M and $\imath : N \hookrightarrow M$ is the inclusion is denoted by dA.

THEOREM. *Let Ω be a domain in M whith compact closure and $f : \bar{\Omega} \to \mathbf{R}$ a function in $C^0(\bar{\Omega}) \cap C^\infty(\Omega)$, with $f|\partial\Omega = 0$. For any regular value t of $|f|$, let*

$$\Gamma(t) = |f|^{-1}(t),$$

and dA_t denote the $(n-1)$-dimensional Riemannian measure induced on $\Gamma(t)$. Then

$$dV_{|\Gamma(t)} = \frac{dA_t dt}{\|\text{grad } f\|}, \tag{1.22}$$

and for any function $\phi \in L^1(\Omega)$

$$\int_\Omega \phi \|\text{grad } f\|\, dV = \int_0^\infty dt \int_{\Gamma(t)} \phi\, dA_t. \tag{1.23}$$

PROOF: The critical values of f have Lebesgue measure equal to zero by Sard's theorem (Bredon [2], Chapter 2, Theorem 6.2). The regular values of f, R_f, are open in $\mathbf{R}$ and for $t \in R_f$ the preimage $f^{-1}(t)$ is an $(n-1)$-submanifold in M with $f^{-1}(t) \cap \Omega$ compact. Let $(\alpha, \beta) \subseteq R_f$ and $\mu \in (\alpha, \beta)$. Then one can construct a diffeomorphism

$$\Psi : f^{-1}(\mu) \times (\alpha, \beta) \to f^{-1}(\alpha, \beta),$$

for which

$$f(\Psi(q,t)) = t \quad \text{for all } (q,t) \in f^{-1}(\mu) \times (\alpha, \beta).$$

This is done as follows. On $f^{-1}(\alpha, \beta)$ define the vector field X by

$$X = \frac{\text{grad } f}{\|\text{grad } f\|^2}.$$

Let ψ^t be the flow determined by X and

$$\Psi(q,t) = \psi^{t-\mu}(q).$$

Then $\partial_t f(\Psi(q,t)) = 1$ and $f(\Psi(q,\mu)) = \mu$ hence Ψ is the desired map. Moreover $\|\partial_t \Psi\| = \|\text{grad } f\|^{-1}$ and is always orthogonal to the level surface $f^{-1}(t)$. Hence if $\{\xi_1, \ldots, \xi_{n-1}, \nu\}$is an orthonormal basis in M_q, $q \in f^{-1}(t)$ with $\nu \perp f^{-1}(t)$ then

$$dV(\xi_1, \ldots, \xi_{n-1}, \nu) = \|\text{grad } f\| dA_t \wedge dt(\xi_1, \ldots, \xi_{n-1}, \partial_t).$$

(1.22) and (1.23) now follow immediately. $\square$

2 Surfaces of revolution

In this section we construct some explicit examples of two dimensional manifolds with short geodesic loops. First we construct a manifold M such that

$$\iint_M dA < \epsilon \quad \text{and} \quad l(M) < \epsilon, \tag{2.1}$$

here dA is the area element of M, $l(M)$ is the length of the shortest simple closed geodesic on M and ϵ is some arbitrary small positive number. Next, we construct a manifold N with

$$\iint_N K^2\, dA < \epsilon \quad \text{and} \quad l(N) < \epsilon. \tag{2.2}$$

The final construction is a sequence of surfaces $\{M_n\}$ satisfying

$$\iint_{M_n} dA < C_1 \quad \text{and} \quad \iint_{M_n} K^2\, dA < C_2, \tag{2.3}$$

where $C_{1,2}$ are constants and,

$$l(M_n) \to 0 \qquad \text{as} \qquad n \to \infty. \tag{2.4}$$

As the title of the chapter suggests all examples constructed are surfaces of revolution. We begin with some general facts about these surfaces.

2.1 Geodesics. Consider a curve $(h(u), 0, k(u)),\ u \in [0,1],\ h \geq 0$ in the xz-plane. When rotated around the z-axis it generates a surface S which can be written on parametric form as

$$(u,v) \mapsto (h(u)\cos v, h(u)\sin v, k(u)), \quad \text{for } (u,v) \in [0,1] \times [0,2\pi). \tag{2.5}$$

In the coordinates (u,v) the metric on S (induced by $\mathbf{R}^3$) is given by

$$ds^2 = \left(h'^2 + k'^2\right) du^2 + h^2\, dv^2. \tag{2.6}$$

Our main concern will be the curves on S of the form

$$(u,v) = \left(u_0, \frac{t}{h(u_0)}\right), \quad t \in [0, 2\pi\, h(u_0)], \quad \text{where} \quad h'(u_0) = 0. \tag{2.7}$$

These curves are geodesics. The easiest way to see that is by turning to analytical mechanics and considering S as the configuration manifold of a free particle with mass one. The Lagrangian L is then given by $L = T - U$ where T is the kinetic energy and U the potential,

$$T = \frac{1}{2}\left(\frac{ds}{dt}\right)^2, \qquad U \equiv 0. \tag{2.8}$$

Hence, along a curve as in (2.7)

$$\frac{\partial L}{\partial u} = (h'h'' + k'k'')\dot{u}^2 + hh'\dot{v}^2 = 0 \quad \text{and} \quad \frac{\partial L}{\partial v} = 0, \tag{2.9}$$

where $'$ and $\dot{}$ denotes u and t differentiation respectively. We also have

$$\frac{\partial L}{\partial \dot{u}} = \left(h'^2 + k'^2\right)\dot{u} \quad \text{and} \quad \frac{\partial L}{\partial \dot{v}} = h^2 \dot{v},$$

implying

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right) = \left(h'^2 + k'^2\right)\ddot{u} + 2\left(h'h'' + k'k''\right)\dot{u}^2 = 0, \tag{2.10}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{v}}\right) = h^2\ddot{v} + 2hh'\dot{u}\dot{v} = 0, \tag{2.11}$$

respectively. By (2.9-11) the Euler-Lagrange equation, $\frac{d}{dt}\left(\frac{\partial L}{\partial(\dot{u},\dot{v})}\right) = \frac{\partial L}{\partial(u,v)}$, is satisfied by the curves which thus represent motions of the system and hence are geodesics, compare Arnold [1], Section 45D. Note that the parameterization in (2.7) is by arclength.

2.2 Explicit formulas. Consider the curvature of a surface S given by (2.5). A straightforward calculation shows that

$$K = \frac{h'k'k'' - k'k'h''}{h(h'^2 + k'^2)^2}. \tag{2.12}$$

Furthermore the area element of S is

$$dA = \sqrt{g}\, dudv = h\sqrt{h'^2 + k'^2}\, dudv. \tag{2.13}$$

Equations (2.12) and (2.13) then gives

$$K^2\, dA = \frac{(h'k'k'' - k'k'h'')^2}{h(h'^2 + k'^2)^{7/2}}\, dudv \tag{2.14}$$

Note that if the curve $(h, 0, k)$ is parameterized by arclength then, $h'^2 + k'^2 = 1$ and hence, $h'h'' = -k'k''$. Equations (2.6), (2.12-13) then simplify to

$$\begin{aligned} ds^2 &= du^2 + h^2 dv^2, \\ K &= \frac{-h''}{h}, \\ dA &= h\, dudv. \end{aligned} \tag{2.15}$$

2.3 Three examples. At this point we can explicitly construct manifolds satisfying (2.1-4). Let $\delta > 0$ be given. For standard sphere of radius $R, \mathbf{S}^2_R$ we have $K \equiv \frac{1}{R^2}$ and $\int_{\mathbf{S}^2_R} dA = 4\pi R^2$. Hence,

$$\int_{\mathbf{S}^2_R} K^2\, dA = \frac{4\pi}{R^2}. \tag{2.16}$$

Also note that $l(\mathbf{S}^2_R) = 2\pi R$. From this it is clear that $\mathbf{S}^2_R$ satisfies (2.1) if R is small enough.

Now, to construct N of (2.2) choose $R > \sqrt{\frac{8\pi}{\epsilon}}$. Consider the circles C_1, C_2 centered at $(0,0,R)$ and $(0,0,-R)$ in the xz-plane. Let γ_1 be the part of C_1 lying in the region $x > 0, z > R$ and γ_2 be the part of C_2 lying in the region $x > 0, z < -R$. When rotated around the z-axis γ_1 and γ_2 generates two spherical caps Γ_1 and Γ_2. Take a curve σ in the xz-plane connecting γ_1 and γ_2 smoothly and passing through the point $(\delta/2\pi, 0, 0)$ in such a manner that ${\sigma_x}' = 0$ there. Revolved around the z-axis σ generates a cylindrical surface Σ. For the surface $N' = \Gamma_1 \cup \Gamma_2 \cup \Sigma$ it follows that

$$\int_{N'} K^2\, dA = \int_{\Gamma_1} K^2\, dA + \int_{\Gamma_2} K^2\, dA + \int_{\Sigma} K^2\, dA \le \frac{\delta}{2} + \int_{\Sigma} K^2\, dA. \tag{2.17}$$

Now let $\sigma(\lambda) = (\sigma_x, 0, \lambda\sigma_z) = (h, 0, \lambda k), \lambda > 1)$ and $\Sigma(\lambda)$ be the corresponding surface of revolution. By using formula (2.14) it follows that

$$\begin{aligned}\int_{\Sigma(\lambda)} K^2\, dA &= \iint \frac{\lambda^4 (h'k'k'' - k'k'h'')^2}{h(h'^2 + \lambda^2 k'^2)^{7/2}}\, dudv \\ &\le \frac{2\pi}{\lambda^3} \int \frac{(h'k'k'' - k'k'h'')^2}{hk'^2}\, du.\end{aligned} \tag{2.18}$$

We can obviously choose σ so that $k' \equiv 1$. Hence, if we let λ approach ∞ the expression in (2.18) tends to zero. The C^1-surface N constructed by putting spherical caps of radius R at the ends of $\Sigma(\lambda)$ for λ large enough then satisfies (2.2) by (2.17-18).

For (2.3-4), choose some $\delta > 0$ and let $\lambda = (u^4, 0, u)$ for $u \in [0, \delta)$ and connect this curve smoothly, with some curve σ, to the curve $\tau : (a + R\cos(u), 0, R\sin(u))$, $u \in [0, \frac{1}{2}\pi], a > \delta, R > \delta^4$. Define $\gamma' = \lambda \cup \sigma \cup \tau = (f, 0, g)$ this curve is smooth and lies in the region $x \ge 0, z > 0$ extend it to a curve γ in $x \ge 0$ by lettnig $\gamma'' = (f, 0, -g)$ and $\gamma = \gamma' \cup \gamma'' = (h, 0, k)$. Finally let $\gamma_n = (h_n, 0, k_n) = (h + \frac{1}{n}, 0, k)$ and M_n be the surface constructed by rotating γ_n around the z-axis and glueing in flat disks of radius $\frac{1}{n}$at the ends. Then clearly

$$\int_{M_n} dA \le \int_{M_1} dA = C_1 < \infty, \tag{2.19}$$

for all n and

$$\begin{aligned}\int_{M_n} K^2\, dA &< 2\pi \int \frac{(h'k'k'' - k'k'h'')^2}{h(h'^2 + k'^2)^{7/2}}\, du \\ &\le 4\pi \Big(\int_{\{u:\gamma(u)\in\tau\}} + \int_{\{u:\gamma(u)\in\sigma\}} + \int_{\{u:\gamma(u)\in\lambda\}} \Big).\end{aligned} \tag{2.20}$$

The first two of the integrals inside the paranthesis are clearly finite and the last one is

$$\int_0^\delta \frac{144u^4}{u^4(16u^6+1)^{\frac{7}{2}}}\, du = \int_0^\delta \frac{144}{(16u^6+1)^{\frac{7}{2}}}\, du < 144\delta. \tag{2.21}$$

Thus if we call the sum in (2.20) C_2 and note that $h'_n = 0$ at the point $(\frac{1}{n}, 0, 0)$ the construction of a sequence of manifolds satisfying (2.3-4) is complete by (2.20-21) and the section on geodesics.

In this chapter 2-manifolds M homeomorphic to $\mathbf{S}^2$ and with non-negative curvature are considered.

3.1 Fermi coordinates. We start by letting $c : [0, L] \to M$ be a unit speed parameterization of some geodesic loop γ in M and N be the unit normal along c. For each $v \in [0, L]$ let $\beta(v)$ be the cut distance from the curve γ along the geodesic $u \mapsto \exp(uN(v))$. Define the map $F(v, u)$ on the set of ordered pairs (v, u) with $v \in [0, L]$ and $0 \le u \le \beta(v)$ by

$$F(v, u) = \exp_{c(v)}(uN(v)) .$$

Then (v, u) are Fermi coordinates on the disk $D \subset M$ bounded by γ and with inner normal N.

Proposition. *In these coordinates the metric ds^2, the Gaussian curvature K and the area element dA are given by*

$$ds^2 = E^2\, dv^2 + du^2, \quad K = \frac{-E_{uu}}{E} \text{ and } \quad dA = E\, dvdu, \tag{3.1}$$

respectively. Moreover, by the fact that γ is a geodesic, E satisfies the initial conditions

$$E(v, 0) \equiv 1, \quad E_u(v, 0) \equiv 0. \tag{3.2}$$

Proof: If v is fixed, then $F(v, u)$ is a geodesic parameterized by arclength. Hence

$$\langle \partial_u F, \partial_u F \rangle = 1.$$

By the Gauss Lemma and the fact that $\partial_u \perp \partial_v$ it follows that

$$\langle \partial_u F, \partial_v F \rangle = 0.$$

By defining E as $\langle \partial_v F, \partial_v F\rangle^{1/2}$ the expressions for the metric and area element follows. To establish the initial conditions satisfied by E we first note that $F(v, 0)$ is a geodesic parameterized by arclength. Therefore $E(v, 0) = 1$. Then we compute

$$\begin{aligned} E_u(v, u) =& \partial_u F \langle \partial_v F, \partial_v F\rangle^{1/2} \\ =& \langle \nabla_u \partial_v F, \partial_v F\rangle / \langle \partial_v F, \partial_v F\rangle^{1/2} \qquad \text{[by (1.3)]} \\ =& \langle \nabla_v \partial_u F, \partial_v F\rangle / \langle \partial_v F, \partial_v F\rangle^{1/2} \qquad \text{[compare (1.13)]} \\ =& \Big(\partial_v F \langle \partial_u F, \partial_v F\rangle - \langle \partial_u F, \nabla_v \partial_v F\rangle \Big) \Big/ \langle \partial_v F, \partial_v F\rangle^{1/2}, \end{aligned}$$

where the last expression clearly vanishes for $u = 0$. Finally we demonstrate the equation involving the curvature by differentiating one more time

$$\begin{aligned}
E_{uu} &= \partial_u F \partial_u F \langle \partial_v F, \partial_v F\rangle^{1/2} \\
&= \partial_u F \Big(\langle \nabla_u \partial_v F, \partial_v F\rangle / \langle \partial_v F, \partial_v F\rangle^{1/2} \Big) \\
&= \Big(\partial_u F \langle \nabla_u \partial_v F, \partial_v F\rangle \Big) \Big/ \ \langle \partial_v F, \partial_v F\rangle^{1/2} - \langle \nabla_u \partial_v F, \partial_v F\rangle^2 / \langle \partial_v F, \partial_v F\rangle^{3/2}.
\end{aligned}$$

Now,

$$\partial_u F \langle \nabla_u \partial_v F, \partial_v F\rangle = \langle \nabla_u \nabla_u \partial_v F, \partial_v F\rangle + \langle \nabla_u \partial_v F, \nabla_u \partial_v F\rangle,$$

and

$$\begin{aligned}
\langle \nabla_u \partial_v F, \nabla_u \partial_v F\rangle &= \langle \nabla_u \partial_v F, \partial_v F\rangle^2 / \langle \partial_v F, \partial_v F\rangle + \langle \nabla_u \partial_v F, \partial_u F\rangle^2 / \langle \partial_u F, \partial_u F\rangle \\
&= \langle \nabla_u \partial_v F, \partial_v F\rangle^2 / \langle \partial_v F, \partial_v F\rangle + (\partial_u F \langle \partial_v F, \partial_u F\rangle)^2 - (\langle \partial_v F, \nabla_u \partial_u F\rangle)^2 \\
&= \langle \nabla_u \partial_v F, \partial_v F\rangle^2 / \langle \partial_v F, \partial_v F\rangle.
\end{aligned}$$

Hence

$$\begin{aligned}
E_{uu} =& \langle \nabla_u \nabla_u \partial_v F, \partial_v F\rangle / \langle \partial_v F, \partial_v F\rangle^{1/2} \\
=& \langle \nabla_u \nabla_v \partial_u F, \partial_v F\rangle / \langle \partial_v F, \partial_v F\rangle^{1/2} \qquad \text{[compare (2.13)]} \\
=& \langle \nabla_u \nabla_v \partial_u F - \nabla_v \nabla_u \partial_u F, \partial_v F\rangle / \langle \partial_v F, \partial_v F\rangle^{1/2} \qquad \text{[since } \nabla_u \partial_u F = 0] \\
=& \langle R(\partial_u F, \partial_v F)\partial_u F, \partial_v F\rangle / \langle \partial_v F, \partial_v F\rangle^{1/2} \qquad \text{[since } [\partial_u, \partial_v] = 0] \\
=& - \langle \partial_v F, \partial_v F\rangle^{1/2} \langle R(\partial_u F, \partial_v F)\partial_v F, \partial_u F\rangle / \langle \partial_v F, \partial_v F\rangle = -EK.
\end{aligned}$$

□

3.2 The L^2-norm of curvature. In this section we show that if M posesses a short geodesic loop the L^2-norm of the curvature on M is large. The expression for the curvature in (3.1) and (3.2) explains the use of the follwing lemma.

LEMMA. *Let $k(t)$ be a non-negative function on $[0, \infty)$ and let $y(t)$ be defined by the initial value problem*

$$y''(t) + k(t)y(t) = 0, \quad y(0) = 1, \quad y'(0) = 0. \tag{3.3}$$

Denote the smallest positive zero of y by α (it may be that $\alpha = \infty$). Then y' is decreasing on $(0, \alpha)$. Furthermore if $\beta \le \alpha$ and if

(a) $\quad - y'(\beta) \le 1$ *we have*

$$\int_0^\beta k^2(t)y(t)\, dt \ge \int_0^\beta k(t)y(t)\, dt + \frac{1}{2}\Big(y^2(\beta) + {y'}^2(\beta) - 1\Big) \tag{3.4}$$

(b) $\quad -y'(\beta) > 1$ *we have*

$$\int_0^\beta k^2(t)y(t)\,dt \geq \int_0^\beta k(t)y(t)\,dt + \frac{1}{2}\left(\frac{y(\beta)^2}{-y'(\beta)} - y'(\beta) - \frac{1}{-y'(\beta)}\right) \tag{3.5}$$

PROOF: Note that $y(t) \geq 0$ on $[0, \alpha)$ which together with (3.3) immediately gives y' decreasing. We now turn to (a) and (b). First

$$\frac{1}{2}\left(y^2 + y'^2\right)' = yy' + y'y'' = (1-k)yy'. \tag{3.6}$$

Then $k(k-1) \geq k-1$. Hence,

$$\int_0^\beta k^2y - ky\,dt \geq \int_0^\beta (k-1)y\,dt. \tag{3.7}$$

Assume, as in (a), that $-y'(\beta) \leq 1$ then (3.7), (3.6) and the fact that y' is decreasing gives

$$\int_0^\beta k^2y - ky\,dt \geq \int_0^\beta (k-1)y\,dt \geq \int_0^\beta (k-1)y(-y')\,dt = \frac{1}{2}\left(y^2 + y'^2\right)\Big|_0^\beta,$$

which is (a). Now, if we instead assume, as in (b), that $-y'(\beta) > 1$ we similarly get

$$\begin{aligned}\int_0^\beta k^2y - ky\,dt &\geq \int_0^\beta (k-1)y\,dt \\ &\geq \frac{1}{-y'(\beta)}\int_0^\beta (k-1)y(-y')\,dt = \frac{1}{-2y'(\beta)}\left(y^2 + y'^2\right)\Big|_0^\beta,\end{aligned}$$

and this is (b). $\square$

By computing $\iint K^2\,dA$ in Fermi coordinates and using (3.4-5) we are able to prove yet another lemma.

LEMMA. *Let M be a manifold homeomorphic to the sphere. If the Gaussian curvature K on M satisfies $K \geq 0$ and there is a closed geodesic in M of length 2π then*

$$\iint_M K^2\,dA \geq \frac{7\pi}{2} \tag{3.8}$$

PROOF: Let γ be the closed geodesic of length 2π. Introduce Fermi coordinates based on γ on the disk $D \subset M$, just as in the begining of this section. Using (3.2) we see that the

v-coordinate ranges from 0 to 2π. By the fact that γ is a geodesic the boundary term in the Gauss-Bonnet theorem vanishes and hence using (3.1-2) we have

$$\begin{aligned} 2\pi = \iint_D K\, dA &= \int_0^{2\pi} \int_0^{\beta(v)} \frac{-E_{uu}}{E} E\, dudv \\ &= \int_0^{2\pi} \int_0^{\beta(v)} -E_{uu}\, dudv = \int_0^{2\pi} -E_u(v, \beta(v))\, dv. \end{aligned} \tag{4.9}$$

Next, divide the interval $[0, 2\pi]$ into the measurable sets
$\Omega_1 = \{v \in [0, 2\pi] : -E_u(v, \beta(v)) \leq 1\}$ and $\Omega_2 = \{v \in [0, 2\pi] : -E_u(v, \beta(v)) > 1\}$. By (3.1) we have that $E(v, u)$ for fixed v satisfies the ordinary differential equation

$$E_{uu} + KE = 0, \tag{3.11}$$

and we may apply (3.4) and (3.5) to see that

$$\begin{aligned} \int_{\Omega_1} \int_0^{\beta(v)} K^2 E\, dudv \geq & \int_{\Omega_1} \int_0^{\beta(v)} KE\, dudv \\ &+ \int_{\Omega_1} \frac{1}{2} \left(E^2(v, \beta(v)) + E_u^2(v, \beta(v)) - 1\right)\, dv, \end{aligned} \tag{3.12}$$

and that

$$\begin{aligned} \int_{\Omega_2} \int_0^{\beta(v)} K^2 E\, dudv \geq & \int_{\Omega_2} \int_0^{\beta(v)} KE\, dudv \\ &+ \int_{\Omega_2} \frac{1}{2} \frac{E^2(v, \beta(v))}{-E_u(v, \beta(v))} - E_u(v, \beta(v)) + \frac{1}{E_u(v, \beta(v))}\, dv. \end{aligned} \tag{3.13}$$

Adding (3.12) and (3.13) we get

$$\begin{aligned} \int_0^{2\pi} \int_0^{\beta(v)} & K^2 E\, dudv \\ & \geq \int_0^{2\pi} \int_0^{\beta(v)} KE\, dudv \\ & \quad + \int_{\Omega_1} \frac{1}{2} \left(E^2(v, \beta(v)) + E_u^2(v, \beta(v)) - 1\right)\, dv \\ & \quad + \int_{\Omega_2} \frac{1}{2} \left(\frac{E^2(v, \beta(v))}{-E_u(v, \beta(v))} - E_u(v, \beta(v)) + \frac{1}{E_u(v, \beta(v))}\right)\, dv \\ & = \iint_D K\, dA \\ & \quad + \frac{1}{2} \left(\int_{\Omega_1} E^2(v, \beta(v))\, dv + \int_{\Omega_2} \frac{E^2(v, \beta(v))}{-E_u(v, \beta(v))}\, dv\right) \\ & \quad + \frac{1}{2} \left(\int_{\Omega_1} E_u^2(v, \beta(v)) - 1\, dv + \int_{\Omega_2} -E_u(v, \beta(v)) + \frac{1}{E_u(v, \beta(v))}\, dv\right) \end{aligned} \tag{3.14}$$

Noting that $E_u^2 \geq -E_u - 1/4$ and remembering the definitions of $\Omega_{1,2}$ we get from (3.14)

$$\begin{aligned}\iint_D K^2\,dA &= \int_0^{2\pi}\int_0^{\beta(v)} K^2 E\,dudv\\ &\geq \iint_D K\,dA\\ &\quad + \frac{1}{2}\left(\int_{\Omega_1} -E_u(v,\beta(v)) - \tfrac{5}{4}\,dv + \int_{\Omega_2} -E_u(v,\beta(v)) - 1\,dv\right)\\ &\geq 2\pi + \frac{1}{2}\left(\int_0^{2\pi} -E_u(v,\beta(v))\,dv - \tfrac{5}{2}\pi\right) = \tfrac{7}{4}\pi.\end{aligned}$$

Here we have used equation (3.9). A corresponding result naturally holds on the other disk bounded by γ. The lemma is proved. □

We are now in a position to prove our main result.

THEOREM. *Let M be a manifold homeomorphic to $\mathbf{S}^2$ on which the Gaussian curvature is non-negative. If $l(M)$ is the length of the shortest geodesic loop in M then*

$$\iint_M K^2\,dA \geq \tfrac{7}{2}\pi\left(\frac{2\pi}{l(M)}\right)^2 \tag{3.15}$$

PROOF: Let g denote the metric on M. Consider the metric

$$g' = \left(\frac{2\pi}{l(M)}\right) g.$$

Let M' be the Riemannian manifold which topologically is M but has the metric g' . Then M' has a closed geodesic of length 2π and non-negative curvature. Hence by the lemma

$$\iint_{M'} K_{M'}^2\,dA \geq \tfrac{7}{2}\pi.$$

However,

$$\iint_M K_M^2\,dA = \left(\frac{2\pi}{l(M)}\right)^2 \iint_{M'} K_{M'}^2\,dA.$$

This proves the theorem. □

3.3 Area. Let γ_0 be a closed geodesic in M. Define the function $f : M \to \mathbf{R}$ by

$$f(p) = d(p,\gamma_0), \tag{3.16}$$

where d is the distance function on M. By expressing f in Fermi coordinates based on γ_0 it is clear that grad f is defined almost everywhere. Note that the cut locus of γ_0 is a set of measure zero, and that $\|\text{grad } f\| = 1$ where it is defined. The curve γ_0 divides M into two parts M_+ and M_-. Define the function $g : M \to \mathbf{R}$ by

$$g(p) = \begin{cases} f(p) & \text{for } p \in M_+ \\ -f(p) & \text{for } p \in M_- \end{cases}, \tag{3.18}$$

then grad $g = \pm$grad f. Moreover if $R_\pm = \sup\{f(p) : p \in M_\pm\}$ it is clear that $g : M \to [-R_-, R_+]$. Let γ_r denote the closed curve (or union of closed curves)

$$\{p \in M : g(p) = r\}. \tag{3.19}$$

Define the function $L : [-R_-, R_+]$ by

$$L(r) = \int_{\gamma_r} ds. \tag{3.20}$$

Using the notation above we prove the following lemma.

LEMMA. *If $K \geq 0$ on M then L has a global maximum for $r = 0$.*

PROOF: Letting $c : [a, b] \to M$ be an arclength parameterization of γ_r and using the theorem on the first variation of arclength we find that

$$L'(r) = -\int_a^b \langle \nabla_t c'(t), F_* \partial_u(t) \rangle \, dt, \tag{3.21}$$

where F_* is the differential of the 'Fermi coordinate map' at the point (v, u) corresponding to $c(t)$. If we define

$$\sigma_r = \{p \in M : r \leq g(p) \leq R_+\}, \tag{3.22}$$

then, σ_r is a union of disks with boundary γ_r. The orientation of $\partial\sigma_r$ is such that $\{\mathbf{n}, c'\}$ is a positively oriented basis in the tangent spaces along γ_r ($\mathbf{n}$ is the outer normal of σ_r). It is then evident that $\imath c' = F_* \partial_u$, hence by (1.19)

$$L'(r) = -\int_{\partial\sigma} \kappa_g \, ds. \tag{3.23}$$

By the Gauss-Bonnet theorem we then have

$$\iint_{\sigma_r} K \, dA + \int_{\partial\sigma} \kappa_g \, ds = 2\pi\chi(\sigma). \tag{3.24}$$

Now, $\sigma_r \subset \sigma_0$ and $K \geq 0$, so

$$\iint_{\sigma_0} K \, dA = 2\pi,$$

implies that

$$\iint_{\sigma_r} K\, dA \leq 2\pi.$$

Moreover, since σ_r is a disjoint union of disks, $\chi(\sigma_r) \geq 1$. Hence (3.24) and (3.23) implies

$$L'(r) \leq 0 \quad \text{for} \quad 0 \leq r \leq R_+.$$

By exactly the same reasoning on M_- we conclude that

$$L'(r) \geq 0 \quad \text{for} \quad 0 \geq r \geq -R_-.$$

Noting that γ_0 is a geodesic it follows from the theorem on first variation of arclength that $L'(0) = 0$. The lemma is thus proven. $\square$

This lemma enables us to prove the following result, that is known in greater generality, see Chavel [3], Corollary 7.2. However, the proof for the more general version requires other techniques than those used here.

THEOREM. *Let M be a manifold, homeomorphic to the sphere and with non-negative Gaussian curvature. If $l(M)$ is the length of the shortest geodesic loop in M and $d(M)$ is the diameter of M then*

$$A \leq 2d(M)\, l(M) \tag{3.25}$$

PROOF: Using the coarea formula we find that

$$A = \iint_M \|\text{grad } g\|\, dA = \int_{-R_-}^{R_+} dr \int_{\gamma_r} ds = \int_{-R_-}^{R_+} L(r)\, dr$$

$$\leq (R_+ + R_-)\, L(0) \leq 2d(M)\, l(M).$$

Here the first inequlity follows from the lemma and the second by the fact that (of course) $R_\pm \leq d(M)$. $\square$

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