Integr. Equ. Oper. Theory
DOI 10.1007/s00020-017-2396-4

Integral Equations and Operator Theory

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Invertibility Properties of Singular Integral Operators Associated with the Lamé and Stokes Systems on Infinite Sectors in Two Dimensions

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Abstract. In this paper we establish sharp invertibility results for the elastostatics and hydrostatics single and double layer potential type operators acting on $L^p(\partial\Omega)$, $1 < p < \infty$, whenever Ω is an infinite sector in $\mathbb{R}^2$. This analysis is relevant to the layer potential treatment of a variety of boundary value problems for the Lamé system of elastostatics and the Stokes system of hydrostatics in the class of curvilinear polygons in two dimensions, such as the Dirichlet, the Neumann, and the Regularity problems. Mellin transform techniques are used to identify the critical integrability indices for which invertibility of these layer potentials fails. Computer-aided proofs are produced to further study the monotonicity properties of these indices relative to parameters determined by the aperture of the sector Ω and the differential operator in question.

Mathematics Subject Classification. Primary 35J25, 35J47, 42A38; Secondary 45E05, 45B05.

Keywords. Lamé system, Stokes system, Mellin transform, Hardy kernel operator, Single layer potential, Double layer potential, Conormal derivative, Pseudo-stress conormal derivative, Infinite sector, Interval analysis, Computer-aided proof, Validated numerics.

1. Introduction

Let Ω be a domain in $\mathbb{R}^n$. Some of the classical boundary value problems associated with the Lamé system in Ω are the Dirichlet, Neumann, and Regularity problems. When these problems are considered in the $L^p(\partial\Omega)$ context, $1 < p < \infty$, one seeks an elastic field $\vec{u} \in \mathcal{C}^2(\Omega)$ such that

This work was supported in part by the SQuaRE program at the American Institute of Mathematics, by the NSF DMS Grant 1458138, the Simons Foundation Grant 318658, and the Swedish Research Council Grant 2008-7510.

Birkhäuser

Published online: 24 August 2017

$$\begin{cases} \mathcal{L}\vec{u} = \vec{0} \quad \text{in} \quad \Omega, \\ \vec{u}|_{\partial\Omega} = \vec{f} \in L^p(\partial\Omega), \\ M(\vec{u}) \in L^p(\partial\Omega), \end{cases} \tag{1.1}$$

in the case of the Dirichlet problem,

$$\begin{cases} \mathcal{L}\vec{u} = \vec{0} \quad \text{in} \quad \Omega, \\ \partial_{\nu_{A(r)}}\vec{u} = \vec{f} \in L^p(\partial\Omega), \\ M(\nabla\vec{u}) \in L^p(\partial\Omega), \end{cases} \tag{1.2}$$

in the case of the Neumann problem, and

$$\begin{cases} \mathcal{L}\vec{u} = \vec{0} \quad \text{in} \quad \Omega, \\ \vec{u}|_{\partial\Omega} = \vec{f} \in L^p_1(\partial\Omega), \\ M(\nabla\vec{u}) \in L^p(\partial\Omega), \end{cases} \tag{1.3}$$

in the case of the Regularity problem. Here $\mathcal{L}$ is the Lamé differential operator from (3.1), $\cdot|_{\partial\Omega}$ denotes the non-tangential restriction to the boundary as in (2.3), M denotes the non-tangential maximal operator introduced in (2.5), $\partial_{\nu_{A(r)}}$ denotes the conormal derivative from (3.8) and (3.9), and the Sobolev space of order one, $L^p_1(\partial\Omega)$, is as in (2.6).

In a similar vein, analogous problems to (1.1)–(1.3) are posed for the linearized, homogeneous, time independent Navier–Stokes equations, i.e., the Stokes system. They reside in looking for a velocity field $\vec{u} \in \mathcal{C}^2(\Omega)$ and a pressure function $\mathbf{p} \in \mathcal{C}^1(\Omega)$ such that

$$\begin{cases} \triangle\vec{u} = \nabla\mathbf{p} \quad \text{in} \quad \Omega, \\ \operatorname{div}\vec{u} = 0 \quad \text{in} \quad \Omega, \\ \vec{u}|_{\partial\Omega} = \vec{f} \in L^p(\partial\Omega), \\ M(\vec{u}), M(\mathbf{p}) \in L^p(\partial\Omega), \end{cases} \tag{1.4}$$

in the case of the Dirichlet problem,

$$\begin{cases} \triangle\vec{u} = \nabla\mathbf{p} \quad \text{in} \quad \Omega, \\ \operatorname{div}\vec{u} = 0 \quad \text{in} \quad \Omega, \\ \partial_{\nu_{A(r)}}\{\vec{u}, \mathbf{p}\} = \vec{f} \in L^p(\partial\Omega), \\ M(\nabla\vec{u}), M(\mathbf{p}) \in L^p(\partial\Omega), \end{cases} \tag{1.5}$$

in the case of the Neumann problem, and

$$\begin{cases} \triangle\vec{u} = \nabla\mathbf{p} \quad \text{in} \quad \Omega, \\ \operatorname{div}\vec{u} = 0 \quad \text{in} \quad \Omega, \\ \vec{u}|_{\partial\Omega} = \vec{f} \in L^p_1(\partial\Omega), \\ M(\nabla\vec{u}), M(\mathbf{p}) \in L^p(\partial\Omega), \end{cases} \tag{1.6}$$

in the case of the Regularity problem. The conormal derivative $\partial_{\nu_{A(r)}}\{\vec{u}, \mathbf{p}\}$ in (1.5) is as introduced in (4.3).

Boundary value problems for the Lamé and Stokes systems in non-smooth domains have been investigated in numerous contexts and the mathematical and engineering literature on these topics is very ample. Some of the classical references are the monographs by Deuring [10], Kupradze et al. [23,24], Ladyzhenskaya [25], and Maz'ya [29]. The case of the Lamé system in Lipschitz domains and domains with isolated singularities has been considered by, among others, Bacuta and Bramble [3], Dahlberg et al. [6–8], Lewis [27], Mayboroda and Mitrea [28], Maz'ya et al. [20,21,29–31], and Shen [41]. Boundary value problems for the Stokes system in non-smooth domains have been treated by Dauge [9], Deuring [11], Fabes et al. [14], Kellogg and Osborn [17], Kilty [18], Kohr and Wendland [19], Maz'ya and collaborators [20–22,29,30], Mitrea and Wright [34], and Shen [41,42].

Considering for instance the Regularity problem, when Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$, with connected boundary and $p = 2$, the well-posedness of the boundary value problem (1.3) has been studied by Dahlberg et al. [7]. Building on the work in [8], the well-posedness of (1.3) in the class of bounded Lipschitz domains in $\mathbb{R}^3$ was further investigated by Dahlberg and Kenig [6] who showed there exists $\varepsilon = \varepsilon(\Omega) > 0$, depending only on the Lipschitz character of the domain Ω, such that the problem (1.3) is well-posed whenever $p \in (1, 2+\varepsilon)$. This integrability range is sharp in the class of bounded Lipschitz domains in $\mathbb{R}^3$. The regularity problem (1.6) for the Stokes system in the class of bounded Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, with connected boundary has been treated by Fabes et al. [14] when $p = 2$. More recently, as a byproduct of their study of the transmission boundary value problem for the Stokes system, Mitrea and Wright established in [34] optimal well-posedness results for (1.4)–(1.6) in Lipschitz domains with arbitrary topology, in all space dimensions.

The focus of this paper is to establish sharp invertibility results for singular integral operators naturally associated with problems (1.1)–(1.3) and (1.4)–(1.6), stated in the class of infinite sectors in two dimensions. Our main result regarding layer potential operators associated with the Lamé system is

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^2$ be an infinite sector of aperture $\theta \in (0, 2\pi)$, assume $p \in (1,\infty)$, and consider the Lamé system of elastostatics in Ω as in (3.1) with Lamé moduli $\mu > 0$ and $\lambda + \mu \geq 0$. Introduce*

$$\kappa := \frac{\mu+\lambda}{3\mu+\lambda}. \tag{1.7}$$

Then $\kappa \in [0,1)$ and the following hold:

(A) *If $\kappa \in (0,1)$, there exist*

$$\begin{aligned} &p_1(\theta,\kappa) \in \left(2, \tfrac{2\pi-\theta}{\pi-\theta}\right) \quad \text{and} \quad p_2(\theta,\kappa) \in \left(\tfrac{2\pi-\theta}{\pi-\theta}, \infty\right), \quad \text{if } \theta \in (0,\pi), \\ &p_3(\theta,\kappa) \in \left(\tfrac{\theta}{\theta-\pi}, \infty\right) \quad \text{and} \quad p_4(\theta,\kappa) \in \left(2, \tfrac{\theta}{\theta-\pi}\right), \quad \text{if } \theta \in (\pi,2\pi), \end{aligned} \tag{1.8}$$

such that

$$p_1(\theta,\kappa) = p_4(2\pi-\theta,\kappa) \quad \text{and} \quad p_2(\theta,\kappa) = p_3(2\pi-\theta,\kappa), \quad \forall\, \theta \in (0,\pi), \tag{1.9}$$

with the following significance.

(A.1) *With $S^{Lamé}$ denoting the single layer potential operator in (3.12), there holds*

$$\begin{gathered} S^{Lamé}: L^p(\partial\Omega) \to \dot{L}^p_1(\partial\Omega) \quad \textit{is invertible} \\ \textit{when } \theta \in (0,\pi) \textit{ if and only if } p \in (1,\infty)\setminus\{p_1(\theta,\kappa), p_2(\theta,\kappa)\}, \end{gathered} \tag{1.10}$$

and

$$\begin{gathered} S^{Lamé}: L^p(\partial\Omega) \to \dot{L}^p_1(\partial\Omega) \quad \textit{is invertible} \\ \textit{when } \theta \in (\pi,2\pi) \textit{ if and only if } p \in (1,\infty)\setminus\{p_3(\theta,\kappa), p_4(\theta,\kappa)\}. \end{gathered} \tag{1.11}$$

(A.2) *With $K^{Lamé}_{\Psi}$ standing for the boundary-to-boundary pseudo-stress double layer potential operator from (3.27), the operators*

$$\begin{gathered} \pm\tfrac{1}{2}I + K^{Lamé}_{\Psi}: L^p(\partial\Omega) \to L^p(\partial\Omega) \quad \textit{are invertible} \\ \textit{when } \theta \in (0,\pi) \textit{ if and only if } p \in (1,\infty)\setminus\{p_1'(\theta,\kappa), p_2'(\theta,\kappa)\}, \end{gathered} \tag{1.12}$$

and the operators

$$\begin{gathered} \pm\tfrac{1}{2}I + K^{Lamé}_{\Psi}: L^p(\partial\Omega) \to L^p(\partial\Omega) \quad \textit{are invertible} \\ \textit{when } \theta \in (\pi,2\pi) \textit{ if and only if } p \in (1,\infty)\setminus\{p_3'(\theta,\kappa), p_4'(\theta,\kappa)\}. \end{gathered} \tag{1.13}$$

Here for each $j \in \{1,\dots,4\}$, $p_j'(\theta,\kappa)$ stands for the conjugate exponent of $p_j(\theta,\kappa)$.

(A.3) *With $\partial_{\nu_\Psi} := \frac{\partial}{\partial_{\nu_\Psi}}$ standing for the pseudo-stress conormal derivative from (3.10), and with $\mathcal{D}^{Lamé}_{\Psi}$ denoting the boundary-to-domain pseudo-stress double layer potential operator from (3.26), one has that*

$$\begin{gathered} \partial_{\nu_\Psi}\mathcal{D}^{Lamé}_{\Psi}: \dot{L}^p_1(\partial\Omega) \to L^p(\partial\Omega) \quad \textit{is invertible} \\ \textit{when } \theta \in (0,\pi) \textit{ if and only if } p \in (1,\infty)\setminus\{p_1(\theta,\kappa), p_2(\theta,\kappa)\}, \end{gathered} \tag{1.14}$$

and

$$\begin{gathered} \partial_{\nu_\Psi}\mathcal{D}^{Lamé}_{\Psi}: \dot{L}^p_1(\partial\Omega) \to L^p(\partial\Omega) \quad \textit{is invertible} \\ \textit{when } \theta \in (\pi,2\pi) \textit{ if and only if } p \in (1,\infty)\setminus\{p_3(\theta,\kappa), p_4(\theta,\kappa)\}. \end{gathered} \tag{1.15}$$

(B) *If $\kappa = 0$ then:*

(B.1) *The operator*

$$\begin{gathered} S^{Lamé}: L^p(\partial\Omega) \to \dot{L}^p_1(\partial\Omega) \quad \textit{is invertible} \\ \textit{when } \theta \in (0,\pi) \textit{ if and only if } p \in (1,\infty)\setminus\left\{\tfrac{2\pi-\theta}{\pi-\theta}\right\} \\ \textit{and when } \theta \in (\pi,2\pi) \textit{ if and only if } p \in (1,\infty)\setminus\left\{\tfrac{\theta}{\theta-\pi}\right\}. \end{gathered} \tag{1.16}$$

(B.2) *The operators*

$$\begin{gathered} \pm\tfrac{1}{2}I + K^{Lamé}_{\Psi}: L^p(\partial\Omega) \to L^p(\partial\Omega) \quad \textit{are invertible} \\ \textit{when } \theta \in (0,\pi) \textit{ if and only if } p \in (1,\infty)\setminus\left\{\tfrac{2\pi-\theta}{\pi}\right\} \\ \textit{and when } \theta \in (\pi,2\pi) \textit{ if and only if } p \in (1,\infty)\setminus\left\{\tfrac{\theta}{\pi}\right\}. \end{gathered} \tag{1.17}$$

(B.3) *The operator*

$$\begin{gathered} \partial_{\nu_\Psi}\mathcal{D}^{Lamé}_{\Psi}: \dot{L}^p_1(\partial\Omega) \to L^p(\partial\Omega) \quad \textit{is invertible} \\ \textit{when } \theta \in (0,\pi) \textit{ if and only if } p \in (1,\infty)\setminus\left\{\tfrac{2\pi-\theta}{\pi-\theta}\right\} \\ \textit{and when } \theta \in (\pi,2\pi) \textit{ if and only if } p \in (1,\infty)\setminus\left\{\tfrac{\theta}{\theta-\pi}\right\}. \end{gathered} \tag{1.18}$$

(C) *For each $\kappa \in [0,1)$ one has:*

(C.1) *The operator*

$$S^{Lamé} : L^p(\partial\Omega) \to \dot{L}_1^p(\partial\Omega) \text{ is invertible} \\ \text{when } \theta = \pi \text{ for all } p \in (1, \infty). \tag{1.19}$$

(C.2) *The operators*

$$\pm\tfrac{1}{2}I + K_\Psi^{Lamé} : L^p(\partial\Omega) \to L^p(\partial\Omega) \text{ are invertible} \\ \text{when } \theta = \pi \text{ for all } p \in (1, \infty). \tag{1.20}$$

(C.3) *The operator*

$$\partial_{\nu_\Psi} \mathcal{D}_\Psi^{Lamé} : \dot{L}_1^p(\partial\Omega) \to L^p(\partial\Omega) \text{ is invertible} \\ \text{when } \theta = \pi \text{ for all } p \in (1, \infty). \tag{1.21}$$

Before stating a similar result regarding the hydrostatics layer potential operators, let us consider the function

$$f : [0, \pi] \longrightarrow \mathbb{R}, \qquad f(\theta) := \sin\theta + (2\pi - \theta) \cdot \cos\theta. \tag{1.22}$$

A simple differentiation shows that $f'(\theta) = -(2\pi - \theta) \cdot \sin\theta < 0$ on $(0, \pi)$, and consequently f is strictly decreasing on $(0, \pi)$. Combined with the fact that $f(\pi/2) = 1$ and $f(2\pi/3) = \frac{\sqrt{3}}{2} - \frac{2\pi}{3} < 0$, we obtain that

$$\text{there exists a unique } \theta_o \in [0, \pi] \\ \text{such that } \sin\theta_o + (2\pi - \theta_o) \cdot \cos\theta_o = 0, \tag{1.23}$$

and

$$\theta_o \in (\pi/2, 2\pi/3). \tag{1.24}$$

In addition

$$f(\theta) > 0 \text{ whenever } \theta \in [0, \theta_o) \text{ and } f(\theta) \leq 0 \text{ whenever } \theta \in [\theta_o, \pi]. \tag{1.25}$$

In fact, using a computer-assisted proof (see Lemma 5.4) it can be shown that

$$\theta_o \in [1.78977584927052, 1.78977584927053]. \tag{1.26}$$

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^2$ be an infinite sector of aperture $\theta \in (0, 2\pi)$, assume $p \in (1, \infty)$, and recall θ_o from (1.23)–(1.25). Then the following hold.*

(A) *Suppose $\theta \in (0, \theta_o) \cup (2\pi - \theta_o, 2\pi)$. Then there exist integrability indexes $p_1(\theta), p_2(\theta), p_3(\theta), p_4(\theta) \in (2, \infty)$ such that*

$$p_1(\theta) \in \left(2, \tfrac{2\pi-\theta}{\pi-\theta}\right) \text{ and } p_2(\theta) \in \left(\tfrac{2\pi-\theta}{\pi-\theta}, \infty\right), \text{ if } \theta \in (0, \theta_o), \\ p_3(\theta) \in \left(\tfrac{\theta}{\theta-\pi}, \infty\right) \text{ and } p_4(\theta) \in \left(2, \tfrac{\theta}{\theta-\pi}\right), \text{ if } \theta \in (2\pi - \theta_o, 2\pi), \tag{1.27}$$

and

$$p_1(\theta) = p_4(2\pi - \theta) \text{ and } p_2(\theta) = p_3(2\pi - \theta), \quad \forall\, \theta \in (0, \theta_o), \tag{1.28}$$

with the following significance.

(A.1) *With S^{Stokes} standing for the operator in (4.12), there holds*

$$S^{Stokes} : L^p(\partial\Omega) \to \dot{L}_1^p(\partial\Omega) \text{ is invertible} \\ \text{when } \theta \in (0, \theta_o) \text{ if and only if } p \in (1, \infty) \backslash \{p_1(\theta), p_2(\theta)\}, \tag{1.29}$$

and

$$\begin{gathered} S^{Stokes} : L^p(\partial\Omega) \to \dot{L}^p_1(\partial\Omega) \quad \textit{is invertible} \\ \textit{when } \theta \in (2\pi - \theta_o, 2\pi) \textit{ if and only if } p \in (1,\infty)\backslash\{p_3(\theta), p_4(\theta)\}. \end{gathered} \tag{1.30}$$

(A.2) *With* K_Ψ^{Stokes} *denoting the boundary-to-boundary pseudo-stress double layer potential operator from* (4.15)*, the operators*

$$\begin{gathered} \pm\tfrac{1}{2}I + K_\Psi^{Stokes} : L^p(\partial\Omega) \to L^p(\partial\Omega) \quad \textit{are invertible} \\ \textit{when } \theta \in (0, \theta_o) \textit{ if and only if } p \in (1,\infty)\backslash\{p_1'(\theta), p_2'(\theta)\}, \end{gathered} \tag{1.31}$$

and the operators

$$\begin{gathered} \pm\tfrac{1}{2}I + K_\Psi^{Stokes} : L^p(\partial\Omega) \to L^p(\partial\Omega) \quad \textit{are invertible} \\ \textit{when } \theta \in (2\pi - \theta_o, 2\pi) \textit{ if and only if } p \in (1,\infty)\backslash\{p_3'(\theta), p_4'(\theta)\}. \end{gathered} \tag{1.32}$$

Here for each $j \in \{1, \dots, 4\}$, $p_j'(\theta)$ *stands for the conjugate exponent of* $p_j(\theta)$.

(A.3) *With* ∂_{ν_Ψ} *standing for the pseudo-stress conormal derivative from* (4.3)–(4.4)*, and with* $\mathcal{D}_\Psi^{Stokes}$ *standing for the boundary-to-domain pseudo-stress double layer potential operator from* (4.14)*, one has that*

$$\begin{gathered} \partial_{\nu_\Psi}\mathcal{D}_\Psi^{Stokes} : \dot{L}^p_1(\partial\Omega) \to L^p(\partial\Omega) \quad \textit{is invertible} \\ \textit{when } \theta \in (0, \theta_o) \textit{ if and only if } p \in (1,\infty)\backslash\{p_1(\theta), p_2(\theta)\}, \end{gathered} \tag{1.33}$$

and

$$\begin{gathered} \partial_{\nu_\Psi}\mathcal{D}_\Psi^{Stokes} : \dot{L}^p_1(\partial\Omega) \to L^p(\partial\Omega) \quad \textit{is invertible} \\ \textit{when } \theta \in (2\pi - \theta_o, 2\pi) \textit{ if and only if } p \in (1,\infty)\backslash\{p_3(\theta), p_4(\theta)\}. \end{gathered} \tag{1.34}$$

(B) *If* $\theta \in [\theta_o, \pi) \cup (\pi, 2\pi - \theta_o]$ *then there exist* $q_1(\theta), q_2(\theta) \in (2,\infty)$ *such that*

$$\begin{aligned} q_1(\theta) &\in \Big(2, \tfrac{2\pi-\theta}{\pi-\theta}\Big) \quad \textit{if} \ \ \theta \in [\theta_o, \pi), \\ q_2(\theta) &\in \Big(2, \tfrac{\theta}{\theta-\pi}\Big), \quad \textit{if} \ \ \theta \in (\pi, 2\pi - \theta_o], \end{aligned} \tag{1.35}$$

and

$$q_1(\theta) = q_2(2\pi - \theta) \quad \forall\, \theta \in [\theta_o, \pi), \tag{1.36}$$

with the following significance.

(B.1) *The operator*

$$\begin{gathered} S^{Stokes} : L^p(\partial\Omega) \to \dot{L}^p_1(\partial\Omega) \quad \textit{is invertible} \\ \textit{when } \theta \in [\theta_o, \pi) \textit{ if and only if } p \in (1,\infty)\backslash\{q_1(\theta)\}, \end{gathered} \tag{1.37}$$

and the operator

$$\begin{gathered} S^{Stokes} : L^p(\partial\Omega) \to \dot{L}^p_1(\partial\Omega) \quad \textit{is invertible} \\ \textit{when } \theta \in (\pi, 2\pi - \theta_o] \textit{ if and only if } p \in (1,\infty)\backslash\{q_2(\theta)\}. \end{gathered} \tag{1.38}$$

(B.2) *The operators*

$$\begin{gathered} \pm\tfrac{1}{2}I + K_\Psi^{Stokes} : L^p(\partial\Omega) \to L^p(\partial\Omega) \quad \textit{are invertible} \\ \textit{when } \theta \in [\theta_o, \pi) \textit{ if and only if } p \in (1,\infty)\backslash\{q_1'(\theta)\}, \end{gathered} \tag{1.39}$$

and the operators

$$\begin{gathered} \pm\tfrac{1}{2}I + K_\Psi^{Lam\acute{e}} : L^p(\partial\Omega) \to L^p(\partial\Omega) \quad \textit{are invertible} \\ \textit{when } \theta \in (\pi, 2\pi - \theta_o] \textit{ if and only if } p \in (1,\infty)\backslash\{q_2'(\theta)\}. \end{gathered} \tag{1.40}$$

Here for each $j \in \{1,2\}$, $q_j'(\theta)$ stands for the conjugate exponent of $q_j(\theta)$.

(B.3) *The operator*

$$\begin{array}{c} \partial_{\nu_\Psi}\mathcal{D}_\Psi^{Stokes} : \dot{L}_1^p(\partial\Omega) \to L^p(\partial\Omega) \ \text{ is invertible} \\ \text{when } \theta \in [\theta_o, \pi) \text{ if and only if } p \in (1,\infty)\setminus\{q_1(\theta)\}, \end{array} \tag{1.41}$$

and the operator

$$\begin{array}{c} \partial_{\nu_\Psi}\mathcal{D}_\Psi^{Stokes} : \dot{L}_1^p(\partial\Omega) \to L^p(\partial\Omega) \ \text{ is invertible} \\ \text{when } \theta \in (\pi, 2\pi-\theta_o] \text{ if and only if } p \in (1,\infty)\setminus\{q_2(\theta)\}. \end{array} \tag{1.42}$$

(C) *The following hold:*

(C.1) *The operator*

$$\begin{array}{c} S^{Stokes} : L^p(\partial\Omega) \to \dot{L}_1^p(\partial\Omega) \ \text{ is invertible} \\ \text{when } \theta = \pi \text{ for all } p \in (1,\infty). \end{array} \tag{1.43}$$

(C.2) *The operators*

$$\begin{array}{c} \pm\tfrac{1}{2}I + K_\Psi^{Stokes} : L^p(\partial\Omega) \to L^p(\partial\Omega) \ \text{ are invertible} \\ \text{when } \theta = \pi \text{ for all } p \in (1,\infty). \end{array} \tag{1.44}$$

(C.3) *The operator*

$$\begin{array}{c} \partial_{\nu_\Psi}\mathcal{D}_\Psi^{Stokes} : \dot{L}_1^p(\partial\Omega) \to L^p(\partial\Omega) \ \text{ is invertible} \\ \text{when } \theta = \pi \text{ for all } p \in (1,\infty). \end{array} \tag{1.45}$$

The methods employed for proving these results are those of pseudo-differential calculus of Mellin type. This is possible since in the current geometrical setting, that of infinite sectors in two dimensions, the operators $\partial_\tau S^{Lamé}$ and $\partial_\tau S^{Stokes}$ can be identified with Mellin convolution type operators. The invertibiliy results established for the operators $\partial_\tau S^{Lamé}$ and $\partial_\tau S^{Stokes}$ yield in turn invertibilty results for the operators $S^{Lamé}$ and S^{Stokes}, and ultimately for the operators, $\pm\frac{1}{2}I + K_\Psi^{Lamé}$ and $\pm\frac{1}{2}I + K_\Psi^{Stokes}$, and $\partial_{\nu_\Psi}\mathcal{D}_\Psi^{Lamé}$ and $\partial_{\nu_\Psi}\mathcal{D}_\Psi^{Stokes}$, via the operator identities (3.266)–(3.267), for the Lamé system, and (4.40)–(4.41), for the Stokes system.

One novel aspect of this work is the realization that interval analysis techniques and computer-aided proofs can be employed to shed further light on the nature of the critical indices from Theorems 1.1 and 1.2. The implementation of this mix of Mellin transform techniques and validated numerics methods is motivated by the fact that the critical indices arise as roots of certain explicit elementary functions dependent however on parameters related to the geometry of the domain and the underlying differential operator, θ and κ respectively. The dependence of the roots on θ and κ is intricate making it difficult to be studied via traditional analytic methods. As such the computer-aided proofs we produce in the second part of the paper help us elucidate at least partially the nature of this dependence. Concretely in the case of the Lamé system we have:

Theorem 1.3. *Let Ω be an infinite sector of aperture $\theta \in (0,2\pi)\setminus\{\pi\}$, assume $\kappa \in (0,1)$, and recall the critical indices $p_i(\theta,\kappa)$, $i \in \{1,\ldots,4\}$, from* Theorem 1.1. *Then, with $\varepsilon = 10^{-6}$ and $\delta = 10^{-4}$, the following hold*

(1) *The critical value* $p_1(\theta,\kappa)$ *is increasing in* θ *and decreasing in* κ *on* $[\varepsilon, \pi-\varepsilon]\times[0,1-\delta]$.
(2) *The critical value* $p_2(\theta,\kappa)$ *is increasing in* θ *and increasing in* κ *on* $[\varepsilon, \pi-\varepsilon]\times[0,1-\delta]$.
(3) *The critical value* $p_3(\theta,\kappa)$ *is decreasing in* θ *and increasing in* κ *on* $[\pi+\varepsilon, 2\pi-\varepsilon]\times[0,1-\delta]$.
(4) *The critical value* $p_4(\theta,\kappa)$ *is decreasing in* θ *and decreasing in* κ *on* $[\pi+\varepsilon, 2\pi-\varepsilon]\times[0,1-\delta]$.

The reason for not being able to take $\varepsilon=\delta=0$ in Theorem 1.3 is that the behavior of $p_1(\theta,\kappa)$ ceases to be strictly monotonic if either $\theta=\pi$ or $\kappa=1$ and a similar phenomenon can be observed for the other critical indices. As our computer-aided proofs are based on set-valued computations, rounding errors are introduced, and we can therefore only prove strict inequalities. We should stress that, even though the proof of Theorem 1.3 is computer-aided, it is rigorous in the mathematical sense (see e.g., [1,35,37]).

Based on (non-rigorous) numerical simulations we conjecture that when $\kappa\in[0,1]$ there holds

$$\begin{aligned}
&p_1(\theta,\kappa) \text{ is increasing in } \theta \text{ and decreasing in } \kappa \text{ on } (0,\pi)\times[0,1],\\
&p_2(\theta,\kappa) \text{ is increasing in } \theta \text{ and increasing in } \kappa \text{ on } (0,\pi)\times[0,1],\\
&p_3(\theta,\kappa) \text{ is decreasing in } \theta \text{ and increasing in } \kappa \text{ on } (\pi,2\pi)\times[0,1],\\
&p_4(\theta,\kappa) \text{ is decreasing in } \theta \text{ and decreasing in } \kappa \text{ on } (\pi,2\pi)\times[0,1].
\end{aligned} \tag{1.46}$$

The remainder of the paper has the following format. Section 2 contains basic definitions, a brief review of the algebra generated by Hardy kernels and the truncated Hilbert transform, and an introduction to the Mellin transform. Section 3 debuts with some background information on the elastic single layer potential $S^{Lamé}$ and in Sect. 3.1 we compute the Mellin symbol of the operator $\partial_\tau S^{Lamé}$ in preparation for the proof of Theorem 1.1, which is presented in Sect. 3.2. A key role in our analysis is played by Lemma 3.7, whose proof relies on a delicate argument by contradiction. In Sect. 4 we treat the case of the Stokes system where we prove Theorem 1.2. Section 5 contains in its first part the computer-aided analysis of the critical indices $p_i(\theta,\kappa)$, $i\in\{1,\dots,4\}$ culminating with the proof of the monotonicity statements made in Theorem 1.3. Section 5.1 briefly discusses relevant computational details of the computer-aided proof approach while Sect. 5.3 provides basic background on the interval analysis method.

2. Preliminaries

In this section we introduce basic notation and review known results that are useful for the remainder of the paper.

Definition 2.1. An open and proper set $\Omega\subseteq\mathbb{R}^2$ is called a graph Lipschitz domain provided there exists a Lipschitz function $\phi:\mathbb{R}\to\mathbb{R}$ such that

$$\Omega=\{X=(X_1,X_2)\in\mathbb{R}^2:\ X_2>\phi(X_1)\}. \tag{2.1}$$

Throughout the paper, given a graph Lipschitz domain $\Omega \subseteq \mathbb{R}^2$, we shall introduce the surface measure $\sigma := \mathscr{H}^1 \lfloor \partial\Omega$, where $\mathscr{H}^1$ stands for the 1-dimensional Hausdorff measure in $\mathbb{R}^2$. Also ν will denote the outward unit normal vector to $\partial\Omega$ which exists almost everywhere with respect to σ. Going further, set $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^2 \backslash \overline{\Omega}$ (where, given a set $E \subseteq \mathbb{R}^2$, $\overline{E}$ stands for the closure of E in $\mathbb{R}^2$). For any $P \in \partial\Omega$, introduce the non-tangential approach regions $\Upsilon^{\pm}(P)$ with vertex at P by setting

$$\Upsilon^{\pm}(P) := \{X \in \Omega_{\pm} : \ |P - X| < \omega \operatorname{dist}(X, \partial\Omega)\}, \tag{2.2}$$

where $\omega > 1$ is a fixed, sufficiently large constant. The regions defined in (2.2) are then used to define non-tangential traces on $\partial\Omega$. Specifically, if $u_{\pm} : \Omega_{\pm} \to \mathbb{R}$ are sufficiently nice functions we let

$$u_{\pm}\big|_{\partial\Omega}(P) := \lim_{\substack{X \in \Upsilon^{\pm}(P) \\ X \to P}} u_{\pm}(X), \quad \text{for a.e.} \quad P \in \partial\Omega, \tag{2.3}$$

and

$$\partial_\nu u_{\pm}(P) := \langle \nu(P), (\nabla u_{\pm})\big|_{\partial\Omega}(P) \rangle, \quad \text{for } \sigma\text{-a.e. } \ P \in \partial\Omega. \tag{2.4}$$

Here and elsewhere $\langle \cdot, \cdot \rangle$ stands for the canonical inner product in $\mathbb{R}^2$. Also, we recall the non-tangential maximal function operator M acting on functions $u_{\pm} : \Omega_{\pm} \to \mathbb{R}$ which is given at each boundary point $P \in \partial\Omega$ by

$$M(u_{\pm})(P) := \sup\{|u_{\pm}(X)| : \ X \in \Upsilon^{\pm}(P)\}. \tag{2.5}$$

For each $1 < p < \infty$, the space $L^p(\partial\Omega)$ is the Lebesgue space of p-integrable functions on $\partial\Omega$ with respect to the surface measure σ, and we denote by $L^p_{loc}(\partial\Omega)$ the local version of this space. Also let

$$L^p_1(\partial\Omega) := \{f \in L^p(\partial\Omega) : \partial_\tau f \in L^p(\partial\Omega)\}, \tag{2.6}$$

and

$$\dot{L}^p_1(\partial\Omega) := \{f \in L^p_{loc}(\partial\Omega) : \ \partial_\tau f \in L^p(\partial\Omega)\}/\mathbb{R}, \tag{2.7}$$

where ∂_τ is the tangential derivative along $\partial\Omega$. Here, if $[g] \in \dot{L}^p_1(\partial\Omega)$ denotes the equivalence class of the function g, we set

$$\|[g]\|_{\dot{L}^p_1(\partial\Omega)} := \|\partial_\tau g\|_{L^p(\partial\Omega)}. \tag{2.8}$$

When understood from the context, we shall not distinguish between $L^p(\partial\Omega)$ and $\big[L^p(\partial\Omega)\big]^m$ with a similar convention for $\dot{L}^p_1(\partial\Omega)$ and $\big[\dot{L}^p_1(\partial\Omega)\big]^m$, for some $m \in \mathbb{N}$. A simple observation is that the operator (also denoted by ∂_τ) given by

$$\begin{gathered} \partial_\tau : \dot{L}^p_1(\partial\Omega) \longrightarrow L^p(\partial\Omega), \quad \partial_\tau([f]) := \partial_\tau f, \\ \text{is well-defined, linear, bounded and invertible for each } p \in (1, \infty). \end{gathered} \tag{2.9}$$

Next we shall discuss Hardy kernel operators on $L^p(\mathbb{R}_+)$, where $\mathbb{R}_+$ stands for the set of non-negative real numbers. We start with the following definition.

Definition 2.2. Let h be a real-valued measurable function on $\mathbb{R}_+ \times \mathbb{R}_+$ and assume that $1 \leq p < \infty$. Then h is called a Hardy kernel for $L^p(\mathbb{R}_+)$ provided that

(1) h is a homogeneous function of degree -1, i.e., for each $\lambda > 0$ and each $s, t \in \mathbb{R}_+$ one has $h(\lambda s, \lambda t) = \lambda^{-1} h(s,t)$;
(2) $\displaystyle\int_0^\infty |h(1,t)| t^{-1/p}\, dt \left(= \int_0^\infty |h(s,1)| s^{1/p-1}\, ds \right) < \infty.$

Given $m \in \mathbb{N}$, a matrix-valued measurable function $h = \big(h_{ij}\big)_{i,j\in\{1,\dots,m\}}$ on $\mathbb{R}_+ \times \mathbb{R}_+$ is called a Hardy kernel for $\big[L^p(\mathbb{R}_+)\big]^m$ provided that h_{ij} is a Hardy kernel for $L^p(\mathbb{R}_+)$ for each $i, j \in \{1, \dots, m\}$,.

Fix $p \in [1, \infty)$ and $m \in \mathbb{N}$ and assume that $h = (h_{ij})_{i,j\in\{1,\dots,m\}}$ is a Hardy kernel for $[L^p(\mathbb{R}_+)]^m$. For any vector-valued function $\vec{f} \in \big[L^p(\mathbb{R}_+)\big]^m$, define the action of the operator T, called a Hardy kernel operator with kernel h, on $\vec{f}$ by setting

$$T\vec{f}(s) := \int_0^\infty h(s,t) \cdot \vec{f}(t)\, dt, \qquad \forall\, s \in \mathbb{R}_+, \tag{2.10}$$

where $\cdot$ denotes matrix multiplication.

Going further, let f be an infinitely differentiable function with compact support in the interval $[0, \infty)$. Then the Mellin transform of f is defined as

$$\mathcal{M}f(z) := \int_0^\infty x^{z-1} f(x)\, dx, \quad z \in \mathbb{C}. \tag{2.11}$$

If f is a measurable function on $\mathbb{R}_+$ and the integral in (2.11) converges absolutely for all z in some non-empty strip $\Gamma_{\alpha,\beta} := \{z \in \mathbb{C} :\ \alpha < \operatorname{Re} z < \beta\}$, $\alpha, \beta \in \mathbb{R}$, then the integral $\mathcal{M}f(z)$ is called the *Mellin transform* of the function f. The strip $\Gamma_{\alpha,\beta}$ is occasionally referred to as a strip of holomorphy for f. It is straightforward to see that for each $z \in \mathbb{C}$ such that $z+1$ belongs to a strip of holomorphy for a function f one has

$$(\mathcal{M}g)(z) = (\mathcal{M}f)(z+1), \quad \text{whenever} \quad g(t) := tf(t). \tag{2.12}$$

Finally, if $\mathcal{X}$ is a Banach space and $T : \mathcal{X} \to \mathcal{X}$ is a linear and continuous operator, the spectrum of T acting on $\mathcal{X}$ is defined as the set

$$\sigma(T; \mathcal{X}) := \{w \in \mathbb{C} \,:\, wI - T \text{ is not invertible on } \mathcal{X}\}, \tag{2.13}$$

where I denotes the identity operator on $\mathcal{X}$. In the above context the spectral radius of the operator T acting on $\mathcal{X}$ is given by

$$\rho(T; \mathcal{X}) := \sup\{|w| : w \in \sigma(T; \mathcal{X})\}. \tag{2.14}$$

In particular, $\rho(T; \mathcal{X})$ is the radius of the smallest closed circular disc centered at the origin containing $\sigma(T; \mathcal{X})$.

The following result found in [4] and [12] allows one to explicitly determine the spectrum of the operator T (as defined in (2.10)) acting on $[L^p(\mathbb{R}_+)]^m$, if its kernel k is a linear combination of the kernel of the Hilbert transform and Hardy kernels for $[L^p(\mathbb{R}_+)]^m$ for some $1 < p < \infty$.

Theorem 2.3. *Let $m \in \mathbb{N}$ and assume that $h = (h_{ij})_{i,j\in\{1,\dots,m\}}$ is a Hardy kernel for $\big[L^p(\mathbb{R}_+)\big]^m$ for some $1 < p < \infty$. Consider $M \in \mathbb{R}^{m\times m}$ a matrix* *with real constant entries and let* $c_1, c_2 \in \mathbb{R}$ *be constants. If an operator* T *acting on* $[L^p(\mathbb{R}_+)]^m$ *is given by*

$$T\vec{f}(s) := \int_0^\infty k(s,t) \cdot \vec{f}(t)\,dt, \qquad a.e.\ s \in \mathbb{R}_+, \tag{2.15}$$

for each $\vec{f} \in \big[L^p(\mathbb{R}_+)\big]^m$, *where*

$$k(s,t) := c_1 \cdot h(s,t) + \frac{c_2}{s-t} \cdot M, \qquad \forall\, s,t \in \mathbb{R}_+, \tag{2.16}$$

then T *is a linear and bounded operator from* $[L^p(\mathbb{R}_+)]^m$ *into itself. Moreover, its spectrum satisfies*

$$\sigma(T; [L^p(\mathbb{R}_+)]^m) = \overline{S}, \tag{2.17}$$

where $\overline{S}$ *denotes the closure of the set* $S \subseteq \mathbb{C}$ *given by*

$$S := \big\{w \in \mathbb{C} :\ \det(wI - \mathcal{M}k(\cdot,1))(1/p + i\xi) = 0,\ \textit{for some } \xi \in \mathbb{R}\big\}, \tag{2.18}$$

with I *standing for the identity operator.*

An immediate corollary of Theorem 2.3 is as follows.

Corollary 2.4. *In the context of* Theorem 2.3, *with* $c_1, c_2 \in \mathbb{R}$, *and* $c_2 \neq 0$ *and* $\det M \neq 0$, *the operator* T *is invertible on* $[L^p(\mathbb{R}_+)]^m$, $1 < p < \infty$, *if and only if the following holds*

$$\det \mathcal{M}k(\cdot,1)(1/p + i\xi) \neq 0 \quad \forall\, \xi \in \mathbb{R}. \tag{2.19}$$

Proof. Start by fixing $1 < p < \infty$. For the direct implication, assume that T is invertible on $[L^p(\mathbb{R}_+)]^m$. Consequently $0 \notin \sigma(T; [L^p(\mathbb{R}_+)]^m)$ and using the characterization (2.17) from Theorem 2.3 we obtain that $0 \notin \overline{S}$ where S is as in (2.18). In particular $0 \notin S$ and thus (2.19) holds.

Turning our attention to the reverse implication, assume that (2.19) is valid and seeking a contradiction, suppose that the operator T is not invertible on $[L^p(\mathbb{R}_+)]^m$. This implies $0 \in \overline{S}$ and, consequently there exist sequences $\{w_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and $\{\xi_j\}_{j\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\lim_{j\to\infty} w_j = 0, \tag{2.20}$$

and

$$\det(w_j I - \mathcal{M}k(\cdot,1))(1/p + i\xi_j) = 0 \quad \text{for each } \ j \in \mathbb{N}. \tag{2.21}$$

Consider first the case when the sequence $\{\xi_j\}_{j\in\mathbb{N}}$ contains a bounded subsequence. Employing the Bolzano-Weierstrass theorem, we can then assume without loss of generality that there exists $\xi^* \in \mathbb{R}$ such that

$$\lim_{j\to\infty} \xi_j = \xi^*. \tag{2.22}$$

Since the application

$$\mathbb{R} \ni \xi \mapsto \mathcal{M}k(\cdot,1)(1/p + i\xi) \ \text{ is continuous,} \tag{2.23}$$

and the determinant function is continuous, based on (2.20) and (2.22) we can then deduce that $\det(\mathcal{M}k(\cdot,1))(1/p + i\xi^*) = 0$, contradicting (2.19).

We are left with considering the scenario when the sequence $\{\xi_j\}_{j\in\mathbb{N}}$ has a subsequence $\{\xi_{j_k}\}_{k\in\mathbb{N}}$ convergent to either $+\infty$ or $-\infty$ as $k\to\infty$. In this case, introduce the space $L^1_*(\mathbb{R}_+)$ by setting

$$L^1_*(\mathbb{R}_+) := \left\{ f:\partial\Omega\to\mathbb{C} :\ f \ \text{ measurable and } \int_{\mathbb{R}_+} |f(x)|\,\frac{dx}{x} < \infty \right\}. \tag{2.24}$$

Using that h is a Hardy kernel for $[L^p(\mathbb{R}_+)]^m$, it follows that the matrix-valued function h_p defined by $h_p(x) := x^{1/p}h(x)$, for each $x\in\mathbb{R}_+$, has all its entries belonging to $L^1_*(\mathbb{R}_+)$. Since the Fourier transform on the Haar group is in fact the Mellin transform (cf. e.g., [39]), the latter condition along with a version of the Riemann-Lebesgue lemma in the Haar group context guarantee that

$$\lim_{\xi\to\pm\infty} \mathcal{M}h(\cdot,1)(1/p+i\xi) = \mathbf{0}, \tag{2.25}$$

where $\mathbf{0}$ stands for the $m\times m$ zero matrix. Combining this with the information that

$$\lim_{\xi\to\pm\infty} \mathcal{M}\left(\frac{1}{\cdot-1}\right)(1/p+i\xi) = -\pi i, \tag{2.26}$$

and (2.16), allows us to conclude that

$$\lim_{\xi\to\pm\infty} \mathcal{M}k(\cdot,1)(1/p+i\xi) = -c_2\pi i\cdot M. \tag{2.27}$$

Passing then to the limit in (2.21) along the subsequence $\{j_k\}_{k\in\mathbb{N}}$, and using (2.20) and (2.27) along with the continuity of the determinant function, we arrive at

$$0 = \lim_{k\to\infty} \det \mathcal{M}k(\cdot,1)(1/p+i\xi_{j_k}) = (-c_2\pi i)^m\cdot\det M. \tag{2.28}$$

Finally this is a contradiction since, by hypotheses, $c_2\neq 0$ and $\det M\neq 0$. This completes the proof of the corollary. $\square$

For the remainder of the paper we will refer to $\mathcal{M}k$ as the Mellin symbol of k, the kernel of the operator T.

3. The Case of the Lamé System

The goal of this section is to investigate invertibility properties of singular integral operators of single and double layer type associated with the Lamé system on infinite sectors in $\mathbb{R}^2$. After recalling some notation, in Sect. 3.1 we compute the Mellin symbol of the kernel of the tangential derivative of the elastic single layer potential operator in infinite sectors. In Sect. 3.2 we present the proof of Theorem 1.1, the main result regarding the Lamé system.

Start by fixing $\Omega\subseteq\mathbb{R}^2$, a graph Lipschitz domain, and denote by $\mathcal{L}$ the Lamé differential operator. Specifically, if $\vec{u}=(u_1,u_2):\Omega\to\mathbb{R}^2$ is a vector-valued function (called displacement) with components in $\mathcal{C}^2(\Omega)$, the action of $\mathcal{L}$ on $\vec{u}$ is given by

$$\mathcal{L}\vec{u} := \mu\Delta\vec{u} + (\lambda+\mu)\nabla\mathrm{div}\,\vec{u}, \tag{3.1}$$

where the constants μ and λ are called the Lamé moduli and they satisfy

$$\mu > 0 \quad\text{and}\quad \lambda + \mu \geq 0. \tag{3.2}$$

It is straightforward to see that for each $r \in \mathbb{R}$, there holds

$$\mathcal{L}\vec{u} = \begin{pmatrix} \mu\Delta u_1 + (\lambda+\mu)\partial_1(\operatorname{div}\vec{u}) \\ \mu\Delta u_2 + (\lambda+\mu)\partial_2(\operatorname{div}\vec{u}) \end{pmatrix} = \begin{pmatrix} a^{1\ell}_{ij}(r)\partial_i\partial_j u_\ell \\ a^{2\ell}_{ij}(r)\partial_i\partial_j u_\ell \end{pmatrix}, \tag{3.3}$$

where

$$a^{k\ell}_{ij}(r) := \mu\delta_{ij}\delta_{k\ell} + (\lambda+\mu-r)\delta_{ik}\delta_{j\ell} + r\delta_{i\ell}\delta_{jk}, \quad \forall\, i,j,k,\ell \in \{1,2\}. \tag{3.4}$$

Above and throughout the paper we use Einstein's convention for summation over repeated indices and $\delta_{k\ell}$ denotes the Kronecker symbol for $k,\ell \in \{1,2\}$. For each $r \in \mathbb{R}$, we shall refer to the collection

$$A(r) := (a^{k\ell}_{ij}(r))_{i,j,k,\ell\in\{1,2\}} \tag{3.5}$$

as the tensor of coefficients associated with the writing of $\mathcal{L}$ as in (3.3)–(3.4).

Moving on, recall the classical, radially-symmetric matrix-valued fundamental solution of the Lamé differential operator $G^{Lamé} := (G^{Lamé}_{ij})_{i,j\in\{1,2\}}$ given by (c.f. e.g., [23, formula (9.2) in Chapter 9] and [33, formula (10.7.1) in Chapter 10])

$$G^{Lamé}_{ij}(X) := C_1\delta_{ij}\log|X|^2 - C_2\frac{X_iX_j}{|X|^2}, \quad \forall\, X = (X_1,X_2) \in \mathbb{R}^2\backslash\{\mathbf{0}\}, \tag{3.6}$$

where

$$C_1 := \frac{3\mu+\lambda}{8\mu(2\mu+\lambda)\pi} \quad\text{and}\quad C_2 := \frac{\mu+\lambda}{4\mu(2\mu+\lambda)\pi}. \tag{3.7}$$

In particular $\mathcal{L}G^{Lamé} = \delta I_{2\times 2}$ as distributions in $\mathbb{R}^2$, where the operator $\mathcal{L}$ acts on the columns of the matrix $G^{Lamé}$, $I_{2\times 2}$ is the 2 by 2 identity matrix, and δ is the Dirac-delta distribution with mass at the origin.

Next, fix $r \in \mathbb{R}$ and consider the tensor of coefficients (3.5), where the $a^{k\ell}_{ij}(r)$'s are as in (3.4). Then, given a suitably smooth vector-valued function $\vec{u} = (u_1,u_2)$ defined in Ω, the conormal derivative of $\vec{u}$ associated to the choice of tensor of coefficients $A(r)$ is given by

$$\frac{\partial\vec{u}}{\partial\nu_{A(r)}} := \left(\Big(\frac{\partial\vec{u}}{\partial\nu_{A(r)}}\Big)^1, \Big(\frac{\partial\vec{u}}{\partial\nu_{A(r)}}\Big)^2\right), \tag{3.8}$$

where, for each $j \in \{1,2\}$,

$$\begin{aligned}\left(\frac{\partial\vec{u}}{\partial\nu_{A(r)}}\right)^j &:= \nu_i a^{j\ell}_{ik}(r)\big(\partial_k u_\ell\big)\Big|_{\partial\Omega} \\ &= \mu\Big\langle\nu, (\nabla u_j)\Big|_{\partial\Omega}\Big\rangle + (\lambda+\mu-r)\nu_j(\operatorname{div}\vec{u})\Big|_{\partial\Omega} + r\nu_i(\partial_j u_i)\Big|_{\partial\Omega}.\end{aligned} \tag{3.9}$$

Above $\nu = (\nu_1,\nu_2)$ is the outward unit normal vector to $\partial\Omega$ and $\Big|_{\partial\Omega}$ denotes non-tangential restriction to $\partial\Omega$ in the sense of (2.3). The conormal derivative $\frac{\partial}{\partial\nu_{A(r)}}$ from (3.8)–(3.9) is called the *pseduo-stress* conormal derivative, denoted by $\partial\nu_\Psi$, when the value of the parameter r is equal to $\mu(\lambda+\mu)/(3\mu+\lambda)$, i.e.,

$$\frac{\partial}{\partial\nu_\Psi} := \frac{\partial}{\partial\nu_{A(r_o)}} \quad\text{where}\quad r_o := \frac{\mu(\lambda+\mu)}{3\mu+\lambda}. \tag{3.10}$$

Also, when $r=\mu$ the conormal derivative $\frac{\partial}{\partial\nu_{A(r)}}$ from (3.8)–(3.9) is called the *traction* or *stress* conormal derivative.

Next, define the elastostatics single layer potential operator $\mathcal{S}^{Lamé}$, and its boundary version $S^{Lamé}$ acting on a vector-valued function $\vec{f}:\partial\Omega\longrightarrow\mathbb{R}^2$, $\vec{f}=\binom{f_1}{f_2}$, by setting

$$\mathcal{S}^{Lamé}\vec{f}(X) := \int_{\partial\Omega} G^{Lamé}(X-Q)\cdot\vec{f}(Q)\,d\sigma(Q), \qquad X\in\mathbb{R}^2\backslash\partial\Omega, \tag{3.11}$$

$$S^{Lamé}\vec{f}(X) := \int_{\partial\Omega} G^{Lamé}(X-Q)\cdot\vec{f}(Q)\,d\sigma(Q), \qquad X\in\partial\Omega, \tag{3.12}$$

where $G^{Lamé} := (G^{Lamé}_{ij})_{i,j\in\{1,2\}}$ is the fundamental solution from (3.6)–(3.7).

We shall also work with double layer potential operators associated with the differential operator $\mathcal{L}$ from (3.1). Specifically, if $r\in\mathbb{R}$ is fixed and the tensor of coefficients $A(r)=(a^{k\ell}_{ij}(r))_{i,j,k,\ell\in\{1,2\}}$ is as in (3.4), then the double layer potential operator associated with $A(r)$ is denoted by $\mathcal{D}^{Lamé}_{A(r)}$ and its action on a vector-valued function $\vec{f}:\partial\Omega\longrightarrow\mathbb{R}^2$ with $\vec{f}=\binom{f_1}{f_2}$ is given by the formula

$$\mathcal{D}^{Lamé}_{A(r)}\vec{f}(X) := \int_{\partial\Omega}\left[\frac{\partial G^{Lamé}}{\partial\nu_{A(r)}}(X-\cdot)\right]^t(Q)\cdot\vec{f}(Q)\,d\sigma(Q), \quad \text{at each point } X\in\mathbb{R}^2\backslash\partial\Omega, \tag{3.13}$$

where the conormal derivative $\frac{\partial}{\partial\nu_{A(r)}}$ is applied to the columns of the fundamental solution $G^{Lamé}$ from (3.6)–(3.7), i.e.,

$$\frac{\partial G^{Lamé}}{\partial\nu_{A(r)}}(X-\cdot) = -\Big(\nu_i(\cdot)a^{k\ell}_{ij}(r)(\partial_j G^{Lamé}_{\ell m})(X-\cdot)\Big)_{k,m\in\{1,2\}}, \tag{3.14}$$

and the superscript t stands for transposition of matrices. The boundary version of $\mathcal{D}^{Lamé}_{A(r)}$ is the operator $K^{Lamé}_{A(r)}$ whose action on $\vec{f}$ as above is defined by setting

$$K^{Lamé}_{A(r)}\vec{f}(X) := p.v.\int_{\partial\Omega}\left[\frac{\partial G^{Lamé}}{\partial\nu_{A(r)}}(X-\cdot)\right]^t(Q)\cdot\vec{f}(Q)\,d\sigma(Q), \quad \text{for } \sigma\text{-a.e. point } X\in\partial\Omega, \tag{3.15}$$

where $p.v.$ denotes principle value. The formal adjoint of the operator $K^{Lamé}_{A(r)}$ is $\big(K^{Lamé}_{A(r)}\big)^*$, whose action on $\vec{f}$ is given by

$$\big(K^{Lamé}_{A(r)}\big)^*\vec{f}(X) := -p.v.\int_{\partial\Omega}\Big[\frac{\partial G^{Lamé}}{\partial\nu_{A(r)}}(\cdot-Q)\Big](X)\cdot\vec{f}(Q)\,d\sigma(Q), \quad \text{for } \sigma\text{-a.e. point } X\in\partial\Omega. \tag{3.16}$$

A basic result which follows from [5] and standard techniques is

Proposition 3.1. *Let Ω be a graph Lipschitz domain in $\mathbb{R}^2$, assume that $r \in \mathbb{R}$ is fixed, and recall the tensor of coefficients $A(r) = (a_{ij}^{k\ell}(r))_{i,j,k,\ell\in\{1,2\}}$ from (3.4). Set $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^2 \setminus \overline{\Omega}$. Then, for each $p \in (1,\infty)$,*

(1) *There holds*

$$S^{Lamé} : L^p(\partial\Omega) \to \dot{L}^p_1(\partial\Omega) \quad \text{is a linear and bounded operator,} \tag{3.17}$$

$$K^{Lamé}_{A(r)} : L^p(\partial\Omega) \to L^p(\partial\Omega) \quad \text{is a linear and bounded operator,} \tag{3.18}$$

$$\big(K^{Lamé}_{A(r)}\big)^* : L^p(\partial\Omega) \to L^p(\partial\Omega) \quad \text{is a linear and bounded operator.} \tag{3.19}$$

(2) *For each $\vec{f} \in L^p(\partial\Omega)$ there holds $M\Big(\mathcal{D}^{Lamé}_{A(r)}\vec{f}\Big) \in L^p(\partial\Omega)$. Moreover there exists a finite constant $C > 0$ depending only on the Lipschitz character of Ω such that*

$$\|M\Big(\mathcal{D}^{Lamé}_{A(r)}\vec{f}\Big)\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{L^p(\partial\Omega)}. \tag{3.20}$$

(3) *For every $\vec{f} \in L^p(\partial\Omega)$ there holds*

$$\mathcal{D}^{Lamé}_{A(r)}\vec{f}\Big|_{\partial\Omega_\pm}(P) = (\pm\tfrac{1}{2}I + K^{Lamé}_{A(r)})\vec{f}(P), \qquad \sigma - a.e.\ P \in \partial\Omega. \tag{3.21}$$

(4) *For every $\vec{f} \in L^p(\partial\Omega)$ one has $M\Big(\nabla\mathcal{S}^{Lamé}\vec{f}\Big) \in L^p(\partial\Omega)$. Moreover there exists a finite constant $C > 0$ depending only on the Lipschitz character of Ω such that*

$$\|M\Big(\nabla\mathcal{S}^{Lamé}\vec{f}\Big)\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{L^p(\partial\Omega)}. \tag{3.22}$$

(5) *For each $\vec{f} \in L^p(\partial\Omega)$, the single layer satisfies*

$$\mathcal{S}^{Lamé}\vec{f}\Big|_{\partial\Omega_+} = \mathcal{S}^{Lamé}\vec{f}\Big|_{\partial\Omega_-} = S^{Lamé}\vec{f}, \tag{3.23}$$

and

$$\partial_\tau\mathcal{S}^{Lamé}\vec{f}\Big|_{\partial\Omega_+} = \partial_\tau\mathcal{S}^{Lamé}\vec{f}\Big|_{\partial\Omega_-} = \partial_\tau S^{Lamé}\vec{f}. \tag{3.24}$$

Moreover, if $(\partial_\tau S^{Lamé})^*$ is the formal adjoint of $\partial_\tau S^{Lamé}$, then

$$\big(\partial_\tau S^{Lamé}\big)^* = -S^{Lamé}\partial_\tau. \tag{3.25}$$

We conclude this section by introducing the notation $\mathcal{D}^{Lamé}_{\Psi}$ and $K^{Lamé}_{\Psi}$ for the boundary-to-domain and boundary-to-boundary double layer potentials associated with the pseudo-stress conormal derivative from (3.10). Concretely we set

$$\mathcal{D}^{Lamé}_{\Psi} := \mathcal{D}^{Lamé}_{A(r_o)}, \quad \text{with} \quad r_o := \frac{\mu(\lambda+\mu)}{3\mu+\lambda}, \tag{3.26}$$

and

$$K^{Lamé}_{\Psi} := K^{Lamé}_{A(r_o)} \quad \text{with} \quad r_o := \frac{\mu(\lambda+\mu)}{3\mu+\lambda}. \tag{3.27}$$

3.1. The Mellin Symbol of the Tangential Derivative of the Single Layer

The main goal of this subsection is to explicitly compute the matrix of Mellin symbols of the operator $\partial_\tau S^{Lamé}$ on infinite angles in $\mathbb{R}^2$. Specifically, we shall assume that Ω is the infinite sector in $\mathbb{R}^2$ of aperture $\theta \in (0, 2\pi)$ that is the upper-graph of the Lipschitz function $\phi : \mathbb{R} \to \mathbb{R}$ given by

$$\phi(x) := |x| \cot(\theta/2), \quad x \in \mathbb{R}. \tag{3.28}$$

Recall the matrix-valued fundamental solution $G^{Lamé} = (G^{Lamé}_{ij})_{i,j\in\{1,2\}}$ of the Lamé system of elastostatics (3.1) from (3.6)–(3.7) and the single layer potential operator $S^{Lamé}$ from (3.12). In the following lemma we compute the formula for the kernel of the operator $\partial_\tau S^{Lamé}$.

Lemma 3.2. *Let $\theta \in (0, 2\pi)$ and assume that $\Omega \subseteq \mathbb{R}^2$ is the upper-graph of the function $\phi : \mathbb{R} \to \mathbb{R}$ from (3.28). Then for each $\vec{f} : \partial\Omega \to \mathbb{R}^2$ such that $\vec{f} \in L^p(\partial\Omega)$ for some $p \in (1, \infty)$, there holds*

$$(\partial_\tau S^{Lamé} \vec{f})(X) = \int_{\partial\Omega} k(X, Q) \cdot \vec{f}(Q)\, d\sigma(Q), \ \forall\, X \in \partial\Omega \backslash \{\mathbf{0}\}, \tag{3.29}$$

with

$$\begin{gathered} k(X, Q) := \begin{pmatrix} A_{11}(X, Q) & A_{12}(X, Q) \\ A_{21}(X, Q) & A_{22}(X, Q) \end{pmatrix}, \\ \forall\ X, Q \in \partial\Omega,\ X \neq Q \ \ \textit{and} \ \ X \neq \mathbf{0}, \end{gathered} \tag{3.30}$$

where the functions

$$A_{ij} : \partial\Omega \times \partial\Omega \backslash \Big(\mathrm{diag}(\partial\Omega) \cup \big(\{\mathbf{0}\} \times \partial\Omega\big)\Big) \longrightarrow \mathbb{R}, \qquad i, j \in \{1, 2\}, \tag{3.31}$$

are as described below. Specifically, if the point $X = (X_1, X_2) \in \partial\Omega \backslash \{\mathbf{0}\}$ and $Q = (Q_1, Q_2) \in \partial\Omega$, $Q \neq X$, then with the vector $\nu(X) = (\nu_1(X), \nu_2(X))$ denoting the outward unit normal to $\partial\Omega$ at the point X, one has

$$\begin{aligned} A_{11}(X, Q) := &-\frac{2\nu_2(X)(X_1 - Q_1)}{|X - Q|^2} \left\{ (C_1 - C_2) + C_2 \frac{(X_1 - Q_1)^2}{|X - Q|^2} \right\} \\ &+ \frac{2\nu_1(X)(X_2 - Q_2)}{|X - Q|^2} \left\{ C_1 + C_2 \frac{(X_1 - Q_1)^2}{|X - Q|^2} \right\}, \end{aligned} \tag{3.32}$$

$$\begin{aligned} A_{12}(X, Q) := &-\frac{C_2\nu_2(X)(X_2 - Q_2)}{|X - Q|^2} \left\{ -1 + 2\frac{(X_1 - Q_1)^2}{|X - Q|^2} \right\} \\ &+ \frac{C_2\nu_1(X)(X_1 - Q_1)}{|X - Q|^2} \left\{ -1 + 2\frac{(X_2 - Q_2)^2}{|X - Q|^2} \right\}, \end{aligned} \tag{3.33}$$

$$\begin{aligned} A_{21}(X, Q) := &-\frac{C_2\nu_2(X)(X_2 - Q_2)}{|X - Q|^2} \left\{ -1 + 2\frac{(X_1 - Q_1)^2}{|X - Q|^2} \right\} \\ &+ \frac{C_2\nu_1(X)(X_1 - Q_1)}{|X - Q|^2} \left\{ -1 + 2\frac{(X_2 - Q_2)^2}{|X - Q|^2} \right\}, \end{aligned} \tag{3.34}$$

and

$$A_{22}(X-Q) := -\frac{2\nu_2(X)(X_1-Q_1)}{|X-Q|^2}\left\{C_1 + C_2\frac{(X_2-Q_2)^2}{|X-Q|^2}\right\} + \frac{2\nu_1(X)(X_2-Q_2)}{|X-Q|^2}\left\{(C_1-C_2) + C_2\frac{(X_2-Q_2)^2}{|X-Q|^2}\right\}. \tag{3.35}$$

Proof. Fix $p \in (1,\infty)$ and assume that $\vec{f} \in L^p(\partial\Omega)$. Using the Lebesgue dominated convergence theorem we may write for each $X \in \partial\Omega\backslash\{\mathbf{0}\}$

$$\partial_\tau S^{Lamé}\vec{f}(X) = \int_{\partial\Omega} \partial_{\tau(X)}[G^{Lamé}(X-Q)]\cdot\vec{f}(Q)\,d\sigma(Q). \tag{3.36}$$

Thus (3.29) holds with

$$k(X,Q) = \begin{pmatrix} \partial_{\tau(X)}[G_{11}^{Lamé}(X-Q)] & \partial_{\tau(X)}[G_{12}^{Lamé}(X-Q)] \\ \partial_{\tau(X)}[G_{21}^{Lamé}(X-Q)] & \partial_{\tau(X)}[G_{22}^{Lamé}(X-Q)] \end{pmatrix}, \tag{3.37}$$

for any $X, Q \in \partial\Omega$ satisfying $X \neq Q$ and $X \neq \mathbf{0}$.

To finish the proof, there remains to show that

$$\begin{gathered} \partial_{\tau(X)}[G_{ij}^{Lamé}(X-Q)] = A_{ij}(X,Q), \\ \forall\, i,j \in \{1,2\} \;\text{ and }\; \forall\, X,Q \in \partial\Omega \;\text{ satisfying }\; X \neq Q \;\text{ and }\; X \neq \mathbf{0}. \end{gathered} \tag{3.38}$$

With this goal in mind fix $i,j \in \{1,2\}$ and let $\nu(X) = (\nu_1(X),\nu_2(X))$ be the outward unit normal vector at $X \in \partial\Omega\backslash\{\mathbf{0}\}$. Then $\tau(X) = (-\nu_2(X),\nu_1(X))$, and consequently

$$\begin{aligned} \partial_{\tau(X)}[G_{ij}^{Lamé}(X-Q)] &= \Big\langle \tau(X), (\nabla G_{ij}^{Lamé})(X-Q)\Big\rangle \\ &= -\nu_2(X)(\partial_1 G_{ij}^{Lamé})(X-Q) \\ &\quad + \nu_1(X)(\partial_2 G_{ij}^{Lamé})(X-Q). \end{aligned} \tag{3.39}$$

Moreover, straightforward calculations based on (3.6)–(3.7) give that whenever $X = (X_1,X_2) \neq \mathbf{0}$ there holds

$$(\partial_1 G_{ij}^{Lamé})(X) = 2C_1\delta_{ij}\frac{X_1}{|X|^2} - C_2\frac{\delta_{i1}X_j + \delta_{1j}X_i}{|X|^2} + 2C_2\frac{X_iX_jX_1}{|X|^4}, \tag{3.40}$$

$$(\partial_2 G_{ij}^{Lamé})(X) = 2C_1\delta_{ij}\frac{X_2}{|X|^2} - C_2\frac{\delta_{i2}X_j + \delta_{2j}X_i}{|X|^2} + 2C_2\frac{X_iX_jX_2}{|X|^4}. \tag{3.41}$$

Then (3.38) follows from (3.39) and (3.40)–(3.41), completing the proof of the lemma. $\square$

Going further, if $\theta \in (0,2\pi)$ and Ω is as in the hypothesis of Lemma 3.2 in what follows we shall denote by $(\partial\Omega)_1$ and $(\partial\Omega)_2$ the left and the right side of the (infinite) angle $\partial\Omega$, respectively. Hence

$$(\partial\Omega)_j = \big\{((-1)^j s\sin\tfrac{\theta}{2}, s\cos\tfrac{\theta}{2}) :\; s \in \mathbb{R}_+\big\} \;\text{ for each }\; j \in \{1,2\}. \tag{3.42}$$

Next observe that one can naturally identify the sides $(\partial\Omega)_j$ for $j = 1,2$ with $\mathbb{R}_+$ via the mapping $(\partial\Omega)_j \ni P \mapsto |P| \in \mathbb{R}_+$. Based on this for each $p \in [1,\infty)$, the space $L^p(\partial\Omega)$ can be identified with $L^p(\mathbb{R}_+)\oplus L^p(\mathbb{R}_+)$. In turn, in light of these identifications the kernel k from (3.30) with entries (3.32)–(3.35) can be regarded as a kernel on $\mathbb{R}_+ \times \mathbb{R}_+$. Specifically the function $k(\cdot,\cdot)$ defined on $\partial\Omega \times \partial\Omega \backslash \mathrm{diag}(\partial\Omega \times \partial\Omega)$ shall be identified with the following 4×4 kernel matrix $\widetilde{k}$ defined on $\mathbb{R}_+ \times \mathbb{R}_+ \backslash \mathrm{diag}(\mathbb{R}_+ \times \mathbb{R}_+)$ given by

$$\widetilde{k} = \begin{pmatrix} \widetilde{k}^{11}_{11} & \widetilde{k}^{11}_{12} & \widetilde{k}^{12}_{11} & \widetilde{k}^{12}_{12} \\ \widetilde{k}^{11}_{21} & \widetilde{k}^{11}_{22} & \widetilde{k}^{12}_{21} & \widetilde{k}^{12}_{22} \\ \widetilde{k}^{21}_{11} & \widetilde{k}^{21}_{12} & \widetilde{k}^{22}_{11} & \widetilde{k}^{22}_{12} \\ \widetilde{k}^{21}_{21} & \widetilde{k}^{21}_{22} & \widetilde{k}^{22}_{21} & \widetilde{k}^{22}_{22} \end{pmatrix}, \tag{3.43}$$

where using notation introduced in Lemma 3.2, for each $i, j \in \{1, 2\}$ and $s, t \in \mathbb{R}_+$ with $s \neq t$ one has

$$\widetilde{k}^{11}_{ij}(s,t) = A_{ij}\big((-s\sin\tfrac{\theta}{2}, s\cos\tfrac{\theta}{2}), (-t\sin\tfrac{\theta}{2}, t\cos\tfrac{\theta}{2})\big), \tag{3.44}$$

$$\widetilde{k}^{12}_{ij}(s,t) = A_{ij}\big((-s\sin\tfrac{\theta}{2}, s\cos\tfrac{\theta}{2}), (t\sin\tfrac{\theta}{2}, t\cos\tfrac{\theta}{2})\big), \tag{3.45}$$

$$\widetilde{k}^{21}_{ij}(s,t) = A_{ij}\big((s\sin\tfrac{\theta}{2}, s\cos\tfrac{\theta}{2}), (-t\sin\tfrac{\theta}{2}, t\cos\tfrac{\theta}{2})\big), \tag{3.46}$$

$$\widetilde{k}^{22}_{ij}(s,t) = A_{ij}\big((s\sin\tfrac{\theta}{2}, s\cos\tfrac{\theta}{2}), (t\sin\tfrac{\theta}{2}, t\cos\tfrac{\theta}{2})\big). \tag{3.47}$$

Indeed, if $i, j \in \{1, 2\}$ and X and Q are such that $X, Q \in \partial\Omega$ satisfying $|X| = s \in \mathbb{R}_+$ and $|Q| = t \in \mathbb{R}_+$ with $s \neq t$, then

$$\begin{aligned} &\widetilde{k}^{11}_{ij}(s,t) = A_{ij}(X,Q), \quad \text{if} \quad X, Q \in (\partial\Omega)_1, \\ &\widetilde{k}^{12}_{ij}(s,t) = A_{ij}(X,Q), \quad \text{if} \quad X \in (\partial\Omega)_1 \quad \text{and} \quad Q \in (\partial\Omega)_2, \\ &\widetilde{k}^{21}_{ij}(s,t) = A_{ij}(X,Q), \quad \text{if} \quad X \in (\partial\Omega)_2 \quad \text{and} \quad Q \in (\partial\Omega)_1, \\ &\widetilde{k}^{22}_{ij}(s,t) = A_{ij}(X,Q), \quad \text{if} \quad X, Q \in (\partial\Omega)_2, \end{aligned} \tag{3.48}$$

from which (3.44)–(3.47) immediately follow.

Our next result establishes an explicit formula and useful properties for the kernel $\widetilde{k}$ introduced in (3.43), with entries as in (3.44)–(3.47).

Lemma 3.3. *Let $\theta \in (0, 2\pi)$, $C_1 \in (0, \infty)$, $C_2 \in [0, \infty)$, and consider the kernel $\widetilde{k} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^{4\times 4}$ introduced in (3.43), with entries given in (3.44)–(3.47). Then, for each $s, t \in \mathbb{R}_+$ such that $s \neq t$ there holds*

$$\widetilde{k}(s,t) = \begin{pmatrix} -\dfrac{2C_1}{s-t} & 0 & -A(s,t) & B(s,t) \\ 0 & -\dfrac{2C_1}{s-t} & B(s,t) & -C(s,t) \\ A(s,t) & B(s,t) & \dfrac{2C_1}{s-t} & 0 \\ B(s,t) & C(s,t) & 0 & \dfrac{2C_1}{s-t} \end{pmatrix}, \tag{3.49}$$

where the functions $A, B, C : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ are given by

$$\begin{aligned} A(s,t) := 2 \cdot & \frac{C_1(s - t\cos\theta) - C_2(s+t)\sin^2(\frac{\theta}{2})}{s^2 - 2st\cos\theta + t^2} \\ & + 2C_2 \cdot \frac{\sin^2(\frac{\theta}{2})(s+t)^2(s - t\cos\theta)}{(s^2 - 2st\cos\theta + t^2)^2}, \end{aligned} \tag{3.50}$$

$$\begin{aligned} B(s,t) := -C_2 \cdot & \frac{s\sin\theta}{s^2 - 2st\cos\theta + t^2} \\ & + C_2 \cdot \frac{(s^2 - t^2)(s - t\cos\theta)\sin\theta}{(s^2 - 2st\cos\theta + t^2)^2}, \end{aligned} \tag{3.51}$$

and

$$\begin{aligned} C(s,t) := 2 \cdot & \frac{C_1(s - t\cos\theta) - C_2(s-t)\cos^2(\frac{\theta}{2})}{s^2 - 2st\cos\theta + t^2} \\ & + 2C_2 \cdot \frac{\cos^2(\frac{\theta}{2})(s-t)^2(s - t\cos\theta)}{(s^2 - 2st\cos\theta + t^2)^2}. \end{aligned} \tag{3.52}$$

In addition, for each $s, t \in \mathbb{R}_+$ such that $s \neq t$, there holds

$$\widetilde{k}(s,t) = h(s,t) + \frac{2C_1}{s-t} \cdot \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.53}$$

where $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^{4\times 4}$ given by

$$h(s,t) := \begin{pmatrix} 0 & 0 & -A(s,t) & B(s,t) \\ 0 & 0 & B(s,t) & -C(s,t) \\ A(s,t) & B(s,t) & 0 & 0 \\ B(s,t) & C(s,t) & 0 & 0 \end{pmatrix}, \quad \forall\, s,t \in \mathbb{R}_+, \tag{3.54}$$

is a Hardy kernel for $[L^p(\mathbb{R}_+) \oplus L^p(\mathbb{R}_+)]^2 \equiv [L^p(\mathbb{R}_+)]^4$.

Proof. Fix $s, t \in \mathbb{R}_+$ such that $s \neq t$ and let $X, Q \in \partial\Omega$ be such that $s = |X|$ and $t = |Q|$. If $X, Q \in (\partial\Omega)_1$, there holds

$$X = \left(-s\sin\tfrac{\theta}{2}, s\cos\tfrac{\theta}{2}\right) \text{ and } Q = \left(-t\sin\tfrac{\theta}{2}, t\cos\tfrac{\theta}{2}\right), \tag{3.55}$$

and

$$\nu(X) = \left(-\cos\tfrac{\theta}{2}, -\sin\tfrac{\theta}{2}\right). \tag{3.56}$$

Appealing to (3.55), (3.32)–(3.35) and (3.48), straightforward calculations give

$$\widetilde{k}^{11}(s,t) := \begin{pmatrix} \widetilde{k}^{11}_{11}(s,t) & \widetilde{k}^{11}_{12}(s,t) \\ \widetilde{k}^{11}_{21}(s,t) & \widetilde{k}^{11}_{22}(s,t) \end{pmatrix} = -\frac{2C_1}{s-t} \cdot I_{2\times 2}. \tag{3.57}$$

Consider next the case when $X \in (\partial\Omega)_1, Q \in (\partial\Omega)_2$. Then,

$$X = \left(-s\sin\tfrac{\theta}{2}, s\cos\tfrac{\theta}{2}\right) \text{ and } Q = \left(t\sin\tfrac{\theta}{2}, t\cos\tfrac{\theta}{2}\right), \tag{3.58}$$

and $\nu(X)$ is as in (3.56). Based on this, (3.58), (3.32)–(3.35), and (3.48) we may write

$$\widetilde{k}^{12}(s,t) := \begin{pmatrix} \widetilde{k}^{12}_{11}(s,t) & \widetilde{k}^{12}_{12}(s,t) \\ \widetilde{k}^{12}_{21}(s,t) & \widetilde{k}^{12}_{22}(s,t) \end{pmatrix} = \begin{pmatrix} -A(s,t) & B(s,t) \\ B(s,t) & -C(s,t) \end{pmatrix}, \tag{3.59}$$

where $A(s,t), B(s,t)$ and $C(s,t)$ are as in (3.50), (3.51) and (3.52), respectively.

Moving on, when $X \in (\partial\Omega)_2$ and $Q \in (\partial\Omega)_1$ we have

$$X = \big(s\sin\tfrac{\theta}{2}, s\cos\tfrac{\theta}{2}\big) \ \text{ and } \ Q = \big(-t\sin\tfrac{\theta}{2}, t\cos\tfrac{\theta}{2}\big), \tag{3.60}$$

and

$$\nu(X) = \big(\cos\tfrac{\theta}{2}, -\sin\tfrac{\theta}{2}\big). \tag{3.61}$$

Thus, algebraic manipulations based on (3.60)–(3.61), (3.32)–(3.35), (3.48) and (3.50)–(3.52) give

$$\widetilde{k}^{21}(s,t) := \begin{pmatrix} \widetilde{k}^{21}_{11}(s,t) & \widetilde{k}^{21}_{12}(s,t) \\ \widetilde{k}^{21}_{21}(s,t) & \widetilde{k}^{21}_{22}(s,t) \end{pmatrix} = \begin{pmatrix} A(s,t) & B(s,t) \\ B(s,t) & C(s,t) \end{pmatrix}. \tag{3.62}$$

Next we shall consider the scenario where $X, Q \in (\partial\Omega)_2$. Then

$$X = \big(s\sin\tfrac{\theta}{2}, s\cos\tfrac{\theta}{2}\big) \ \text{ and } Q = \big(t\sin\tfrac{\theta}{2}, t\cos\tfrac{\theta}{2}\big), \tag{3.63}$$

and $\nu(X)$ is as in (3.61). This, (3.63), (3.32)–(3.35), (3.48), and straightforward algebra yield

$$\widetilde{k}^{22}(s,t) := \begin{pmatrix} \widetilde{k}^{22}_{11}(s,t) & \widetilde{k}^{22}_{12}(s,t) \\ \widetilde{k}^{22}_{21}(s,t) & \widetilde{k}^{22}_{22}(s,t) \end{pmatrix} = \frac{2C_1}{s-t} \cdot I_{2\times 2}. \tag{3.64}$$

Combining (3.57), (3.59), (3.62) and (3.64) immediately gives (3.49), as desired.

Turning our attention to proving the last statement in the lemma, notice that on grounds of (3.49), the formula (3.53) holds with h as in (3.54). Thus, it remains to establish that the function $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^{4\times 4}$ given in (3.54) is a Hardy kernel for $[L^p(\mathbb{R}_+)]^4$, or equivalently that each of the functions A, B, C given in (3.50)–(3.52) is a Hardy kernel for $L^p(\mathbb{R}_+)$. With this goal in mind, we start with the observation that, based on (3.50)–(3.52), the functions A, B, C are homogeneous of degree -1. In addition, note that

$$1 - 2t\cos\theta + t^2 \neq 0 \ \text{ for any } \ \theta \in (0, 2\pi) \ \text{ and any } \ t \in \mathbb{R}_+. \tag{3.65}$$

Indeed, since $1 - 2t\cos\theta + t^2 = (t - \cos\theta)^2 + \sin^2\theta \geq \sin^2\theta$ then (3.65) follows immediately when $\theta \neq \pi$. When $\theta = \pi$ the expression $1 - 2t\cos\theta + t^2$ becomes $(t+1)^2$, which is > 0 for $t \in \mathbb{R}_+$. In particular, (3.65) in concert with (3.50)–(3.52) yield

$$A(1,\cdot), B(1,\cdot), \ \text{ and } \ C(1,\cdot) \text{ are continuous functions on } \ [0,\infty), \tag{3.66}$$

and

$$|A(1,t)|, |B(1,t)|, \ \text{ and } \ |C(1,t)| \ \text{ are } \ \mathcal{O}\Big(\frac{1}{t}\Big) \ \text{ as } \ t \to \infty. \tag{3.67}$$

From (3.66) and (3.67) it easily follows that for each $p \in (1, \infty)$ one has

$$\begin{aligned} &\int_0^\infty |A(1,t)| t^{-1/p}\, dt < \infty, \\ &\int_0^\infty |B(1,t)| t^{-1/p}\, dt < \infty, \\ &\int_0^\infty |C(1,t)| t^{-1/p}\, dt < \infty, \end{aligned} \tag{3.68}$$

and consequently A, B, and C are Hardy kernels for $L^p(\mathbb{R}_+)$ in the sense of Definition 2.2. The proof of the lemma is now complete. $\square$

Lemma 3.4. *Consider $\theta \in (0, 2\pi)$, $C_1 \in (0, \infty)$, $C_2 \in [0, \infty)$, and assume that the function $\widetilde{k} : \mathbb{R}_+ \times \mathbb{R}_+ \backslash \mathrm{diag}(\mathbb{R}_+ \times \mathbb{R}_+) \longrightarrow \mathbb{R}^{4\times 4}$ is as introduced in (3.43), with its entries given in (3.44)–(3.47). Then, for each $z \in \mathbb{C}$ with the property that $\operatorname{Re} z \in (0, 1)$ there holds*

$$\mathcal{M}(\widetilde{k}(\cdot, 1))(z) := \begin{pmatrix} -v(z) & 0 & -a(z) & b(z) \\ 0 & -v(z) & b(z) & -c(z) \\ a(z) & b(z) & v(z) & 0 \\ b(z) & c(z) & 0 & v(z) \end{pmatrix}, \tag{3.69}$$

where, with $\gamma := \pi - \theta$ and the constants C_1, C_2 as in (3.7),

$$v(z) := -2C_1\pi \cdot \frac{\cos(\pi z)}{\sin(\pi z)}, \tag{3.70}$$

$$a(z) := -\frac{2C_1\pi}{\sin(\pi z)} \cos(\gamma z + \theta) + \frac{C_2\pi(z-1)\sin\theta}{\sin(\pi z)} \sin(\gamma z + \theta), \tag{3.71}$$

$$b(z) := -\frac{C_2\pi(z-1)\sin\theta}{\sin(\pi z)} \cos(\gamma z + \theta), \tag{3.72}$$

$$c(z) := -\frac{2C_1\pi}{\sin(\pi z)} \cos(\gamma z + \theta) - \frac{C_2\pi(z-1)\sin\theta}{\sin(\pi z)} \sin(\gamma z + \theta). \tag{3.73}$$

Proof. Fix an angle $\theta \in (0, 2\pi) \backslash \{\pi\}$, pick a complex number $z \in \mathbb{C}$ satisfying $\operatorname{Re} z \in (0, 1)$, and consider the functions $g, h : \mathbb{R}_+ \longrightarrow \mathbb{R}$ given by

$$g(s) := \frac{1}{s^2 - 2s\cos\theta + 1} \quad \text{and} \quad h(s) := sg(s), \qquad \forall\, s \in \mathbb{R}_+. \tag{3.74}$$

Using (3.65) we have that $g, h \in \mathcal{C}(\mathbb{R}_+)$ and elementary calculations give

$$h'(s) = \frac{1 - s^2}{(s^2 - 2s\cos\theta + 1)^2}, \qquad \forall\, s \in \mathbb{R}_+. \tag{3.75}$$

Based on (3.74)–(3.75) and (3.50)–(3.52), we obtain that for each $s \in \mathbb{R}_+$ there holds

$$A(s, 1) = 2C_1(s - \cos\theta)g(s) - 2C_2(1 + \cos\theta)\sin^2(\tfrac{\theta}{2})h'(s), \tag{3.76}$$

$$B(s, 1) = -C_2 \sin\theta \Big(sg(s) + (s - \cos\theta)h'(s) \Big), \tag{3.77}$$

and

$$C(s, 1) = 2C_1(s - \cos\theta)g(s) + 2C_2(1 - \cos\theta)\cos^2(\tfrac{\theta}{2})h'(s). \tag{3.78}$$

Using (3.49) and (3.76)–(3.78) we may therefore write

$$\widetilde{k}(s,1) = \begin{pmatrix} -V(s,1) & 0 & -A(s,1) & B(s,1) \\ 0 & -V(s,1) & B(s,1) & -C(s,1) \\ A(s,1) & B(s,1) & V(s,1) & 0 \\ B(s,1) & C(s,1) & 0 & V(s,1) \end{pmatrix}, \qquad \forall\, s \in \mathbb{R}_+ \backslash \{1\}, \tag{3.79}$$

where

$$V(s,1) := \frac{2C_1}{s-1}, \qquad \forall\, s \in \mathbb{R}_+ \backslash \{1\}. \tag{3.80}$$

The next step is to compute the Mellin transform of each of the entries in the matrix in (3.79) at the point z. Employing formula 2.12 on p.14 in [38] (recall that $\operatorname{Re} z \in (0,1)$) and (3.70) we get

$$\mathcal{M}V(\cdot,1)(z) = -2C_1\pi\cot(\pi z) = v(z). \tag{3.81}$$

Next, based on (3.76)–(3.78) and (3.74), we also have

$$\begin{aligned} \mathcal{M}A(\cdot,1)(z) &= 2C_1 \cdot \mathcal{M}h(z) - 2C_1\cos\theta \cdot \mathcal{M}g(z) \\ &\qquad -2C_2(1+\cos\theta)\sin^2(\tfrac{\theta}{2}) \cdot \mathcal{M}h'(z), \end{aligned} \tag{3.82}$$

$$\mathcal{M}B(\cdot,1)(z) = -C_2\sin\theta \cdot \Big(\mathcal{M}h(z) + \mathcal{M}h'(z+1) - \cos\theta \cdot \mathcal{M}h'(z)\Big), \tag{3.83}$$

and

$$\begin{aligned} \mathcal{M}C(\cdot,1)(z) &= 2C_1 \cdot \mathcal{M}h(z) - 2C_1\cos\theta \cdot \mathcal{M}g(z) \\ &\qquad +2C_2(1-\cos\theta)\cos^2(\tfrac{\theta}{2}) \cdot \mathcal{M}h'(z). \end{aligned} \tag{3.84}$$

Going further, our goal is to compute the Mellin transforms $\mathcal{M}g(z)$, $\mathcal{M}h(z)$, $\mathcal{M}h'(z)$, $\mathcal{M}h'(z+1)$, and the value of $\mathcal{M}h(z) - \cos\theta \cdot \mathcal{M}g(z)$. First, employing formula 2.54 on p.23 in [38] (which requires that $\operatorname{Re} z \in (0,2)$ and $\theta \in (0,2\pi)$, conditions that are satisfied in the current setting) we have

$$\mathcal{M}g(z) = \pi\csc\theta \cdot \csc(\pi z) \cdot \sin[(\pi-\theta)z + \theta] = \pi \cdot \frac{\sin(\gamma z + \theta)}{\sin\theta \cdot \sin(\pi z)}, \tag{3.85}$$

where $\gamma := \pi - \theta$. Also, formula 1.3 on p.11 in [38] and formula (3.85) (the latter applied for $z+1$ which still satisfies $\operatorname{Re}(z+1) \in (0,2)$ as required) give that

$$\begin{aligned} \mathcal{M}h(z) = \mathcal{M}g(z+1) &= \pi\csc\theta \cdot \csc(\pi z) \cdot \sin[(\pi-\theta)z] \\ &= \pi \cdot \frac{\sin(\gamma z)}{\sin\theta \cdot \sin(\pi z)}. \end{aligned} \tag{3.86}$$

Based on (3.85) and (3.86) we obtain

$$\begin{aligned} \mathcal{M}h(z) - \cos\theta \cdot \mathcal{M}g(z) &= \frac{\pi}{\sin\theta \cdot \sin(\pi z)} \cdot \Big(\sin(\gamma z) - \cos\theta \cdot \sin(\gamma z + \theta)\Big) \\ &= -\pi \cdot \frac{\cos(\gamma z + \theta)}{\sin(\pi z)}, \end{aligned} \tag{3.87}$$

where the last equality above follows from the elementary trigonometric identity $\sin(\gamma z) - \cos\theta \cdot \sin(\gamma z + \theta) = -\sin\theta \cdot \cos(\gamma z + \theta)$.

Moving on, based on the definition of the function h from (3.74) it is straightforward to check that

$$\lim_{s\to 0^+} s^{z-1}h(s)=0 \quad \text{and} \quad \lim_{s\to\infty} s^{z-1}h(s)=0 \qquad \text{whenever} \quad z\in\mathbb{C} \ \text{ satisfies } \ \operatorname{Re} z\in(0,3). \tag{3.88}$$

In turn, (3.88), formula 1.9 on p.11 in [38] (which requires the properties in (3.88)), and the first identity in (3.86) guarantee that

$$\mathcal{M}h'(z)=-(z-1)\cdot\mathcal{M}h(z-1)=-(z-1)\cdot\mathcal{M}g(z). \tag{3.89}$$

Combining this with (3.85) yields

$$\mathcal{M}h'(z)=-\pi(z-1)\cdot\frac{\sin(\gamma z+\theta)}{\sin\theta\cdot\sin(\pi z)}. \tag{3.90}$$

Next, appealing again to (3.88) and formula 1.9 on p.11 in [38], this time with $z+1$ in place of z (note that in our setting the condition $\operatorname{Re}(z+1)\in(0,3)$ is still satisfied), we deduce that

$$\begin{aligned}\mathcal{M}h'(z+1)&=-z\cdot\mathcal{M}h(z)=-z\pi\cdot\csc\theta\cdot\csc(\pi z)\cdot\sin[(\pi-\theta)z]\\ &=-z\pi\cdot\frac{\sin(\gamma z)}{\sin\theta\cdot\sin(\pi z)}.\end{aligned} \tag{3.91}$$

Having established (3.85), (3.86), (3.87), (3.90) and (3.91), these identities in combination with (3.82)–(3.84) give that

$$\begin{aligned}\mathcal{M}A(\cdot,1)(z)&=a(z),\\ \mathcal{M}B(\cdot,1)(z)&=b(z),\\ \mathcal{M}C(\cdot,1)(z)&=c(z),\end{aligned} \tag{3.92}$$

where a,b,c are as in (3.71)–(3.73). Thus, the conclusion (3.69) of the lemma holds whenever $\theta\in(0,2\pi)\backslash\{\pi\}$.

There remains to treat the case when $\theta=\pi$ and to this end we start by picking $z\in\mathbb{C}$ with $\operatorname{Re} z\in(0,1)$. On the one hand (3.76)–(3.78) give that

$$A(s,1)=C(s,1)=\frac{2C_1}{s+1} \quad \text{and} \quad B(s,1)=0, \qquad \forall\, s\in\mathbb{R}_+. \tag{3.93}$$

On the other hand, thanks to (3.71)–(3.73) and the fact that $\theta=\pi$, we obtain that

$$a(z)=c(z)=\frac{2C_1\pi}{\sin(\pi z)} \quad \text{and} \quad b(z)=0. \tag{3.94}$$

Then the identities in (3.92) continue to hold, since due to formula 2.4 on p.13 in [38] one has

$$\mathcal{M}\Big(\frac{1}{\cdot+1}\Big)(z)=\frac{\pi}{\sin(\pi z)} \quad \text{if} \quad z\in\mathbb{C} \ \text{ and } \ \operatorname{Re} z\in(0,1). \tag{3.95}$$

The proof of the lemma is now complete. $\square$

The next result will be useful in computing the determinant of the matrix in (3.69).

Lemma 3.5. *Let $n \in \mathbb{N}$ and assume that M, N, S, T are $n \times n$ matrices with complex entries satisfying the property that $MS = SM$. Then*

$$\det \begin{pmatrix} M & N \\ S & T \end{pmatrix} = \det (MT - SN). \tag{3.96}$$

Proof. Assume first that the matrix M is invertible and denote by M^{-1} its inverse. Then, with O standing for the $n \times n$ matrix with zero entries, we clearly have

$$\det \begin{pmatrix} I & O \\ SM^{-1} & -I \end{pmatrix} = (-1)^n, \tag{3.97}$$

and

$$\begin{pmatrix} I & O \\ SM^{-1} & -I \end{pmatrix} \cdot \begin{pmatrix} M & N \\ S & T \end{pmatrix} = \begin{pmatrix} M & N \\ O & SM^{-1}N - T \end{pmatrix}. \tag{3.98}$$

Thus, taking the determinant in each side of (3.98) and using (3.97), we obtain

$$\begin{aligned} (-1)^n \cdot \det \begin{pmatrix} M & N \\ S & T \end{pmatrix} &= \det M \cdot \det (SM^{-1}N - T) \\ &= \det (MSM^{-1}N - MT) \\ &= (-1)^n \cdot \det(MT - SN), \end{aligned} \tag{3.99}$$

where, in the last equality above, we have used that $MS = SM$. From (3.99) the identity (3.96) easily follows.

The case when the matrix M is not invertible follows from the fact that the set of invertible matrices is a dense subset of the set of $n \times n$ matrices with complex entries. Indeed, for M as in the hypothesis and for each $t \in \mathbb{C}$ introduce

$$M_t := M + tI_{n\times n}. \tag{3.100}$$

Then $\det M_t = \det(M + tI_{n\times n}) = p_M(t)$, where p_M is a polynomial of degree n in the variable $t \in \mathbb{C}$. Consequently, there exist disjoint values $\ell_1, \dots, \ell_N \in \mathbb{C}$ with $N \leq n$ such that $p_M(t) = 0$ if and only if $t \in \{\ell_1, \dots, \ell_N\}$ and as such

$$M_t \quad \text{is invertible for each} \quad t \in \mathbb{C}\backslash\{\ell_1, \dots, \ell_N\}. \tag{3.101}$$

Next, consider a sequence $\{t_j\}_{j\in\mathbb{N}}$ satisfying

$$\{t_j\}_{j\in\mathbb{N}} \subseteq \mathbb{C}\backslash\{\ell_1, \dots, \ell_N\} \quad \text{and} \quad \lim_{j\to\infty} t_j = 0. \tag{3.102}$$

From the first part of (3.102) and (3.101) we obtain that M_{t_j} is an invertible matrix for each $j \in \mathbb{N}$. Using this and the fact that $SM_{t_j} = M_{t_j}S$ for each $j \in \mathbb{N}$ (an immediate consequence of the fact that S and M commute and the definition of M_t), based on the first part of the proof we may therefore write

$$\det \begin{pmatrix} M_{t_j} & N \\ S & T \end{pmatrix} = \det(M_{t_j}T - SN). \tag{3.103}$$

Finally, using (3.103) and the continuity of the determinant function the desired equality (3.96) then follows. $\square$

Corollary 3.6. *Let $\theta \in (0, 2\pi)$, $C_1 \in (0, \infty)$ and $C_2 \in [0, \infty)$, and recall the function $\widetilde{k}$ from (3.43) with entries as in (3.44)–(3.47) where for each $i, j \in \{1, 2\}$ the functions A_{ij} are as in (3.32)–(3.35). Then $z \in \mathbb{C}$ with the property that $\operatorname{Re} z \in (0, 1)$ satisfies $\det \mathcal{M}(\widetilde{k}(\cdot, 1))(z) = 0$ if and only if one of the following identities holds*

$$\kappa(z-1)\sin\theta = \sin[(2\pi-\theta)(z-1)], \tag{3.104}$$

$$\kappa(z-1)\sin\theta = -\sin[(2\pi-\theta)(z-1)], \tag{3.105}$$

$$\kappa(z-1)\sin\theta = \sin[\theta(z-1)], \tag{3.106}$$

$$\kappa(z-1)\sin\theta = -\sin[\theta(z-1)], \tag{3.107}$$

where

$$\kappa := \frac{C_2}{2C_1}. \tag{3.108}$$

Proof. Fix a complex number $z \in \mathbb{C}$ such that $\operatorname{Re} z \in (0, 1)$. In light of (3.69) from Lemmas 3.4 and 3.5, applied for the choice of matrices $M := -v(z) \cdot I_{2\times 2}$, $T := v(z) \cdot I_{2\times 2}$, and

$$N := \begin{pmatrix} -a(z) & b(z) \\ b(z) & -c(z) \end{pmatrix} \quad \text{and} \quad S := \begin{pmatrix} a(z) & b(z) \\ b(z) & c(z) \end{pmatrix}, \tag{3.109}$$

with $v(z), a(z), b(z)$ and $c(z)$ as in (3.70)–(3.73), elementary algebraic manipulations give that

$$\det \mathcal{M}(\widetilde{k}(\cdot, 1))(z) = \det \begin{pmatrix} -v^2(z) + a^2(z) - b^2(z) & -b(z)\big[a(z) - c(z)\big] \\ b(z)\big[a(z) - c(z)\big] & -v^2(z) - b^2(z) + c^2(z) \end{pmatrix}. \tag{3.110}$$

Thus

$$\begin{aligned} \det \mathcal{M}(\widetilde{k}(\cdot, 1))(z) &= \big[-v^2(z) + a^2(z) - b^2(z)\big]\big[-v^2(z) - b^2(z) + c^2(z)\big] \\ &\quad + b^2(z)\big[a(z) - c(z)\big]^2 \\ &= \big[v^2(z) + b^2(z) - a(z)c(z)\big]^2 - v^2(z)\big[a(z) - c(z)\big]^2, \end{aligned} \tag{3.111}$$

where the last equality follows from straightforward algebra. Using (3.111) we can therefore conclude that

$$\begin{gathered} \det \mathcal{M}(\widetilde{k}(\cdot, 1))(z) = 0 \ \text{ if and only if} \\ v^2(z) + b^2(z) - a(z)c(z) = \pm v(z)\big[a(z) - c(z)\big]. \end{gathered} \tag{3.112}$$

Next, due to (3.71) and (3.73) there holds

$$a(z)c(z) = \frac{\pi^2}{\sin^2(\pi z)} \cdot \big[4C_1^2 \cdot \cos^2(\gamma z + \theta) - C_2^2 \cdot (z-1)^2 \cdot \sin^2\theta \cdot \sin^2(\gamma z + \theta)\big], \tag{3.113}$$

where as before $\gamma := \pi - \theta$. In turn, (3.113) combined with (3.70) and (3.72) gives that

$$v^2(z) + b^2(z) - a(z)c(z) = \frac{\pi^2}{\sin^2(\pi z)} \cdot \Big[4C_1^2 \cdot \Big(\cos^2(\pi z) - \cos^2(\gamma z + \theta)\Big) + C_2^2 \cdot (z-1)^2 \cdot \sin^2\theta\Big], \tag{3.114}$$

and

$$v(z)[a(z) - c(z)] = -\frac{4C_1C_2\pi^2}{\sin^2(\pi z)} \cdot (z-1)\cos(\pi z) \cdot \sin\theta \cdot \sin(\gamma z + \theta). \tag{3.115}$$

Next, based on the Pythagorean Theorem we write the following sequence of trigonometric identities

$$\begin{aligned} \cos^2(\pi z) - \cos^2(\gamma z + \theta) &= \sin^2(\gamma z + \theta) - \sin^2(\pi z) \\ &= \sin^2(\gamma z+\theta)\cos^2(\pi z) + \sin^2(\pi z)\big(\sin^2(\gamma z+\theta) - 1\big) \\ &= \sin^2(\gamma z + \theta)\cos^2(\pi z) - \cos^2(\gamma z + \theta)\sin^2(\pi z). \end{aligned} \tag{3.116}$$

Thus, using (3.116), the notation introduced in (3.108), and (3.114), we obtain

$$\begin{aligned} &v^2(z) + b^2(z) - a(z)c(z) \\ &\quad= \frac{4C_1^2\pi^2}{\sin^2(\pi z)} \cdot \Big[\sin^2(\gamma z + \theta)\cos^2(\pi z) - \cos^2(\gamma z + \theta)\sin^2(\pi z)\Big] \\ &\qquad+ \frac{4C_1^2\pi^2}{\sin^2(\pi z)} \cdot \kappa^2 \cdot (z-1)^2 \cdot \sin^2\theta. \end{aligned} \tag{3.117}$$

Based on this and (3.115), cancel $\dfrac{4C_1^2\pi^2}{\sin^2(\pi z)}$ from both sides of the identity $v^2(z) + b^2(z) - a(z)c(z) = \pm v(z)\big[a(z) - c(z)\big]$ to obtain that $v^2(z) + b^2(z) - a(z)c(z) = \pm v(z)\big[a(z) - c(z)\big]$ if and only if

$$\begin{gathered} \sin^2(\gamma z + \theta)\cos^2(\pi z) - \cos^2(\gamma z + \theta)\sin^2(\pi z) + \kappa^2(z-1)^2\sin^2\theta \\ = \pm 2\kappa(z-1)\sin\theta\sin(\gamma z + \theta)\cos(\pi z). \end{gathered} \tag{3.118}$$

In turn, (3.118) can be rewritten as

$$\Big(\sin(\gamma z + \theta)\cos(\pi z) \pm \kappa(z-1)\sin\theta\Big)^2 = \sin^2(\pi z)\cos^2(\gamma z + \theta). \tag{3.119}$$

At this point, (3.112) and (3.119) give that

$$\begin{gathered} \det \mathcal{M}\widetilde{k}(\cdot, 1)(z) = 0 \ \text{ if and only if} \\ \sin(\gamma z + \theta)\cos(\pi z) \pm \kappa(z-1)\sin\theta = \pm\sin(\pi z)\cos(\gamma z + \theta), \end{gathered} \tag{3.120}$$

where the choices of sign $\pm$ in the left-hand side and right-hand side of (3.120) are independent of one another. In light of the following useful identities

$$\begin{aligned}
&-\sin(\gamma z+\theta)\cos(\pi z)+\sin(\pi z)\cos(\gamma z+\theta)\\
&\quad=\sin(\pi z-\gamma z-\theta)\\
&\quad=\sin[\theta(z-1)],
\end{aligned}\tag{3.121}$$

and

$$\begin{aligned}
&-\sin(\gamma z+\theta)\cos(\pi z)-\sin(\pi z)\cos(\gamma z+\theta)\\
&\quad=-\sin(\pi z+\gamma z+\theta)\\
&\quad=-\sin[(2\pi-\theta)(z-1)],
\end{aligned}\tag{3.122}$$

statement (3.120) becomes

$$\begin{aligned}
&\det \mathcal{M}\widetilde{k}(\cdot,1)(z)=0\\
&\Longleftrightarrow \kappa(z-1)\sin\theta=\begin{cases}\pm\sin[\theta(z-1)]\\ \text{or}\\ \pm\sin[(2\pi-\theta)(z-1)].\end{cases}
\end{aligned}\tag{3.123}$$

This finishes the proof of Corollary 3.6. □

Our next goal is to identify those complex numbers $z\in\mathbb{C}$ with the property that $\operatorname{Re} z\in(0,1)$ and which also satisfy (3.123). An important ingredient in achieving this is the following result.

Lemma 3.7. *Let $\theta\in(0,2\pi)$ and assume that the constants $C_1\in(0,\infty)$ and $C_2\in[0,\infty)$ are such that*

$$\kappa:=\frac{C_2}{2C_1}\in[0,1].\tag{3.124}$$

Then the following implication holds:

$$\begin{gathered}\textit{if}\ \ z\in\mathbb{C}\ \ \textit{is such that}\ \operatorname{Re} z\in(0,1)\\ \textit{and one of the identities (3.104)–(3.107) holds, then}\ \ \operatorname{Im} z=0.\end{gathered}\tag{3.125}$$

Proof. First note that changing θ to $2\pi-\theta$ in any one of the Eqs. (3.104), (3.105), (3.106) or (3.107) yields one of the other three equations. Consequently, it suffices to restrict our analysis to the case when $\theta\in(0,\pi]$. Going further, since for any $w\in\mathbb{C}$ one has

$$\sin(\overline{w})=\overline{\sin(w)},\tag{3.126}$$

where the bar denotes conjugation of complex numbers, a quick inspection of (3.104)–(3.107) shows that if $z\in\mathbb{C}$ satisfies one of the Eqs. (3.104)–(3.107) then so does $\overline{z}$. In this light, (3.125) follows as soon as we establish that

$$\begin{gathered}\text{if}\ \theta\in(0,\pi]\ \ \text{and}\ \ z\in\mathbb{C},\ \ \operatorname{Re} z\in(0,1)\ \text{and}\ \operatorname{Im} z\in[0,\infty)\\ \text{and one of the identities (3.104)–(3.107) holds, then}\ \ \operatorname{Im} z=0.\end{gathered}\tag{3.127}$$

First we will show that the implication (3.127) is true in the case when $\theta=\pi$ or $\kappa=0$. Indeed, if $\theta=\pi$ or $\kappa=0$, then the left-hand sides of (3.104)–(3.107) are all equal to zero and having any one of these equations satisfied requires that

$$\text{either} \quad \sin[(2\pi-\theta)(z-1)] = 0 \quad \text{or} \quad \sin[\theta(z-1)] = 0. \tag{3.128}$$

However, since all the zeros of the sine function lie on the real line, it follows that in the current case $z-1 \in \mathbb{R}$ and hence $\operatorname{Im} z = 0$ as desired.

Therefore it remains to consider the implication (3.127) when

$$\begin{gathered} \theta \in (0,\pi), \ \ \kappa \in (0,1], \\ \text{and} \quad z \in \mathbb{C} \ \text{ is such that } \operatorname{Re} z \in (0,1) \text{ and } \operatorname{Im} z \in [0,\infty), \end{gathered} \tag{3.129}$$

which follows immediately as soon as we establish that

$$\begin{gathered} \text{if} \ \ \theta \in (0,\pi), \ \ \kappa \in (0,1], \\ \text{and} \quad z \in \mathbb{C} \ \text{ is such that } \operatorname{Re} z \in (0,1) \text{ and } \operatorname{Im} z \in (0,\infty), \\ \text{then none of the Eqs. (3.104)–(3.107) is satisfied.} \end{gathered} \tag{3.130}$$

Indeed, if any of the Eqs. (3.104)–(3.107) are satisfied (with θ, κ and z as in (3.129)) then, using (3.130) necessarily $\operatorname{Im} z = 0$.

With the goal of establishing (3.130) fix $\theta \in (0,\pi)$ and $\kappa \in (0,1]$. We shall treat each of the four Eqs. (3.104)–(3.107) as a separate case. Before proceeding with this plan, let us recall the Taylor series expansions of the functions sinh and cosh,

$$\sinh t = \sum_{j=0}^{\infty} \frac{t^{2j+1}}{(2j+1)!} \quad \text{and} \quad \cosh t = \sum_{j=0}^{\infty} \frac{t^{2j}}{(2j)!}, \qquad t \in \mathbb{R}. \tag{3.131}$$

Case 1. *If z is as in (3.130) then*

$$\kappa(z-1)\sin\theta \neq \sin[(2\pi-\theta)(z-1)], \tag{3.132}$$

i.e., Eq. (3.104) is not satisfied.

We shall argue by contradiction and to this end assume that

$$\begin{gathered} \exists\, z = x_0 + iy_0 \in \mathbb{C} \ \text{ with } \ x_0 \in (0,1) \ \text{ and } \ y_0 \in (0,\infty) \\ \text{such that (3.104) holds.} \end{gathered} \tag{3.133}$$

Introduce the functions $G, H : [0,1] \times (0,\infty) \to \mathbb{R}$ given by

$$G(x,y) := \kappa(x-1)\cdot\sin\theta - \sin[(2\pi-\theta)(x-1)]\cdot\cosh[(2\pi-\theta)y], \tag{3.134}$$

$$H(x,y) := \kappa y\cdot\sin\theta - \cos[(2\pi-\theta)(x-1)]\cdot\sinh[(2\pi-\theta)y]. \tag{3.135}$$

By taking the real and imaginary parts in (3.104), under assumption (3.133) we obtain that the system of two equations with two unknowns x and y,

$$\begin{cases} G(x,y) = 0, \\ H(x,y) = 0, \end{cases} \tag{3.136}$$

has $(x_0, y_0) \in (0,1) \times (0,\infty)$ as a solution. Since for $y_0 > 0$ and $\theta \in (0,\pi)$ the hyperbolic trigonometric functions in (3.134)–(3.135) have positive values, it is necessary that

$$\sin[(2\pi-\theta)(x_0-1)] < 0 \quad \text{and} \quad \cos[(2\pi-\theta)(x_0-1)] > 0. \tag{3.137}$$

In turn, conditions (3.137) along with the fact that $x_0 \in (0,1)$ and $\theta \in (0,\pi)$ force the membership $(2\pi-\theta)(x_0-1) \in (-\pi/2, 0)$, i.e.,

$$x_0 \in I_1 := \left(\frac{3\pi - 2\theta}{2(2\pi-\theta)}, 1\right). \tag{3.138}$$

Note that for $\theta \in (0,\pi)$ the inequality $1 > (3\pi-\theta)/(2(2\pi-\theta)) > 0$ holds and consequently $I_1 \subset (0,1)$. In addition,

$$\sin[(2\pi-\theta)(x-1)] < 0 \ \text{ and } \ \cos[(2\pi-\theta)(x-1)] > 0, \quad \forall\, x \in I_1. \tag{3.139}$$

Therefore,

$$\exists\,(x_0,y_0) \in I_1 \times (0,\infty) \ \text{ such that } \ G(x_0,y_0) = H(x_0,y_0) = 0. \tag{3.140}$$

Going further, using the Taylor expansion for the hyperbolic sine function given in (3.131) we obtain that for each $x \in [0,1]$ and $y \in (0,\infty)$ there holds

$$H(x,y) = h_1(x)\cdot y + \sum_{j=1}^{\infty} h_{2j+1}(x)\cdot y^{2j+1}, \tag{3.141}$$

where the functions $h_{2j+1} : [0,1] \to \mathbb{R}$, for $j \in \mathbb{N}\cup\{0\}$ are given by

$$h_1(x) := \kappa\cdot\sin\theta - \cos[(2\pi-\theta)(x-1)]\cdot(2\pi-\theta), \tag{3.142}$$

and

$$h_{2j+1}(x) := -\cos[(2\pi-\theta)(x-1)]\cdot\frac{(2\pi-\theta)^{2j+1}}{(2j+1)!}, \quad j \in \mathbb{N}. \tag{3.143}$$

Thanks to the second inequality in (3.139) and the definition (3.143), we have that $h_{2j+1}(x) < 0$ for all $x \in I_1$ and $j \geq 1$. In particular, $h_{2j+1}(x_0) < 0$ for all $j \geq 1$. Thus $H(x_0,y_0) = 0$ necessarily requires that $h_1(x_0) > 0$. Notice from (3.142) that the function h_1 is continuous, and hence

$$\begin{gathered}\exists\,\varepsilon > 0 \ \text{ such that } (x_0-\varepsilon, x_0+\varepsilon) \subset I_1 \\ \text{and } \ h_1(x) > 0 \ \text{ for all } \ x \in (x_0-\varepsilon, x_0+\varepsilon).\end{gathered} \tag{3.144}$$

Next, using again (3.142) and the first inequality in (3.139), we obtain

$$h_1'(x) = \sin[(2\pi-\theta)(x-1)]\cdot(2\pi-\theta)^2 < 0, \qquad \forall\, x \in I_1. \tag{3.145}$$

Therefore, the function h_1 is monotonically decreasing on the interval I_1, which further combined with (3.144) yields

$$h_1(x) > 0 \ \text{ for all } \ x \in \left(\frac{3\pi-2\theta}{2(2\pi-\theta)}, x_0+\varepsilon\right). \tag{3.146}$$

Reasoning similarly, this time based on the Taylor expansion of the cosh function from (3.131) we obtain that for each $x \in [0,1]$ and $y \in (0,\infty)$ there holds

$$G(x,y) = g_0(x) + \sum_{j=1}^{\infty} g_{2j}(x)\cdot y^{2j}, \tag{3.147}$$

where $g_{2j} : [0,1] \to \mathbb{R}$, for $j \in \mathbb{N}\cup\{0\}$, are given by

$$g_0(x) := \kappa(x-1)\cdot\sin\theta - \sin[(2\pi-\theta)(x-1)], \tag{3.148}$$

and

$$g_{2j}(x) := -\sin[(2\pi-\theta)(x-1)] \cdot \frac{(2\pi-\theta)^{2j}}{(2j)!}, \quad j \in \mathbb{N}. \tag{3.149}$$

Upon recalling the first inequality in (3.137) it follows that $g_{2j}(x_0) > 0$ for all $j \in \mathbb{N}$. Consequently, since $G(x_0, y_0) = 0$ and $y_0 \in (0, \infty)$, we obtain on the one hand that

$$g_0(x_0) < 0. \tag{3.150}$$

On the other hand, based on (3.148) and (3.142), we may write

$$g_0'(x) = \kappa \cdot \sin\theta - \cos[(2\pi-\theta)(x-1)] \cdot (2\pi-\theta) = h_1(x), \qquad \forall x \in [0,1]. \tag{3.151}$$

Thus (3.151) and (3.146) imply that

$$g_0'(x) > 0 \ \text{ for all } \ x \in \left(\frac{3\pi-2\theta}{2(2\pi-\theta)}, x_0+\varepsilon\right), \tag{3.152}$$

and in particular the function g_0 is increasing on $\left(\frac{3\pi-2\theta}{2(2\pi-\theta)}, x_0+\varepsilon\right)$. A simple inspection of (3.148) shows that g_0 is also continuous on the interval $[0,1]$. Since $x_0 \in \left(\frac{3\pi-2\theta}{2(2\pi-\theta)}, x_0+\varepsilon\right)$ we may therefore conclude that

$$g_0(x_0) > g_0\left(\frac{3\pi-2\theta}{2(2\pi-\theta)}\right) = 1 - \frac{\kappa\pi}{2(2\pi-\theta)} \cdot (\sin\theta) > 0, \tag{3.153}$$

where the last inequality follows using $\kappa \in (0,1]$, that $\pi/(2(2\pi-\theta)) < 1$, and $\sin\theta \in (0,1]$ whenever $\theta \in (0,\pi)$. However, (3.153) contradicts (3.150) and finishes the argument by contradiction. Consequently the assumption (3.133) is violated and this establishes the statement made at the beginning of Case 1 completing our analysis in this case.

Case 2. *If z is as in (3.130) then*

$$\kappa(z-1)\sin\theta \neq -\sin[(2\pi-\theta)(z-1)], \tag{3.154}$$

i.e., Eq. (3.105) is not satisfied.

Again we shall argue by contradiction and as such we assume that

$$\begin{gathered} \exists\, z = x_0 + iy_0 \in \mathbb{C} \ \text{ with } \ x_0 \in (0,1) \ \text{ and } \ y_0 \in (0,\infty) \\ \text{such that (3.105) holds.} \end{gathered} \tag{3.155}$$

Introducing the functions $M, N : [0,1] \times (0,\infty) \to \mathbb{R}$ given by

$$M(x,y) := \kappa(x-1) \cdot \sin\theta + \sin[(2\pi-\theta)(x-1)] \cdot \cosh[(2\pi-\theta)y], \tag{3.156}$$

$$N(x,y) := \kappa y \cdot \sin\theta + \cos[(2\pi-\theta)(x-1)] \cdot \sinh[(2\pi-\theta)y], \tag{3.157}$$

and taking the real and imaginary parts of (3.105) we obtain that the following system of two equations with two unknowns x and y

$$\begin{cases} M(x,y) = 0, \\ N(x,y) = 0, \end{cases} \tag{3.158}$$

has $(x_0, y_0) \in (0,1) \times (0,\infty)$ as a solution. An inspection of the signs of the terms involved in the expressions in (3.156) and (3.157) shows that if $(x_0, y_0) \in (0,1) \times (0,\infty)$ is a solution of the system (3.158), then

$$\sin[(2\pi-\theta)(x_0-1)] > 0 \ \text{ and } \ \cos[(2\pi-\theta)(x_0-1)] < 0. \tag{3.159}$$

In turn, (3.159) along with the fact that $x_0 \in (0,1)$ and $\theta \in (0,\pi)$ force that $(2\pi-\theta)(x_0-1) \in (-3\pi/2, -\pi)$. Consequently, $x_0 \in \left(\frac{\pi-2\theta}{2(2\pi-\theta)}, \frac{\pi-\theta}{2\pi-\theta}\right) \cap (0,1)$, which further yields

$$x_o \in \begin{cases} \left(\frac{\pi-2\theta}{2(2\pi-\theta)}, \frac{\pi-\theta}{2\pi-\theta}\right) =: I_2, \text{ if } \theta \in \left(0, \frac{\pi}{2}\right], \\ \left(0, \frac{\pi-\theta}{2\pi-\theta}\right) =: I_3, \text{ if } \theta \in \left(\frac{\pi}{2}, \pi\right). \end{cases} \tag{3.160}$$

Note that one has

$$\begin{gathered} \sin[(2\pi-\theta)(x-1)] > 0 \quad \text{and} \quad \cos[(2\pi-\theta)(x-1)] < 0 \\ \text{whenever} \quad x \in \left(\frac{\pi-2\theta}{2(2\pi-\theta)}, \frac{\pi-\theta}{2\pi-\theta}\right) \cap (0,1). \end{gathered} \tag{3.161}$$

Going further, thanks to the first identity in (3.131), for each $x \in [0,1]$ and each $y \in (0,\infty)$ there holds

$$N(x,y) = \eta_1(x) \cdot y + \sum_{j=1}^{\infty} \eta_{2j+1}(x) \cdot y^{2j+1}, \tag{3.162}$$

where the functions $\eta_{2j+1} : [0,1] \to \mathbb{R}$, for $j \in \mathbb{N} \cup \{0\}$, are given by

$$\eta_1(x) := \kappa \cdot \sin\theta + \cos[(2\pi-\theta)(x-1)] \cdot (2\pi-\theta), \tag{3.163}$$

and

$$\eta_{2j+1}(x) := \cos[(2\pi-\theta)(x-1)] \cdot \frac{(2\pi-\theta)^{2j+1}}{(2j+1)!}, \quad j \in \mathbb{N}. \tag{3.164}$$

Appealing to the second inequality in (3.159) and (3.164), we obtain that

$$\eta_{2j+1}(x_0) < 0 \quad \text{for each} \quad j \in \mathbb{N}. \tag{3.165}$$

Therefore (3.162) and (3.165) in conjunction with the vanishing assumption $N(x_0, y_0) = 0$ imply that $\eta_1(x_0) > 0$. Moreover, thanks to the continuity of the function η_1 and the fact that the intersection of two open intervals is an open set, we may further conclude that there exists $\varepsilon > 0$ with the property that

$$(x_0 - \varepsilon, x_0 + \varepsilon) \subseteq \left(\frac{\pi-2\theta}{2(2\pi-\theta)}, \frac{\pi-\theta}{2\pi-\theta}\right) \cap (0,1), \tag{3.166}$$

and

$$\eta_1(x) > 0 \quad \text{for all} \quad x \in (x_0 - \varepsilon, x_0 + \varepsilon). \tag{3.167}$$

Next, differentiating in (3.163) yields

$$\begin{gathered} \eta_1'(x) = -\sin[(2\pi-\theta)(x-1)] \cdot (2\pi-\theta)^2 < 0 \\ \text{for each} \quad x \in \left(\frac{\pi-2\theta}{2(2\pi-\theta)}, \frac{\pi-\theta}{2\pi-\theta}\right) \cap (0,1), \end{gathered} \tag{3.168}$$

where the inequality above follows from the first inequality in (3.161). In particular, using (3.160) we obtain that the function η_1 is decreasing on the interval I_2 when $\theta \in (0, \pi/2]$, and that the function η_1 is decreasing on the interval I_3 when $\theta \in (\pi/2, \pi)$. These facts, combined with (3.166)–(3.167), guarantee that

$$\eta_1(x) > 0 \;\text{ for all }\; x \in \left(\frac{\pi - 2\theta}{2(2\pi - \theta)}, x_0 + \varepsilon\right) \;\text{ whenever }\; \theta \in \left(0, \frac{\pi}{2}\right], \tag{3.169}$$

and

$$\eta_1(x) > 0 \;\text{ for all }\; x \in (0, x_0 + \varepsilon) \;\text{ whenever }\; \theta \in \left(\frac{\pi}{2}, \pi\right). \tag{3.170}$$

Turning our attention to the function M, based on the second identity in (3.131) for each $x \in [0,1]$ and each $y \in (0, \infty)$ we may write

$$M(x,y) = \xi_0(x) + \sum_{j=1}^{\infty} \xi_{2j}(x) \cdot y^{2j}, \tag{3.171}$$

where the functions $\xi_0, \xi_{2j} : [0,1] \to \mathbb{R}$, $j \in \mathbb{N}$, are given by

$$\xi_0(x) := \kappa(x-1) \cdot \sin\theta + \sin[(2\pi - \theta)(x-1)], \tag{3.172}$$

and

$$\xi_{2j}(x) := \sin[(2\pi - \theta)(x-1)] \cdot \frac{(2\pi - \theta)^{2j}}{(2j)!}, \quad j \in \mathbb{N}. \tag{3.173}$$

Thanks to the first inequality in (3.159), one has $\xi_{2j}(x_0) > 0$ for all $j \in \mathbb{N}$. Since $M(x_0, y_0) = 0$, this and (3.171) further force that

$$\xi_0(x_0) < 0. \tag{3.174}$$

On the other hand, differentiating in (3.172) and using (3.163) yields

$$\begin{gathered} \xi_0'(x) = \kappa \cdot \sin\theta + \cos[(2\pi - \theta)(x-1)] \cdot (2\pi - \theta) = \eta_1(x) \\ \text{for each }\; x \in [0,1]. \end{gathered} \tag{3.175}$$

Using (3.175) along with properties (3.169) and (3.170) it follows that the function ξ_0 is increasing on the interval $((\pi - 2\theta)/(2(2\pi - \theta)), x_0 + \varepsilon)$ when $\theta \in (0, \pi/2]$, and that the function ξ_0 is increasing on the interval $(0, x_0 + \varepsilon)$ when $\theta \in (\pi/2, \pi)$. Based on this and the continuity of ξ_0 on $[0,1]$ it follows that when $\theta \in (0, \pi/2]$ one has

$$\xi_0(x_0) > \xi_0\left(\frac{\pi - 2\theta}{2(2\pi - \theta)}\right) = 1 - \frac{3\kappa\pi}{2(2\pi - \theta)} \cdot \sin\theta \geq 0, \tag{3.176}$$

where the last inequality follows from the fact that $\kappa \in (0,1]$ and in the current case one has $\sin\theta \in (0,1]$ and $\frac{3\pi}{2(2\pi-\theta)} \in (0,1]$. Similarly, in the case when $\theta \in (\pi/2, \pi)$, there holds

$$\xi_0(x_0) > \xi_0(0) = (1 - \kappa) \cdot \sin\theta \geq 0, \tag{3.177}$$

where the inequality above follows immediately since $\kappa \in (0,1]$ and the aperture $\theta \in (\pi/2, \pi)$. However (3.176)–(3.177) contradict (3.174) and this completes the argument in this case.

Case 3. *If z is as in (3.130) then*

$$\kappa(z-1)\sin\theta \neq \sin[\theta(z-1)], \tag{3.178}$$

i.e., Eq. (3.106) is not satisfied.

As in the previous two cases, in order to prove the claim above we shall argue by contradiction. To this end assume that

$$\begin{gathered}\exists\, z = x_0 + i y_0 \in \mathbb{C} \ \text{ with } \ x_0 \in (0,1) \ \text{ and } \ y_0 \in (0,\infty) \\ \text{such that (3.106) holds.}\end{gathered} \tag{3.179}$$

Introducing the functions $R, S : [0,1] \times (0,\infty) \to \mathbb{R}$ given by

$$R(x,y) := \kappa(x-1)\cdot\sin\theta - \sin[\theta(x-1)]\cdot\cosh(\theta y), \tag{3.180}$$

$$S(x,y) := \kappa y\cdot\sin\theta - \cos[\theta(x-1)]\cdot\sinh(\theta y), \tag{3.181}$$

and taking the real and imaginary parts of both sides of the identity (3.106) we obtain that the pair $(x_0, y_0) \in (0,1) \times (0,\infty)$ is a solution of the following system of two equations

$$\begin{cases} R(x,y) = 0, \\ S(x,y) = 0. \end{cases} \tag{3.182}$$

An inspection of the sign of each of the terms appearing in (3.180) and (3.181) shows that necessarily

$$\sin[\theta(x_0-1)] < 0 \ \text{ and } \ \cos[\theta(x_0-1)] > 0, \tag{3.183}$$

and consequently $\theta(x_0 - 1) \in (-\pi/2, 0)$. However, this further implies that

$$x_0 \in \Big(1 - \frac{\pi}{2\theta}\Big) \cap (0,1) = \begin{cases} (0,1), & \text{if } \ \theta \in \big(0, \frac{\pi}{2}\big], \\ \big(1 - \frac{\pi}{2\theta}, 1\big) =: I_4, & \text{if } \ \theta \in \big(\frac{\pi}{2}, \pi\big). \end{cases} \tag{3.184}$$

We shall also find it useful to observe that, in fact,

$$\begin{gathered}\sin[\theta(x-1)] < 0 \ \text{ and } \ \cos[\theta(x-1)] > 0, \\ \text{whenever } \ x \in \big(1 - \tfrac{\pi}{2\theta}\big) \cap (0,1).\end{gathered} \tag{3.185}$$

Going further, based on the first identity in (3.131), for each $x \in [0,1]$ and each $y \in (0,\infty)$ we may write

$$S(x,y) = s_1(x)\cdot y + \sum_{j=1}^{\infty} s_{2j+1}(y)\cdot y^{2j+1}, \tag{3.186}$$

where the functions $s_1, s_{2j+1} : [0,1] \to \mathbb{R}$, $j \in \mathbb{N}$, are given by

$$s_1(x) := \kappa\cdot\sin\theta - \cos[\theta(x-1)]\cdot\theta, \tag{3.187}$$

and

$$s_{2j+1}(x) := -\cos[\theta(x-1)]\cdot\frac{\theta^{2j+1}}{(2j+1)!}, \quad j \in \mathbb{N}. \tag{3.188}$$

Next, thanks to the second inequality in (3.183) we have $s_{2j+1}(x_0) < 0$ for all $j \in \mathbb{N}$. Since $S(x_0, y_0) = 0$ and $y_0 \in (0,\infty)$, (3.186) implies that $s_1(x_0) > 0$. The function s_1 introduced in (3.187) is continuous, thus this further implies there exists $\varepsilon > 0$ such that

$$(x_0 - \varepsilon, x_0 + \varepsilon) \subset \Big(1 - \frac{\pi}{2\theta}\Big) \cap (0,1), \tag{3.189}$$

and

$$s_1(x) > 0 \ \text{ for all } \ x \in (x_0 - \varepsilon, x_0 + \varepsilon). \tag{3.190}$$

Next, differentiating (3.187) and using the first inequality in (3.185) yields

$$s_1'(x) = \sin[\theta(x-1)] \cdot \theta^2 < 0, \qquad \forall\, x \in \Big(1 - \frac{\pi}{2\theta}\Big) \cap (0,1). \tag{3.191}$$

Recalling (3.184), we may therefore conclude that the function s_1 is decreasing on the interval $(0,1)$ whenever $\theta \in (0,\pi/2]$, and that s_1 is decreasing on the interval I_4 whenever $\theta \in (\pi/2,\pi)$. Combining this information with (3.190) gives that

$$s_1(x) > 0 \;\text{ for all }\; x \in (0, x_0 + \varepsilon) \quad \text{when} \quad \theta \in (0,\pi/2], \tag{3.192}$$

and

$$s_1(x) > 0 \;\text{ for all }\; x \in \Big(1 - \frac{\pi}{2\theta}, x_0 + \varepsilon\Big) \quad \text{when} \quad \theta \in (\pi/2,\pi). \tag{3.193}$$

We now turn our attention to the function $R(\cdot,\cdot)$. Appealing to the second identity in (3.131), for each $x \in [0,1]$ and each $y \in (0,\infty)$ we may write

$$R(x,y) = r_0(x) + \sum_{j=1}^{\infty} r_{2j}(x) \cdot y^{2j}, \tag{3.194}$$

where the functions $r_0, r_{2j} : [0,1] \longrightarrow \mathbb{R}$, $j \in \mathbb{N}$, are given by

$$r_0(x) := \kappa(x-1) \cdot \sin\theta - \sin[\theta(x-1)], \tag{3.195}$$

and

$$r_{2j}(x) := -\sin[\theta(x-1)] \cdot \frac{\theta^{2j}}{(2j)!}, \quad j \in \mathbb{N}. \tag{3.196}$$

Thanks to the first inequality in (3.183), for each $j \in \mathbb{N}$ one has $r_{2j}(x_0) > 0$. Since $R(x_0,y_0) = 0$ and $y_0 \in (0,\infty)$, based on (3.194) we may deduce that

$$r_0(x_0) < 0. \tag{3.197}$$

Next, differentiating in (3.195) and using (3.187) gives

$$r_0'(x) = \kappa \cdot \sin\theta - \sin[\theta(x-1)] \cdot \theta = s_1(x), \qquad \forall\, x \in [0,1]. \tag{3.198}$$

Using (3.192)–(3.193) and (3.198) we obtain that the function r_0 is increasing on the interval $(0, x_0+\varepsilon)$ when $\theta \in (0,\pi/2]$, and that r_0 is increasing on the interval $(1 - \frac{\pi}{2\theta}, x_0 + \varepsilon)$ in the case when $\theta \in (\pi/2,\pi)$. Based on this and the continuity of the function r_0, when $\theta \in (0,\pi/2]$ we may deduce

$$r_0(x_0) > r_0(0) = \sin\theta \cdot (1-\kappa) \geq 0, \tag{3.199}$$

granted that $\kappa \in (0,1]$. On the other hand, when $\theta \in (\pi/2,\pi)$, we have

$$r_0(x_0) > r_0\Big(1 - \frac{\pi}{2\theta}\Big) = 1 - \frac{\kappa\pi}{2\theta} \cdot \sin\theta > 0, \tag{3.200}$$

as $\kappa \in (0,1]$ and when $\theta > \pi/2$ one has $\pi/2\theta < 1$. However, (3.199)–(3.200) contradict (3.197) and this finishes the proof of the statement made at the beginning of Case 3.

Case 4. *If z is as in (3.130) then*

$$\kappa(z-1) \cdot \sin\theta \neq -\sin[\theta(z-1)], \tag{3.201}$$

i.e., Eq. (3.107) is not satisfied.

Assume again by contradiction that the claim made above is false, i.e.,

$$\begin{gathered}\exists\, z = x_0 + iy_0 \in \mathbb{C} \;\; \text{with} \;\; x_0 \in (0,1) \;\; \text{and} \;\; y_0 \in (0,\infty) \\ \text{such that (3.107) holds.}\end{gathered} \tag{3.202}$$

Taking the real and imaginary parts in (3.107) we obtain that

$$\kappa(x_0 - 1)\cdot\sin\theta = -\sin[\theta(x_0-1)]\cdot\cosh(\theta y_0), \tag{3.203}$$

$$\kappa y_0\cdot\sin\theta = -\cos[(\theta(x_0-1)]\cdot\sinh(\theta y_0). \tag{3.204}$$

However $\theta \in (0,\pi)$ and $x_0 \in (0,1)$ imply that $\theta(x_0-1) \in (-\pi, 0)$ and thus $\sin[\theta(x_0-1)] < 0$. This violates (3.203), as its left-hand side is negative while the right-hand side is positive. Consequently the claim made at the beginning of this case holds and this finishes Case 4 and the proof of the lemma. □

Lemma 3.8. *Fix $\theta \in (0,\pi)$ and $\kappa \in (0,1]$ and recall θ_o from (1.23) (see also (1.26)). Then the following hold.*

(i) *The equation*

$$\kappa(x-1)\cdot\sin\theta = \sin[(2\pi-\theta)(x-1)] \tag{3.205}$$

has a unique solution in the interval $(0,1)$, and denoting this by $x_1(\theta,\kappa)$ there holds

$$x_1(\theta,\kappa) \in \left(\frac{\pi-\theta}{2\pi-\theta}, \frac{1}{2}\right). \tag{3.206}$$

(ii) *If $\kappa \in (0,1)$, the equation*

$$\kappa(x-1)\cdot\sin\theta = -\sin[(2\pi-\theta)(x-1)] \tag{3.207}$$

has a unique solution in the interval $(0,1)$, and denoting this by $x_2(\theta,\kappa)$ there holds

$$x_2(\theta,\kappa) \in \left(0, \frac{\pi-\theta}{2\pi-\theta}\right). \tag{3.208}$$

If $\kappa = 1$ and $\theta \in (0,\theta_o)$, Eq. (3.207) has a unique solution in the interval $(0,1)$, and denoting this by $x_2(\theta,1)$ there holds

$$x_2(\theta,1) \in \left(0, \frac{\pi-\theta}{2\pi-\theta}\right). \tag{3.209}$$

Finally, if $\kappa = 1$ and $\theta \in [\theta_o,\pi)$ Eq. (3.207) has no solution in the interval $(0,1)$.

(iii) *The equations*

$$\kappa(x-1)\cdot\sin\theta = \sin[\theta(x-1)], \tag{3.210}$$

$$\kappa(x-1)\cdot\sin\theta = -\sin[\theta(x-1)], \tag{3.211}$$

have no solutions in the interval $(0,1)$.

Proof. We begin by examining Eq. (3.205) and we claim that

$$\left.\begin{array}{r} x \in (0,1) \\ \text{and} \\ \kappa(x-1)\cdot\sin\theta = \sin[(2\pi-\theta)(x-1)] \end{array}\right\} \Longrightarrow x \in \mathcal{I}_1, \tag{3.212}$$

where

$$\mathcal{I}_1 := \left(\frac{\pi-\theta}{2\pi-\theta}, 1\right). \tag{3.213}$$

Indeed, if $x \in (0,1)$ then $\kappa(x-1)\cdot\sin\theta < 0$ and thus $\sin[(2\pi-\theta)(x-1)] < 0$. This, together with the fact that $\theta \in (0,\pi)$ and $x \in (0,1)$ guarantees that $(2\pi-\theta)(x-1) \in (-2\pi, 0)$. This further implies that $(2\pi-\theta)(x-1) \in (-\pi, 0)$ and thus $x \in \mathcal{I}_1$ as desired. In particular, based on (3.212) we may deduce that

$$\text{Equation (3.205) has no solution in the interval } \left(0, \frac{\pi-\theta}{2\pi-\theta}\right]. \tag{3.214}$$

We also find it useful to record that

$$x \in \mathcal{I}_1 \implies \sin[(2\pi-\theta)(x-1)] < 0, \tag{3.215}$$

as $x \in \mathcal{I}_1$ immediately implies that $(2\pi-\theta)(x-1) \in (-\pi, 0)$.

Going further, consider the function $T : [0,1] \to \mathbb{R}$ given by

$$T(x) := \kappa(x-1)\cdot\sin\theta - \sin[(2\pi-\theta)(x-1)], \tag{3.216}$$

and first note that $T(1) = 0$. Second,

$$T'(x) = \kappa\cdot\sin\theta - \cos[(2\pi-\theta)(x-1)]\cdot(2\pi-\theta), \quad \forall\, x \in [0,1], \tag{3.217}$$

and

$$T''(x) = \sin[(2\pi-\theta)(x-1)]\cdot(2\pi-\theta)^2, \qquad \forall\, x \in [0,1]. \tag{3.218}$$

Using (3.215) we may deduce that $T''(x) < 0$ whenever $x \in \mathcal{I}_1$. Therefore the function T' is decreasing on the interval $\mathcal{I}_1$. In addition,

$$\begin{array}{c} T'\big(\frac{\pi-\theta}{2\pi-\theta}\big) = \kappa\cdot\sin\theta + (2\pi-\theta) > 0 \ \text{ and} \\ T'(1) = \kappa\cdot\sin\theta - (2\pi-\theta) < 0, \end{array} \tag{3.219}$$

where the last inequality follows from the fact that $\kappa \in (0,1]$ and $\sin\theta < 2\pi-\theta$ for $\theta \in (0,\pi)$. Combining (3.219) with the monotonicity of T' on the interval $\mathcal{I}_1$ and the fact that this function is continuous on $[0,1]$ we obtain that

$$\text{there exists a unique } x_0 \in \mathcal{I}_1 \text{ such that } T'(x_0) = 0, \tag{3.220}$$

and in addition $T' > 0$ on the interval $\big(\frac{\pi-\theta}{2\pi-\theta}, x_0\big)$ and $T' < 0$ on the interval $(x_0, 1)$. In particular,

$$\begin{array}{c} T \text{ is increasing on the interval } \big(\frac{\pi-\theta}{2\pi-\theta}, x_0\big) \\ \text{and decreasing on the interval } (x_0, 1). \end{array} \tag{3.221}$$

Next, note that if $\theta \in (0,\pi)$ then $0 < \frac{\pi-\theta}{2\pi-\theta} < \frac{1}{2}$ and consequently $\frac{1}{2} \in \mathcal{I}_1$. Evaluating the function T at the points $\frac{\pi-\theta}{2\pi-\theta}$ and $\frac{1}{2}$ gives

$$T\left(\frac{\pi-\theta}{2\pi-\theta}\right) = -\frac{\pi\kappa}{2\pi-\theta}\cdot\sin\theta < 0, \tag{3.222}$$

and

$$T(1/2) = -\frac{\kappa\cdot\sin\theta}{2} + \sin\Big(\frac{\theta}{2}\Big) = \sin\Big(\frac{\theta}{2}\Big)\cdot\Big(1 - \kappa\cdot\cos\Big(\frac{\theta}{2}\Big)\Big) > 0, \tag{3.223}$$

where the first inequality above is obvious and the second one follows from the fact that $\sin(\frac{\theta}{2}) > 0$ on $(0, \pi)$ and $\kappa \cos(\frac{\theta}{2}) < 1$ when $\kappa \in (0, 1]$. In particular, by the intermediate value theorem

$$\exists\ x_1(\theta, \kappa) \in \Big(\frac{\pi - \theta}{2\pi - \theta}, \frac{1}{2}\Big) \ \text{ such that } \ T(x_1(\theta, \kappa)) = 0, \tag{3.224}$$

and property (3.221) guarantees that $x_1(\theta, \kappa)$ as above is unique. Thus

$$\text{(3.205) has a unique solution in the interval } \big(\tfrac{\pi-\theta}{2\pi-\theta}, \tfrac{1}{2}\big). \tag{3.225}$$

Going further, using (3.222) and (3.221) in concert with the fact $T(1) = 0$ we conclude that the function T does not vanish on $[\frac{1}{2}, 1)$ and as such (3.205) has no solution in $[\frac{1}{2}, 1)$. This together with (3.214) and (3.225) completes the proof of (i).

We now turn our attention to (ii). A quick inspection of the signs of the left- and right-hand sides of (3.207) shows that a necessary condition for $x \in (0, 1)$ to be a solution of (3.207) is that $\sin[(2\pi - \theta)(x - 1)] > 0$. In particular

$$x \in (0, 1) \ \text{ is a solution of (3.207)} \implies x \in \mathcal{I}_2 := \left(0, \frac{\pi - \theta}{2\pi - \theta}\right), \tag{3.226}$$

and consequently

$$\text{Equation (3.207) has no solution in the interval } \Big[\frac{\pi - \theta}{2\pi - \theta}, 1\Big). \tag{3.227}$$

Also it is useful to record that, as simple manipulations show,

$$x \in \mathcal{I}_2 \implies \sin[(2\pi - \theta)(x - 1)] > 0. \tag{3.228}$$

Going further, consider the function $U : [0, 1] \longrightarrow \mathbb{R}$ given by

$$U(x) := \kappa(x - 1) \cdot \sin\theta + \sin[(2\pi - \theta)(x - 1)], \tag{3.229}$$

and observe that

$$U'(x) = \kappa \cdot \sin\theta + \cos[(2\pi - \theta)(x - 1)] \cdot (2\pi - \theta), \qquad \forall\, x \in [0, 1], \tag{3.230}$$

$$U''(x) = -\sin[(2\pi - \theta)(x - 1)] \cdot (2\pi - \theta)^2, \qquad \forall\, x \in [0, 1]. \tag{3.231}$$

Thus, using (3.228) we obtain that

$$U'' < 0 \text{ on the interval } \mathcal{I}_2 \tag{3.232}$$

and hence

$$\text{the function } U' \text{ is decreasing on } \mathcal{I}_2. \tag{3.233}$$

We shall analyze first the case when $\kappa \in (0, 1)$. In this scenario

$$U(0) = -\kappa \cdot \sin\theta - \sin(2\pi - \theta) = \sin\theta \cdot \big(-\kappa + 1\big) > 0, \tag{3.234}$$

since $\kappa \in (0, 1)$ and $\sin\theta > 0$. At the right endpoint of the interval $\mathcal{I}_2$ we compute

$$U\left(\frac{\pi - \theta}{2\pi - \theta}\right) = -\frac{\pi\kappa}{2\pi - \theta} \cdot \sin\theta < 0, \tag{3.235}$$

and as before, for $\theta \in (0, \pi)$,

$$U'\left(\frac{\pi-\theta}{2\pi-\theta}\right) = \kappa \cdot \sin\theta - (2\pi - \theta) < 0. \tag{3.236}$$

Keeping in mind that $U'' < 0$ on $\mathcal{I}_2$, there are two possible scenarios. One is that $U'(x) < 0$ whenever $x \in \mathcal{I}_2$. In this case U is monotonically decreasing on this interval and by (3.234) and (3.235) and the intermediate value theorem we conclude that

$$\exists\ x_2(\theta,\kappa) \in \left(0, \frac{\pi-\theta}{2\pi-\theta}\right) \ \text{ such that } \ U(x_2(\theta,\kappa)) = 0. \tag{3.237}$$

Moreover, since $U' < 0$ on $\mathcal{I}_2$, we can conclude that $x_2(\theta,\kappa)$ as above is unique, and thus

$$\text{(3.207) has a unique solution in the interval } \big(0, \tfrac{\pi-\theta}{2\pi-\theta}\big). \tag{3.238}$$

The second alternative is that there exists a unique number $x_3 \in \mathcal{I}_2$ such that $U'(x) > 0$ on $(0, x_3)$, $U'(x_3) = 0$, and $U'(x) < 0$ on $\big(x_3, \frac{\pi-\theta}{2\pi-\theta}\big)$. However, this case yields the same conclusions (3.237) and (3.238). This completes the proof of (ii) when $\kappa \in (0, 1)$.

Moving on, let $\kappa = 1$ in (3.207), and recall the conclusions (3.226) and (3.227). With the function U as introduced in (3.229), now with $\kappa = 1$, i.e.

$$U : [0,1] \longrightarrow \mathbb{R}, \quad U(x) := (x-1) \cdot \sin\theta + \sin[(2\pi - \theta)(x-1)], \tag{3.239}$$

we have that (3.232) and (3.233) hold. In this case, as compared to (3.234), we have

$$U(0) = 0, \tag{3.240}$$

as well as the following inequalities, corresponding to (3.235) and (3.236) when $\kappa = 1$,

$$\begin{aligned} U\left(\frac{\pi-\theta}{2\pi-\theta}\right) &= -\frac{\pi}{2\pi-\theta} \cdot \sin\theta < 0, \\ U'\left(\frac{\pi-\theta}{2\pi-\theta}\right) &= \sin\theta - (2\pi - \theta) < 0. \end{aligned} \tag{3.241}$$

Also

$$U'(0) = \sin\theta + (2\pi - \theta) \cdot \cos\theta. \tag{3.242}$$

At this point we recall the angle θ_o from (1.23). By (1.25) it immediately follows that

$$U'(0) > 0 \text{ whenever } \theta \in (0, \theta_o) \text{ and } U'(0) \leq 0 \text{ whenever } \theta \in [\theta_o, \pi). \tag{3.243}$$

Keeping in mind that $U'' < 0$ on $\mathcal{I}_2$ and using (3.243) we may deduce that when $\theta \in [\theta_o, \pi)$ the function U' is strictly negative on the interval $\mathcal{I}_2$. Combining this with (3.240), we obtain that the function U has no zeroes in $\mathcal{I}_2$, and therefore, by (3.227), in $(0, 1)$. Also, when $\theta \in (0, \theta_o)$ we obtain that there exists a unique $x_2(\theta, 1) \in \mathcal{I}_2$ such that $U(x_2(\theta, 1)) = 0$. This finishes the analysis of (ii) when $\kappa = 1$, and completes its proof.

Next we focus on the statement (iii). A necessary condition for the identity (3.210) to hold is that

$$\sin[\theta(x-1)] < 0. \tag{3.244}$$

Introduce $V : [0,1] \longrightarrow \mathbb{R}$,

$$V(x) := \kappa(x-1)\cdot\sin\theta - \sin[\theta(x-1)], \tag{3.245}$$

so that

$$V'(x) = \kappa\cdot\sin\theta - \cos[\theta(x-1)]\cdot\theta, \qquad \forall\, x\in[0,1], \tag{3.246}$$

$$V''(x) = \sin[\theta(x-1)]\cdot\theta^2, \qquad \forall\, x\in[0,1]. \tag{3.247}$$

Note that, due to (3.244) one has $V''<0$ and consequently $V'(x)$ is monotonically decreasing on $(0,1)$. For each $\kappa\in(0,1]$ and $\theta\in(0,\pi)$ we have

$$V(0) = \big(-\kappa+1\big)\cdot\sin\theta \geq 0, \tag{3.248}$$

and

$$V(1) = 0, \tag{3.249}$$

which, due to the concavity property of V, guarantees that $V>0$ for all $x\in(0,1)$. Thus (3.210) has no solutions for the values of the parameters involved as stated in the hypotheses.

Finally, we consider (3.211). A simple inspection shows that the left-hand side of the equation is always negative while the right-hand side is always positive. Thus (3.211) has no solutions. □

The following result describes the roots of the Eqs. (3.205)–(3.211) in the case $\kappa=0$. Its proof is immediate and we omit it.

Lemma 3.9. *Fix $\theta\in(0,\pi)$. Then the following hold.*

(i) *The equation*

$$\sin[(2\pi-\theta)(x-1)] = 0, \tag{3.250}$$

has a unique solution in the interval $(0,1)$, and denoting this by $x_1(\theta)$ there holds

$$x_1(\theta) = \frac{\pi-\theta}{2\pi-\theta} \in \left(0,\frac{1}{2}\right). \tag{3.251}$$

(ii) *The equation*

$$\sin[\theta(x-1)] = 0, \tag{3.252}$$

has no solution in the interval $(0,1)$.

3.2. Proof the Main Result

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $p\in(1,\infty)$ and $\kappa\in(0,1)$. Using Lemma 3.2, after the identification of $(\partial\Omega)_j$ with $\mathbb{R}_+$ for each $j\in\{1,2\}$, the operator $\partial_\tau S^{Lamé}$ is invertible on $L^p(\partial\Omega)$ if and only if the integral operator T given by

$$T\vec{f}(s) := \int_0^\infty \widetilde{k}(s,t)\cdot\vec{f}(t)\,dt, \quad \text{a.e. } s\in\mathbb{R}_+ \text{ and } \forall\,\vec{f}\in\big[L^p(\mathbb{R}_+)\big]^4, \tag{3.253}$$

with integral kernel $\widetilde{k}$ as in (3.43)–(3.47) is invertible on $\big[L^p(\mathbb{R}_+)\big]^4$. According to Lemma 3.3, and using that $C_1>0$, the operator T satisfies the hypothesis of Corollary 2.4. As such, the operator T is invertible on $\big[L^p(\mathbb{R}_+)\big]^4$ if and only if

$$\mathcal{M}\widetilde{k}(\cdot,1)(1/p+i\xi)\neq 0 \quad \forall\,\xi\in\mathbb{R}. \tag{3.254}$$

We invoke next Corollary 3.6 and Lemma 3.7 to conclude that T is not invertible on $\big[L^p(\mathbb{R}_+)\big]^4$ if and only if one of the following equalities hold

$$\kappa(\tfrac{1}{p}-1)\sin\theta=\sin[(2\pi-\theta)(\tfrac{1}{p}-1)], \tag{3.255}$$

$$\kappa(\tfrac{1}{p}-1)\sin\theta=-\sin[(2\pi-\theta)(\tfrac{1}{p}-1)], \tag{3.256}$$

$$\kappa(\tfrac{1}{p}-1)\sin\theta=\sin[\theta(\tfrac{1}{p}-1)], \tag{3.257}$$

$$\kappa(\tfrac{1}{p}-1)\sin\theta=-\sin[\theta(\tfrac{1}{p}-1)]. \tag{3.258}$$

Note that if $\theta=\pi$ the left-hand sides in Eqs. (3.255)–(3.258) are equal to zero while the right-hand sides are different from zero (here we use that $p\in(1,\infty)$ and as such $1-\frac{1}{p}\in(0,1)$). In conclusion the operator $\partial_\tau S^{Lamé}$ is invertible on $L^p(\partial\Omega)$ for each $p\in(1,\infty)$ when $\theta=\pi$. Combining this with (2.9) gives (1.19), proving (C.1) in the statement of the theorem.

We turn our attention to the statement made in part (A.1). Consider first the case when $\theta\in(0,\pi)$. A direct application of Lemma 3.8 yields that Eq. (3.255) has a unique solution denoted by $p_1(\theta,\kappa)$ and this satisfies $p_1(\theta,\kappa)\in\Big(2,\frac{2\pi-\theta}{\pi-\theta}\Big)$. Furthermore, Eq. (3.256) has a unique solution denoted by $p_2(\theta,\kappa)$ and this satisfies $p_2(\theta,\kappa)\in\Big(\frac{2\pi-\theta}{\pi-\theta},\infty\Big)$ while Eqs. (3.257)–(3.258) have no solutions for $p\in(1,\infty)$. In conclusion, the operator

$$\begin{gathered}\partial_\tau S^{Lamé}\text{ is invertible on }L^p(\partial\Omega)\\ \text{for each }p\in(1,\infty)\backslash\{p_1(\theta,\kappa),p_2(\theta,\kappa)\}\text{ when }\theta\in(0,\pi).\end{gathered} \tag{3.259}$$

Using (3.259) and (2.9) the statement (1.10) in Theorem 1.1 immediately follows.

Next, let $\theta\in(\pi,2\pi)$ and let $\gamma:=2\pi-\theta\in(0,\pi)$. In this notation, Eqs. (3.255) and (3.256) become

$$\kappa(\tfrac{1}{p}-1)\sin\gamma=\mp\sin[\gamma(\tfrac{1}{p}-1)], \tag{3.260}$$

and by Lemma 3.8 they have no solutions for $p\in(1,\infty)$. Going further, (3.257) gives

$$\kappa(\tfrac{1}{p}-1)\sin\gamma=-\sin[(2\pi-\gamma)(\tfrac{1}{p}-1)]. \tag{3.261}$$

Using Lemma 3.8 the Eq. (3.261) has a unique solution

$$p_3(\theta,\kappa)\in\Big(\frac{2\pi-\gamma}{\pi-\gamma},\infty\Big)=\Big(\frac{\theta}{\theta-\pi},\infty\Big). \tag{3.262}$$

Similarly, Eq. (3.258) becomes

$$\kappa(\tfrac{1}{p}-1)\sin\gamma=\sin[(2\pi-\gamma)(\tfrac{1}{p}-1)], \tag{3.263}$$

and appealing one last time to Lemma 3.8 this has a unique solution

$$p_4(\theta,\kappa)\in\Big(2,\frac{2\pi-\gamma}{\pi-\gamma}\Big)=\Big(2,\frac{\theta}{\theta-\pi}\Big). \tag{3.264}$$

As such, the operator

$$\begin{gathered}\partial_\tau S^{Lam\acute{e}} \text{ is invertible on } L^p(\partial\Omega)\\ \text{for each } p\in(1,\infty)\backslash\{p_3(\theta,\kappa),p_4(\theta,\kappa)\} \text{ when } \theta\in(\pi,2\pi).\end{gathered} \tag{3.265}$$

As before, (3.265) and (2.9) imply the statement made in (1.11). This finishes the proof of (A.1).

The statement in (B.1) (corresponding to $\kappa=0$) is treated similarly, this time appealing to Lemma 3.9.

Moving on, the statements made in (A.2), (B.2) and (C.2) follow from (A.1), (B.1) and (C.1), (2.9), duality, and the two dimensional identity proved in [2] to the effect that

$$\begin{gathered}\big(\partial_\tau S^{Lam\acute{e}}\big)^2=\big(\tfrac12 I+(K_\Psi^{Lam\acute{e}})^*\big)\circ\big(-\tfrac12 I+(K_\Psi^{Lam\acute{e}})^*\big) \text{ on } L^p(\partial\Omega),\\ \text{for each integrability index } \ p\in(1,\infty),\end{gathered} \tag{3.266}$$

where $(K_\Psi^{Lam\acute{e}})^*$ denotes the dual of the operator $K_\Psi^{Lam\acute{e}}$.

Finally, the statements (A.3), (B.3) and (C.3) are a consequence of (A.2), (B.2) and (C.2), respectively, duality, (A.1), (B.1) and (C.1), and the operator identity (valid in all dimensions)

$$\begin{gathered}\partial_{\nu_\Psi}\mathcal{D}_\Psi^{Lam\acute{e}}\circ S=\big(\tfrac12 I+(K_\Psi^{Lam\acute{e}})^*\big)\circ\big(-\tfrac12 I+(K_\Psi^{Lam\acute{e}})^*\big) \text{ on } \dot{L}_1^p(\partial\Omega),\\ \text{for each integrability index } \ p\in(1,\infty).\end{gathered} \tag{3.267}$$

This completes the proof of the theorem. □

4. The Case of the Stokes System

In this section, we discuss the invertibility of *hydrostatic layer potentials*. To this end, consider the linearized, homogeneous, time independent Navier–Stokes equations, i.e., the Stokes system

$$\begin{cases}\triangle\vec{u}=\nabla\mathbf{p},\\ \operatorname{div}\vec{u}=0,\end{cases} \tag{4.1}$$

in an open set in $\mathbb{R}^2$, where $\vec{u}$ is the velocity field and $\mathbf{p}$ is the pressure function. If we define the matrix $A=A(r):=(a_{ij}^{k\ell}(r))_{i,j,k,\ell\in\{1,2\}}$ by

$$a_{ij}^{k\ell}=a_{ij}^{k\ell}(r):=\delta_{ij}\delta_{k\ell}+r\,\delta_{i\ell}\delta_{jk},\quad \text{for } r\in\mathbb{R}, \tag{4.2}$$

then $a_{ij}^{k\ell}\partial_i\partial_j u_\ell=\triangle u_k+r\partial_k(\operatorname{div}\vec{u})$. Hence, any solution $\vec{u}$, $\mathbf{p}$ of the Stokes system (4.1) satisfies

$$a_{ij}^{k\ell}\partial_i\partial_j u_\ell=\partial_k\mathbf{p}.$$

As before, let $\Omega\subset\mathbb{R}^2$ be an infinite angle of aperture $\theta\in(0,2\pi)$ and let $\nu=(\nu_1,\nu_2)$ be the outward unit normal vector defined at each point on $\partial\Omega$

with the exception of the vertex. The conormal derivative associated with the tensor of coefficients $A(r) := (a_{ij}^{k\ell}(r))_{i,j,k,\ell\in\{1,2\}}$, for $r \in \mathbb{R}$, is defined as

$$\left(\frac{\partial}{\partial \nu_{A(r)}}\{\vec{u}, \mathbf{p}\}\right)^j := \nu_i a_{ik}^{j\ell}(r)\partial_k u_\ell - \nu_j \mathbf{p}, \quad \text{where } j = 1, 2. \tag{4.3}$$

The special choice $r := 1$ gives rise to the so-called stress conormal derivative (see also, e.g., [7,25]). This derivative has a physical interpretation and it is known as the *slip condition* when imposed at the boundary and we shall denote this for the remaining part of the presentation by ∂_{ν_Ψ}. Thus

$$\partial_{\nu_\Psi} := \frac{\partial}{\partial_{\nu_{A(1)}}}. \tag{4.4}$$

Parenthetically we note that

$$1 = \lim_{\lambda\to\infty} \frac{\mu(\lambda+\mu)}{3\mu+\lambda}\Big|_{\mu=1}. \tag{4.5}$$

Going further, denote by $G^{Stokes} = (G_{ij}^{Stokes})_{i,j\in\{1,2\}}$ the Kelvin matrix-valued, radially symmetric fundamental solution for the system of hydrostatics in $\mathbb{R}^2$ given by

$$G_{ij}^{Stokes}(X) := C_1\delta_{ij}\log|X|^2 - C_2\frac{X_iX_j}{|X|^2}, \quad \forall\, X = (X_1, X_2) \in \mathbb{R}^2\backslash\{\mathbf{0}\}, \tag{4.6}$$

where $i, j \in \{1, 2\}$, and

$$C_1 := \frac{1}{8\pi}, \quad \text{and} \quad C_2 := \frac{1}{4\pi}, \tag{4.7}$$

(see e.g., [33, formula (10.7.2) in Chapter 10]). Note that the constants C_1, C_2 in (4.7) satisfy

$$C_1 = \lim_{\lambda\to\infty}\frac{3\mu+\lambda}{8\mu(2\mu+\lambda)\pi}\Big|_{\mu=1} \quad \text{and} \quad C_2 = \lim_{\lambda\to\infty}\frac{\mu+\lambda}{4\mu(2\mu+\lambda)\pi}\Big|_{\mu=1}. \tag{4.8}$$

Consider next the pressure vector $\vec{\mathbf{q}} : \mathbb{R}^2\backslash\{0\} \longrightarrow \mathbb{R}^2$ given by

$$\vec{\mathbf{q}}(X) = (\mathbf{q}_1(X), \mathbf{q}_2(X)) := -\frac{1}{2\pi}\frac{X}{|X|^2}, \quad \forall\, X \in \mathbb{R}^2\backslash\{0\}. \tag{4.9}$$

Then, for each $i, j \in \{1, 2\}$ there holds

$$\Delta G_{ij}^{Stokes} = \Delta G_{ji}^{Stokes} = \partial_i \mathbf{q}_j = \partial_j \mathbf{q}_i \quad \text{on} \quad \mathbb{R}^2\backslash\{0\}. \tag{4.10}$$

Moving on, the boundary-to-domain single layer potential operator is introduced as

$$\mathcal{S}^{Stokes}\vec{f}(X) := \int_{\partial\Omega} G^{Stokes}(X-Y)\cdot\vec{f}(Y)\,d\sigma(Y), \quad X \in \mathbb{R}^2\backslash\partial\Omega, \tag{4.11}$$

and the boundary-to-boundary single layer hydrostatic operator S^{Stokes} is given by

$$S^{Stokes}\vec{f}(X) := \int_{\partial\Omega} G^{Stokes}(X-Y)\cdot\vec{f}(Y)\,d\sigma(Y), \quad X \in \partial\Omega. \tag{4.12}$$

We shall also introduce the double layer potential operators associated with the system (4.1). Specifically, if $r \in \mathbb{R}$ is fixed and the tensor of coefficients

$A(r) = (a_{ij}^{k\ell}(r))_{i,j,k,\ell\in\{1,2\}}$ is as in (4.2), then the double layer potential operator associated with $A(r)$ is denoted by $\mathcal{D}_{A(r)}^{Stokes}$ and its action on a vector-valued function $\vec{f}: \partial\Omega \longrightarrow \mathbb{R}^2$ with $\vec{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ is defined by setting

$$\mathcal{D}_{A(r)}^{Stokes}\vec{f}(X) := \int_{\partial\Omega} \left[\frac{\partial}{\partial\nu_{A(r)}}\{G^{Stokes}, \vec{\mathbf{q}}\}(X-\cdot)\right]^t (Q)\cdot \vec{f}(Q)\, d\sigma(Q) \quad \text{for each point} \quad X \in \mathbb{R}^2\backslash\partial\Omega, \tag{4.13}$$

where $\dfrac{\partial}{\partial\nu_{A(r)}}\{G^{Stokes}, \vec{\mathbf{q}}\}$ is defined as the matrix obtained by applying the conormal derivative from (4.3) to each pair consisting of the j-th column of the fundamental solution G^{Stokes} from (4.6) and the j-th component of the vector $\vec{\mathbf{q}}$. Also, the superscript t stands for transposition of matrices. In the sequel we shall use the notation

$$\mathcal{D}_{\Psi}^{Stokes} := \mathcal{D}_{A(1)}^{Stokes}, \tag{4.14}$$

to denote the slip double boundary-to-domain double layer potential operator. For each $r \in \mathbb{R}$, the boundary version of $\mathcal{D}_{A(r)}^{Stokes}$ is the operator $K_{A(r)}^{Stokes}$ whose action on $\vec{f}$ as above is defined by setting

$$K_{A(r)}^{Stokes}\vec{f}(X) = p.v. \int_{\partial\Omega} \left[\frac{\partial}{\partial\nu_{A(r)}}\{G^{Stokes}, \vec{\mathbf{q}}\}(X-\cdot)\right]^t (Q)\cdot \vec{f}(Q)\, d\sigma(Q) \quad \text{for} \quad \sigma-\text{a.e.} \quad X \in \partial\Omega, \tag{4.15}$$

where $p.v.$ denotes principle value. We set

$$K_{\Psi}^{Stokes} := K_{A(1)}^{Stokes}. \tag{4.16}$$

For each $r \in \mathbb{R}$, the formal adjoint of the operator $K_{A(r)}^{Stokes}$ is denoted by $\left(K_{A(r)}^{Stokes}\right)^*$ and $(K_{\Psi}^{Stokes})^*$ denotes the adjoint of K_{Ψ}^{Stokes}. A similar result to Proposition 3.1 holds in the case of the layer potentials associated with the Stokes system (this follows again from the work in [5]). Concretely we have the following result.

Proposition 4.1. *Assume that Ω is a graph Lipschitz domain in $\mathbb{R}^2$, and fix $r \in \mathbb{R}$. Recall the tensor of coefficients $A(r) = (a_{ij}^{k\ell}(r))_{i,j,k,\ell\in\{1,2\}}$ from (4.2). Set $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^2 \setminus \overline{\Omega}$. Then, for each $p \in (1,\infty)$,*

(1) *There holds*

$$S^{Stokes}: L^p(\partial\Omega) \to L_1^p(\partial\Omega) \quad \textit{is a linear and bounded operator}, \tag{4.17}$$

$$K_{A(r)}^{Stokes}: L^p(\partial\Omega) \to L^p(\partial\Omega) \quad \textit{is a linear and bounded operator}, \tag{4.18}$$

$$\left(K_{A(r)}^{Stokes}\right)^*: L^p(\partial\Omega) \to L^p(\partial\Omega) \quad \textit{is a linear and bounded operator}, \tag{4.19}$$

where $\left(K_{A(r)}^{Stokes}\right)^*$ denotes the adjoint of the operator $K_{A(r)}^{Stokes}$.

(2) *For each* $\vec{f} \in L^p(\partial\Omega)$ *there holds* $M\big(\mathcal{D}^{Stokes}_{A(r)}\vec{f}\big) \in L^p(\partial\Omega)$. *Moreover there exists a finite constant* $C > 0$ *depending only on the Lipschitz character of* Ω *such that*

$$\|M\big(\mathcal{D}^{Stokes}_{A(r)}\vec{f}\big)\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{L^p(\partial\Omega)}. \tag{4.20}$$

(3) *For every* $\vec{f} \in L^p(\partial\Omega)$ *there holds*

$$\mathcal{D}^{Stokes}_{A(r)}\vec{f}\Big|_{\partial\Omega_\pm}(P) = (\pm\tfrac{1}{2}I + K^{Stokes}_{A(r)})\vec{f}(P), \qquad \sigma - a.e.\ P \in \partial\Omega. \tag{4.21}$$

(4) *For every* $\vec{f} \in L^p(\partial\Omega)$ *one has* $M\Big(\nabla\mathcal{S}^{Stokes}\vec{f}\Big) \in L^p(\partial\Omega)$. *Moreover there exists a finite constant* $C > 0$ *depending only on the Lipschitz character of* Ω *such that*

$$\|M\Big(\nabla\mathcal{S}^{Stokes}\vec{f}\Big)\|_{L^p(\partial\Omega)} \leq C\|\vec{f}\|_{L^p(\partial\Omega)}. \tag{4.22}$$

(5) *For each* $\vec{f} \in L^p(\partial\Omega)$, *the single layer satisfies*

$$\mathcal{S}^{Stokes}\vec{f}|_{\partial\Omega_+} = \mathcal{S}^{Stokes}\vec{f}|_{\partial\Omega_-} = S^{Stokes}\vec{f}, \tag{4.23}$$

and

$$\partial_\tau\mathcal{S}^{Stokes}\vec{f}\Big|_{\partial\Omega_+} = \partial_\tau\mathcal{S}^{Stokes}\vec{f}\Big|_{\partial\Omega_-} = \partial_\tau S^{Stokes}\vec{f}. \tag{4.24}$$

Moreover, if $(\partial_\tau S^{Stokes})^*$ *is the formal adjoint of* $\partial_\tau S^{Stokes}$, *then*

$$\big(\partial_\tau S^{Stokes}\big)^* = -S^{Stokes}\partial_\tau. \tag{4.25}$$

In light of the observation made in (4.8), the computations carried out in Sect. 3 for the Mellin symbol of the operator $\partial_\tau S^{Lamé}$ for the Lamé system of elastostatics can now be reworked in the case of the Stokes system by changing the values of C_1 and C_2 as in (4.7). This immediately yields the following results.

Lemma 4.2. *Let* $\Omega \subset \mathbb{R}^2$ *be an infinite sector of aperture* $\theta \in (0, 2\pi)$. *Consider* $X = (X_1, X_2)$, $Q = (Q_1, Q_2) \in \partial\Omega$ *and recall* $G^{Stokes} = (G^{Stokes}_{ij})_{i,j\in\{1,2\}}$ *from* (4.6). *The kernel of the operator* $\partial^{Stokes}_\tau S$ *is the matrix*

$$k(X,Q) = \begin{pmatrix} \partial_{\tau(X)}G^{Stokes}_{11}(X-Q) & \partial_{\tau(X)}G^{Stokes}_{12}(X-Q) \\ \partial_{\tau(X)}G^{Stokes}_{21}(X-Q) & \partial_{\tau(X)}G^{Stokes}_{22}(X-Q) \end{pmatrix}, \tag{4.26}$$

where

$$\partial_{\tau(X)} G_{11}^{Stokes}(X-Q) = -\frac{\nu_2(X)(X_1-Q_1)}{2\pi|X-Q|^2}\left\{-\frac{1}{2}+\frac{(X_1-Q_1)^2}{|X-Q|^2}\right\} + \frac{\nu_1(X)(X_2-Q_2)}{2\pi|X-Q|^2}\left\{\frac{1}{2}+\frac{(X_1-Q_1)^2}{|X-Q|^2}\right\}, \tag{4.27}$$

$$\partial_{\tau(X)} G_{12}^{Stokes}(X-Q) = -\frac{\nu_2(X)(X_2-Q_2)}{2\pi|X-Q|^2}\left\{-\frac{1}{2}+\frac{(X_1-Q_1)^2}{|X-Q|^2}\right\} + \frac{\nu_1(X)(X_1-Q_1)}{2\pi|X-Q|^2}\left\{-\frac{1}{2}+\frac{(X_1-Q_1)^2}{|X-Q|^2}\right\}, \tag{4.28}$$

$$\partial_{\tau(X)} G_{21}^{Stokes}(X-Q) = -\frac{\nu_2(X)(X_2-Q_2)}{2\pi|X-Q|^2}\left\{-\frac{1}{2}+\frac{(X_1-Q_1)^2}{|X-Q|^2}\right\} + \frac{\nu_1(X)(X_1-Q_1)}{2\pi|X-Q|^2}\left\{-\frac{1}{2}+\frac{(X_2-Q_2)^2}{|X-Q|^2}\right\}, \tag{4.29}$$

and

$$\partial_{\tau(X)} G_{22}^{Stokes}(X-Q) = -\frac{\nu_2(X)(X_1-Q_1)}{2\pi|X-Q|^2}\left\{\frac{1}{2}+\frac{(X_2-Q_2)^2}{|X-Q|^4}\right\} + \frac{\nu_1(X)(X_2-Q_2)}{2\pi|X-Q|^2}\left\{-\frac{1}{2}+\frac{(X_2-Q_2)^2}{|X-Q|^2}\right\}. \tag{4.30}$$

Going further, recall the identification of $(\partial\Omega)_j \equiv \mathbb{R}_+$ for each $j \in \{1,2\}$ and the manner in which the kernel $\widetilde{k}$ in (3.43) was associated with k from (3.32)–(3.35). Following this recipe from Sect. 3, denote by $\widetilde{k}$ the kernel associated with k from (4.26)–(4.30). We then have the following result.

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^2$ be the domain consisting of the interior of an infinite sector of aperture $\theta \in (0,2\pi)$ and $\widetilde{k}$ be as in the preamble of this result. Then, for any $z \in \mathbb{C}$ with* $\operatorname{Re} z \in (0,1)$, *there holds*

$$\mathcal{M}\widetilde{k}(\cdot,1)(z) = \begin{pmatrix} -v(z) & 0 & -a(z) & b(z) \\ 0 & -v(z) & b(z) & -c(z) \\ a(z) & b(z) & v(z) & 0 \\ b(z) & c(z) & 0 & v(z) \end{pmatrix} \tag{4.31}$$

where, with $\gamma := \pi - \theta$,

$$v(z) := -\frac{1}{4}\cdot\frac{\cos(\pi z)}{\sin(\pi z)}, \tag{4.32}$$

$$a(z) := -\frac{1}{4\sin(\pi z)}\cos(\gamma z+\theta) + \frac{(z-1)\sin\theta}{4\sin(\pi z)}\sin(\gamma z+\theta), \tag{4.33}$$

$$b(z) := -\frac{(z-1)\sin\theta}{4\sin(\pi z)}\cos(\gamma z+\theta), \tag{4.34}$$

$$c(z) := -\frac{1}{4\sin(\pi z)}\cos(\gamma z+\theta) - \frac{(z-1)\sin\theta}{4\sin(\pi z)}\sin(\gamma z+\theta). \tag{4.35}$$

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^2$ be the domain consisting of the interior of an infinite sector of aperture $\theta \in (0, 2\pi)$ and let $\widetilde{k}$ be as in the preamble of Lemma 4.3. Then* $\det \mathcal{M}\widetilde{k}(\cdot, 1)(z) = 0$ *for some $z \in \mathbb{C}$ with* $\mathrm{Re}\, z \in (0, 1)$*, if and only if one of the following equalities holds*

$$(z-1)\sin\theta = \sin[(2\pi-\theta)(z-1)], \tag{4.36}$$

$$(z-1)\sin\theta = -\sin[(2\pi-\theta)(z-1)], \tag{4.37}$$

$$(z-1)\sin\theta = \sin[\theta(z-1)], \tag{4.38}$$

$$(z-1)\sin\theta = -\sin[\theta(z-1)]. \tag{4.39}$$

Furthermore, if any one of the identities (4.36), (4.37), (4.38) or (4.39) hold for some $\theta \in (0, 2\pi)$ and $z = x + iy$, with $x \in (0, 1)$ and $y \in \mathbb{R}$, then $y = 0$.

Proof. The proof of the if and only if statement follows immediately from Corollary 3.6 where, in the case of the Stokes system of hydrostatics, we have that $\kappa = C_2/(2C_1) = 1$. The proof that if $z \in \mathbb{C}$ with $\mathrm{Re}\, z \in (0, 1)$ satisfies one of the Eqs. (4.36)–(4.39) than z must be a real number is treated in Lemma 3.7 in the case $\kappa = 1$. □

With these tools in hand the proof of Theorem 1.2 follows in a similar fashion to that of Theorem 1.1, making use this time of the following operator identities (see again [2])

$$\begin{aligned} &\big(\partial_\tau S^{Stokes}\big)^2 = \big(\tfrac{1}{2}I + (K_\Psi^{Stokes})^*\big) \circ \big(-\tfrac{1}{2}I + (K_\Psi^{Stokes})^*\big) \\ &\text{on } L^p(\partial\Omega), \text{ for each integrability index } \ p \in (1, \infty), \end{aligned} \tag{4.40}$$

valid in two dimensions and

$$\begin{aligned} &\partial_{\nu_\Psi} \mathcal{D}_\Psi^{Stokes} \circ S = \big(\tfrac{1}{2}I + (K_\Psi^{Stokes})^*\big) \circ \big(-\tfrac{1}{2}I + (K_\Psi^{Stokes})^*\big) \\ &\text{on } \ \dot{L}_1^p(\partial\Omega), \quad \text{for each integrability index } \ p \in (1, \infty), \end{aligned} \tag{4.41}$$

valid in all dimensions.

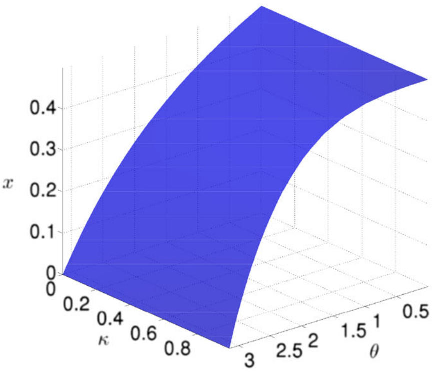
5. On the Critical Indices via Computer Aided Proofs

In this section, we focus on the behavior of the critical indices $p_i(\theta, \kappa)$ for $i \in \{1, \dots, 4\}$ from Theorem 1.1 by analyzing their dependence on the angle θ and the parameter κ.

Our main goal is to prove Theorem 1.3. Recalling Lemma 3.8, the first step is to show that each of the two Eqs. (3.205) and (3.207) implicitly defines a surface $x = x(\theta, \kappa)$ that is monotone with respect to its parameters θ and κ; see Fig. 1.

Proposition 5.1. *Let $\varepsilon = 10^{-6}$ and $\delta = 10^{-4}$. Then the following hold,*

1. *Equation (3.205) implicitly defines a surface $x_1 = x_1(\theta, \kappa)$ whenever $(\theta, \kappa) \in [\varepsilon, \pi - \varepsilon] \times [0, 1 - \delta]$ which is decreasing in θ (when κ is fixed) and increasing in κ (when θ is fixed).*
2. *Equation (3.207) implicitly defines a surface $x_2 = x_2(\theta, \kappa)$ whenever $(\theta, \kappa) \in [\varepsilon, \pi - \varepsilon] \times [0, 1 - \delta]$ which is decreasing in θ (when κ is fixed) and decreasing in κ (when θ is fixed).*



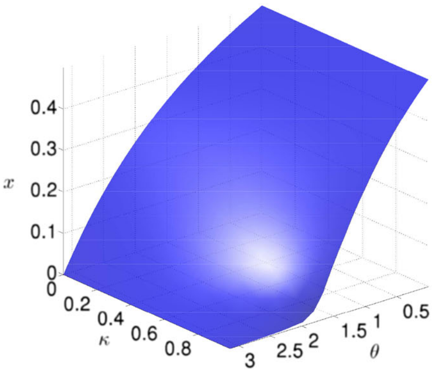


FIGURE 1. The implicit surfaces for Eq. (3.205) (*left*) and Eq. (3.207) (*right*)

Proof. We start with the proof of the statement made in item (1) and introduce the function $f:(0,\pi)\times[0,1]\times[0,1]\longrightarrow\mathbb{R}$ given by

$$f(\theta,\kappa,x):=\kappa(x-1)\sin\theta-\sin[(2\pi-\theta)(x-1)]. \tag{5.1}$$

In this notation (3.205) becomes

$$f(\theta,\kappa,x)=0. \tag{5.2}$$

Employing Lemma 3.8 (and Lemma 3.9 for the case $\kappa=0$), the Eq. (5.2) has *exactly* one solution $x_1=x_1(\theta,\kappa)$ for each pair $(\theta,\kappa)\in(0,\pi)\times[0,1]$. The goal is to use the implicit function theorem for the function f with respect to its dependence in the variable x. Since f is real analytic, matters reduce to proving that $\frac{\partial f}{\partial x}$ is bounded and does not vanish at the points $(\theta,\kappa,x_1(\theta,\kappa))$. Then, by the implicit function theorem, the function

$$[\varepsilon,\pi-\varepsilon]\times[0,1-\delta]\ni(\theta,\kappa)\mapsto x_1(\theta,\kappa) \tag{5.3}$$

is well-defined and as regular as f. With this in hand and using implicit differentiation in (5.2) we obtain

$$\begin{aligned}\frac{\partial x_1}{\partial\theta}(\theta,\kappa)&=-\frac{\dfrac{\partial f}{\partial\theta}(\theta,\kappa,x_1(\theta,\kappa))}{\dfrac{\partial f}{\partial x}(\theta,\kappa,x_1(\theta,\kappa))},\\ \frac{\partial x_1}{\partial\kappa}(\theta,\kappa)&=-\frac{\dfrac{\partial f}{\partial\kappa}(\theta,\kappa,x_1(\theta,\kappa))}{\dfrac{\partial f}{\partial x}(\theta,\kappa,x_1(\theta,\kappa))}.\end{aligned} \tag{5.4}$$

Monotonicity now follows by verifying that $\frac{\partial f}{\partial\theta}$ and $\frac{\partial f}{\partial\kappa}$ in (5.4) do not vanish.

Summarizing, item *(1)* follows as soon as we prove that all partial derivatives of f (that is $\frac{\partial f}{\partial\theta}$, $\frac{\partial f}{\partial\kappa}$, and $\frac{\partial f}{\partial x}$) are bounded and non-zero on the solution set $\mathfrak{A}$ of (5.2), where

$$\mathfrak{A}:=\Big\{(\theta,\kappa,x)\in[\varepsilon,\pi-\varepsilon]\times[0,1-\delta]\times[0,1/2]:\,f(\theta,\kappa,x)=0\Big\}. \tag{5.5}$$

A comment is in order here, vis-a-vis the third component of $\mathfrak{A}$, namely $[0,\frac{1}{2}]$. For each $(\theta,\kappa)\in(0,\pi)\times[0,1]$ Lemma 3.8 provides bounds for the function $x_1(\theta,\kappa):=\frac{1}{p(\theta,\kappa)}$ via (3.206). In particular $x_1(\theta,\kappa)\in[0,\frac{1}{2}]$.

One obstacle to overcome is that Lemma 3.8 provides only a crude bound on $\mathfrak{A}$ that is not sufficient for our needs. To address this, since $\mathfrak{A}$ is a two-dimensional subset of the product of the domains of θ, κ and x, we will enclose it by a finite union of closed, axis-parallel parallelepipeds, referred to as boxes. Specifically,

$$\mathfrak{A}\subset \boldsymbol{B}=\bigcup_{i=1}^{N}\boldsymbol{B}_i. \tag{5.6}$$

The computer-aided part of the proof will produce this finite enclosure by an adaptive bisection procedure and—once we have a sufficiently tight enclosure of $\mathfrak{A}$—prove that all partial derivatives of f are bounded and non-zero on a neighbourhood of $\mathfrak{A}$.

As an initial step, consider the interval extension F of the function f from (5.1); see Sect. 5.3 for an introduction to set-valued numerics. For computational reasons we will find it useful to revisit parts of the proof of Lemma 3.8, and we shall do this in a sequence of four steps.

Step 1. Here our goal is to generate a finite set of boxes

$$\boldsymbol{B}_i:=I_i\times[0,1-\delta]\times[0,\tfrac{1}{2}]\subseteq[\varepsilon,\pi-\varepsilon]\times[0,1-\delta]\times[0,\tfrac{1}{2}],\quad i=1,\dots,N, \tag{5.7}$$

with the property that $I_i:=[\underline{\theta}_i,\overline{\theta}_i]$, for each $i\in\{1,\dots,N\}$, have disjoint interiors,

$$\bigcup_{i=1}^{N}I_i=[\varepsilon,\pi-\varepsilon], \tag{5.8}$$

and, for each $i\in\{1,\dots,N\}$,

$$\begin{gathered}\text{the intervals } F(I_i\times[0,1-\delta]\times\{0\}) \text{ and } F(I_i\times[0,1-\delta]\times\{\tfrac{1}{2}\})\\ \text{reside on opposite sides of the origin.}\end{gathered} \tag{5.9}$$

The construction of the family of boxes $\{\boldsymbol{B}_i\}_{i=1,\dots,N}$ is the result of a computer program which also rigorously verifies that the function f has opposite signs on the two surfaces $\boldsymbol{S}^-=[\varepsilon,\pi-\varepsilon]\times[0,1-\delta]\times\{0\}$ and $\boldsymbol{S}^+=[\varepsilon,\pi-\varepsilon]\times[0,1-\delta]\times\{\frac{1}{2}\}$. All remaining computations will be performed on the family of boxes $\{\boldsymbol{B}_i\}_{i=1,\dots,N}$.

Step 2. In this step we implement an algorithm whose goal is to tighten the enclosure of the solution set $\mathfrak{A}$ obtained as a result of the algorithm in Step 1. This is done by performing a rigorous line search (as described in Sect. 5.3) along each of the four vertical edges of every box $\boldsymbol{B}_i$. For each $i\in\{1,\dots,N\}$, the vertical edges are given by

$$\begin{aligned}\boldsymbol{\ell}_{i,1}&=\{\underline{\theta}_i\}\times\{0\}\times[0,\tfrac{1}{2}]\\ \boldsymbol{\ell}_{i,2}&=\{\overline{\theta}_i\}\times\{0\}\times[0,\tfrac{1}{2}]\\ \boldsymbol{\ell}_{i,3}&=\{\overline{\theta}_i\}\times\{1-\delta\}\times[0,\tfrac{1}{2}]\\ \boldsymbol{\ell}_{i,4}&=\{\underline{\theta}_i\}\times\{1-\delta\}\times[0,\tfrac{1}{2}].\end{aligned} \tag{5.10}$$

We will use interval-bisection in the x-coordinate (the third) to enclose the zeros of the function f along each edge $\boldsymbol{\ell}_{i,j}$. The result of this procedure is

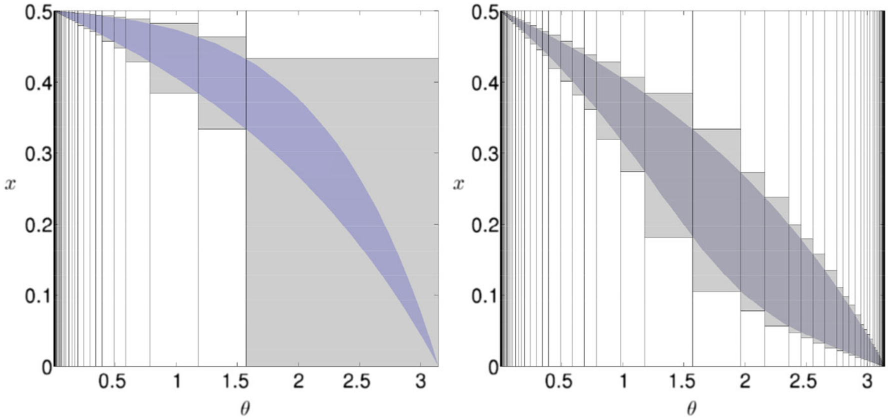


FIGURE 2. The effect of Step 2 in the proof. The *white* (full height) *boxes* are from Step 1, where only the θ-domain is subdivided. The *gray*, *contracted boxes* are the results of the bisection in Step 2. Projecting the implicit surface shows that the contraction is near-optimal. Equation (3.205) appears in the *left*, and Eq. (3.207) in the *right*

that each vertical edge $\boldsymbol{\ell}_{i,j}$ is shrunk to a very small set $\widetilde{\boldsymbol{\ell}}_{i,j}$ which contains the unique zero of f restricted to $\boldsymbol{\ell}_{i,j}$,

$$\begin{aligned} \widetilde{\boldsymbol{\ell}}_{i,1} &= \{\underline{\theta}_i\} \times \{0\} \times [\underline{x}_{i,1}, \overline{x}_{i,1}] \\ \widetilde{\boldsymbol{\ell}}_{i,2} &= \{\overline{\theta}_i\} \times \{0\} \times [\underline{x}_{i,2}, \overline{x}_{i,2}] \\ \widetilde{\boldsymbol{\ell}}_{i,3} &= \{\overline{\theta}_i\} \times \{1-\delta\} \times [\underline{x}_{i,3}, \overline{x}_{i,3}] \\ \widetilde{\boldsymbol{\ell}}_{i,4} &= \{\underline{\theta}_i\} \times \{1-\delta\} \times [\underline{x}_{i,4}, \overline{x}_{i,4}], \end{aligned} \tag{5.11}$$

where, for each $j \in \{1, \ldots, 4\}$ we have $0 \leq \underline{x}_{i,j} < \overline{x}_{i,j} \leq 1/2$. Next, for each $i \in \{1, \ldots, N\}$ consider the box formed by taking the hull of the four contracted vertical edges $\widetilde{\boldsymbol{\ell}}_{i,j}$, $j \in \{1, \ldots, 4\}$,

$$\begin{gathered} \widetilde{\boldsymbol{B}_i} = [\underline{\theta}_i, \overline{\theta}_i] \times [0, 1-\delta] \times [m_i, M_i], \\ \text{where } \ m_i := \min_{j \in \{1,\ldots,4\}} \underline{x}_{i,j} \ \text{ and } \ M_i := \max_{j \in \{1,\ldots,4\}} \overline{x}_{i,j}. \end{gathered} \tag{5.12}$$

Under the additional assumption that the mapping

$$\begin{gathered} [\varepsilon, \pi-\varepsilon] \times [0, 1-\delta] \ni (\theta, \kappa) \mapsto x_1(\theta, \kappa) \\ \text{is monotone in the variables } \theta \text{ and } \kappa, \end{gathered} \tag{5.13}$$

for each $i \in \{1, \ldots, N\}$ we have

$$\boldsymbol{B}_i \cap \mathfrak{A} \subset \widetilde{\boldsymbol{B}_i}. \tag{5.14}$$

Consequently, the family of boxes $\{\widetilde{\boldsymbol{B}_i}\}_{i \in \{1,\ldots,N\}}$ cover $\mathfrak{A}$. In Step 4 we will describe how we verify that assumption (5.13) is indeed satisfied and as such, the family $\{\widetilde{\boldsymbol{B}_i}\}_{i \in \{1,\ldots,N\}}$ obtained in this step is a tighter enclosure of $\mathfrak{A}$ (as compared to $\{\boldsymbol{B}_i\}_{i \in \{1,\ldots,N\}}$).

Step 3. The aim of this step is to ensure the applicability of the implicit function theorem as discussed at the beginning of the proof. To achieve this we implement an algorithm showing that $\frac{\partial f}{\partial x}$ is bounded and non-zero on the enclosure $\widetilde{\boldsymbol{B}} := \bigcup_{i=1}^{N} \widetilde{\boldsymbol{B}_i}$ obtained in the previous step. This reduces to checking that for each $i \in \{1, \dots, N\}$ we have that

$$\operatorname{diam}\Big(\frac{\partial F}{\partial x}(\boldsymbol{B}_i)\Big) \text{ is finite and } 0 \notin \frac{\partial F}{\partial x}(\widetilde{\boldsymbol{B}_i}). \tag{5.15}$$

As a consequence of (5.15), the implicit function theorem is applicable on $\widetilde{\boldsymbol{B}}$, and thus the solution set $\mathfrak{A}$ is a surface.

Step 4. The goal of this step is to prove (5.13). This can be done by verifying that

$$\frac{\partial x_1}{\partial \theta}(\cdot,\cdot) \text{ and } \frac{\partial x_1}{\partial \kappa}(\cdot,\cdot) \text{ do not vanish on } [\varepsilon, \pi-\varepsilon] \times [0, 1-\delta]. \tag{5.16}$$

With an eye towards proving (5.16) we justify the implicit differentiation in (5.2) (here we use Step 3) which led to (5.4). As a consequence, for each $i \in \{1, \dots, N\}$, the following inclusions hold:

$$\frac{\partial x_1}{\partial \theta}\Big|_{[\underline{\theta}_i, \overline{\theta}_i]\times[0,1-\delta]} \subseteq -\frac{\frac{\partial F}{\partial \theta}(\widetilde{\boldsymbol{B}}_i)}{\frac{\partial F}{\partial x}(\widetilde{\boldsymbol{B}}_i)}, \tag{5.17}$$

and

$$\frac{\partial x_1}{\partial \kappa}\Big|_{[\underline{\theta}_i, \overline{\theta}_i]\times[0,1-\delta]} \subseteq -\frac{\frac{\partial F}{\partial \kappa}(\widetilde{\boldsymbol{B}}_i)}{\frac{\partial F}{\partial x}(\widetilde{\boldsymbol{B}}_i)}. \tag{5.18}$$

Next, we appeal to the second part in (5.15) in Step 3 to ensure that for each $i \in \{1, \dots, N\}$ the right-hand sides of (5.17) and (5.18) are meaningful. Since $\frac{\partial f}{\partial \kappa}(\theta, \kappa, x) = (x-1)\sin\theta < 0$ on $(0,\pi) \times [0,1] \times [0, \frac{1}{2}]$ we deduce that $\frac{\partial x_1}{\partial \kappa}(\cdot,\cdot)$ does not vanish on $[\varepsilon, \pi-\varepsilon] \times [0, 1-\delta]$. Matters are therefore reduced to checking that, for each $i \in \{1, \dots, N\}$ we have

$$0 \notin \frac{\partial F}{\partial \theta}(\widetilde{\boldsymbol{B}}_i). \tag{5.19}$$

We achieve this by implementing an algorithm that computes the intervals $\frac{\partial F}{\partial \theta}(\widetilde{\boldsymbol{B}}_i)$ for $i \in \{1, \dots, N\}$ and check that they are bounded away from zero. This establishes the monotonicity of the surface in (5.3) and, at the same time, justifies the tightening process in Step 2.

This finishes the proof of item (1); item (2) follows from a similar treatment, completing the proof of the proposition. $\square$

Remark 5.2. We have hand-coded the partial derivatives of f needed in (5.17). For more complicated functions, one may utilize *automatic differentiation*, which only requires the explicit formula for f. For a concise introduction to this technique, see [15]. Note also that, in Steps 1 and 2, we never subdivide along the κ-component. For general functions f, however, this may have to be done.

Table 1. Computational information

Equations	Boxes	Time (ms)
(3.205)	16,458	4940
(3.207)	39,896	17,300

We are now ready to present the proof of Theorem 1.3.

Proof of Theorem 1.3. Items (1) and (2) are a direct consequence of Proposition 5.1. The case of items (3) and (4), when $\theta \in [\pi+\varepsilon, 2\pi-\varepsilon]$, follows immediately from (1.9) and Proposition 5.1. □

5.1. Computational Results

The actual verifications needed in the proof of Proposition 5.1 were carried out on a single thread on an eight core Intel i7 processor running at 2.67GHz. The operating system was Ubuntu 14.04 with the gcc compiler (version 4.8.2) and the interval analysis package CXSC, version 2.5.4, see [16]. The total computing time was roughly 23 seconds.

In Fig. 1, we illustrate the surfaces $x_1(\cdot,\cdot)$ and $x_2(\cdot,\cdot)$. Note how the surface $x_2(\cdot,\cdot)$ corresponding to Eq. (3.207) is very flat when $(\theta,\kappa) \approx (\pi, 1)$. Similarly, the surface $x_1(\cdot,\cdot)$ corresponding to Eq. (3.205) is flat when $(\theta,\kappa) \approx (0, 1)$. This makes all steps of the computer aided proof very hard to perform near these regions, which is apparent in Fig. 2 where the partitions of the domain are visible.

In Table 1, we present some computational information from the proof. The first column indicates the equation under study. The second column lists the number of boxes produced in Step 1 of the proof. The third column lists the CPU time (in milliseconds) required to complete the entire proof.

5.2. Stokes System

As Proposition 5.1 does not treat the case $\kappa = 1$, we address this situation in what follows. We are interested in the two equations

$$f_\sigma(\theta, x) = (x-1)\sin\theta - \sigma \sin\left[(2\pi-\theta)(x-1)\right] = 0, \qquad \sigma \in \{-1, +1\}. \tag{5.20}$$

We want to know if (5.20) implicitly defines a curve $x_\sigma = x_\sigma(\theta)$, and, if so, for what domain. We are also intersted in monotonicity properties of the curve (Fig. 3).

Proposition 5.3. *The following hold:*

1. *Equation* (5.20) *with* $\sigma = +1$ *implicitly defines a curve* $x_1 = x_1(\theta)$ *for* $\theta \in [10^{-4}, \pi)$ *which is decreasing in* θ.
2. *Equation* (5.20) *with* $\sigma = -1$ *implicitly defines a curve* $x_2 = x_2(\theta)$ *for* $\theta \in (0, 1.78977]$, *which is decreasing in* θ.

The proof uses Lemma 3.8; more precisely, we use the fact that there is at most one solution $x_\sigma(\theta)$ to (5.20) for $\theta \in (0,\pi)$. Based on this, we start by computing an approximation to the curve x_σ at a finite number of grid

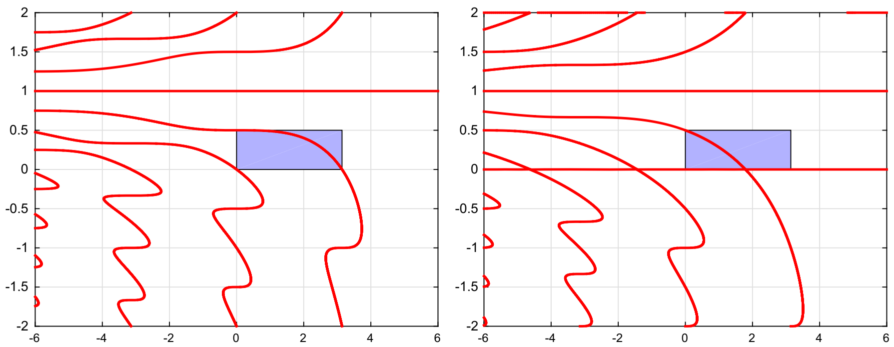


Figure 3. An illustration of Proposition 5.3. The *small boxes* are the maximal domains of interest $(\theta, x) \in (0, \pi) \times (0, 1/2)$. When $\sigma = +1$, the implicit curve $x(\theta)$ (*left*) extends over the entire domain. When $\sigma = -1$, the implicit curve $x(\theta)$ (*right*) exits the domain at $\theta \approx 1.78975$

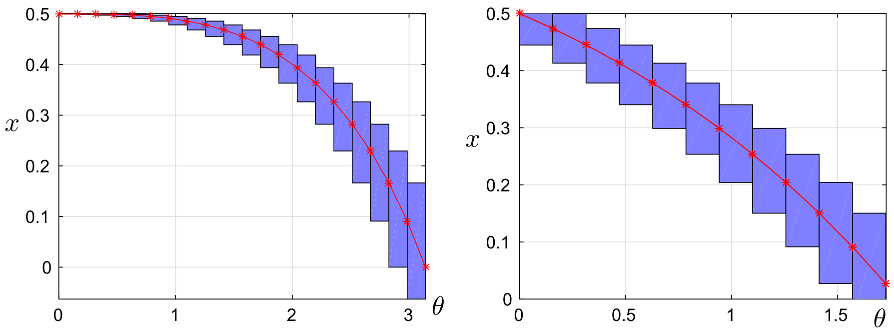


Figure 4. An illustration of the enclosing cover of the graph of $x_\sigma(\theta)$. The *rectangles* of the cover (*blue*) are centered on the approximate graph (*red*), ensuring that the exact solution to (5.20) never comes close to the *horizontal edges* of the *rectangles*. The case $\sigma = +1$ is presented in the *left figure*. The case $\sigma = -1$ is presented in the *right figure* (color figure online)

points θ_i, $i = 1, \dots, N$. Next, we cover the approximate curve with $N - 1$ rectangles as illustrated in Fig. 4. We construct the cover in such a way that the approximate curve extends horizontally across each rectangle. We verify that the partial derivatives of f_σ are bounded and non-zero on each rectangle; this allows us to invoke the implicit function theorem (and to prove monotonicity). Finally, we verify that the function f_σ assumes different signs on the two horizontal edges of each rectangle. This ensures that the graph of the implicit function $x_\sigma(\theta)$ is well-defined and is enclosed by the cover.

In the computer-aided proof of Proposition 5.3, 500000 (2848598) rectangles were used in the cover. The computations took ca 780 (4400) ms for each case $\sigma = -1$ (or $\sigma = 1$). The reported bound 1.78977 in Proposition 5.3 is a lower estimate of the number θ_o introduced in (1.23). In fact, we can enclose this number as accurately as we wish: it is simply a matter of using sufficiently high precision in our computations.

Lemma 5.4. *The equation* $\sin\theta + (2\pi - \theta)\cdot\cos\theta = 0$ *has a unique solution* θ_o *in* $[0, \pi]$ *which satisfies* $\theta_o \in [1.78977584927052, 1.78977584927053]$.

The computer-assisted proof is based on the techniques and algorithms described in Sect. 5.3.

5.3. Interval Analysis

The foundation of most computer-aided proofs dealing with continuous problems is the ability to compute with set-valued functions. This allows for all rounding errors to be taken into account, and even more importantly, all discretization errors. Here, we will briefly describe the fundamentals of interval analysis (for a concise reference on this topic, see e.g., [1,35,37]).

Let $\mathbb{IR}$ denote the set of closed intervals. For any element $\boldsymbol{x} \in \mathbb{IR}$, we adopt the notation $\boldsymbol{x} = [\underline{\boldsymbol{x}}, \overline{\boldsymbol{x}}]$, where $\underline{\boldsymbol{x}}, \overline{\boldsymbol{x}} \in \mathbb{R}$. If $\star$ is one of the operators $+, -, \times, \div$, we define the arithmetic on elements of $\mathbb{IR}$ by

$$\boldsymbol{x} \star \boldsymbol{y} = \{a \star b \colon a \in \boldsymbol{x}, b \in \boldsymbol{y}\},$$

except that $\boldsymbol{x} \div \boldsymbol{y}$ is undefined if $0 \in \boldsymbol{y}$. Working exclusively with closed intervals, we can describe the resulting interval in terms of the endpoints of the operands:

$$\begin{aligned}
\boldsymbol{x} + \boldsymbol{y} &= [\underline{\boldsymbol{x}} + \underline{\boldsymbol{y}}, \overline{\boldsymbol{x}} + \overline{\boldsymbol{y}}],\\
\boldsymbol{x} - \boldsymbol{y} &= [\underline{\boldsymbol{x}} - \overline{\boldsymbol{y}}, \overline{\boldsymbol{x}} - \underline{\boldsymbol{y}}],\\
\boldsymbol{x} \times \boldsymbol{y} &= [\min(\underline{\boldsymbol{x}}\underline{\boldsymbol{y}}, \underline{\boldsymbol{x}}\overline{\boldsymbol{y}}, \overline{\boldsymbol{x}}\underline{\boldsymbol{y}}, \overline{\boldsymbol{x}}\overline{\boldsymbol{y}}), \max(\underline{\boldsymbol{x}}\underline{\boldsymbol{y}}, \underline{\boldsymbol{x}}\overline{\boldsymbol{y}}, \overline{\boldsymbol{x}}\underline{\boldsymbol{y}}, \overline{\boldsymbol{x}}\overline{\boldsymbol{y}})],\\
\boldsymbol{x} \div \boldsymbol{y} &= \boldsymbol{x} \times [1/\overline{\boldsymbol{y}}, 1/\underline{\boldsymbol{y}}], \quad \text{if } 0 \notin \boldsymbol{y}.
\end{aligned} \tag{5.21}$$

Note that the identities (5.21) reduce to ordinary real arithmetic when the intervals are *thin*, i.e., when $\underline{\boldsymbol{x}} = \overline{\boldsymbol{x}}$ and $\underline{\boldsymbol{y}} = \overline{\boldsymbol{y}}$. When computing with finite precision, however, directed rounding must also be taken into account, see e.g., [35,36].

A key feature of interval arithmetic is that it is *inclusion monotonic*, i.e., if $\boldsymbol{x} \subseteq \widehat{\boldsymbol{x}}$, and $\boldsymbol{y} \subseteq \widehat{\boldsymbol{y}}$, then

$$\boldsymbol{x} \star \boldsymbol{y} \subseteq \widehat{\boldsymbol{x}} \star \widehat{\boldsymbol{y}}, \tag{5.22}$$

where we demand that $0 \notin \widehat{\boldsymbol{y}}$ for division.

One of the main reasons for passing to interval arithmetic is that this approach provides a simple way of enclosing the range of real-valued elementary functions f over simple domains. In what follows, we will use the notation $\operatorname{range}(f; \boldsymbol{x}) := \{f(x) \colon x \in \boldsymbol{x}\}$. Except for the most trivial cases, classical mathematics provides few tools to accurately bound the range of a function. To achieve this latter goal, we extend the real functions to *interval functions* which take and return intervals rather than real numbers. Based on (5.21) we extend a given representation of a real-valued rational function to its interval version by simply substituting all occurrences of the real variable x with the interval variable $\boldsymbol{x}$ (and the real arithmetic operators with their interval counterparts). This produces a rational *interval* function $F\colon \mathbb{IR} \cap D_f \to \mathbb{IR}$, called the *natural interval extension* of $f\colon D_f \to \mathbb{R}$, where $D_f \subseteq \mathbb{R}$ is the domain of the function f. As long as all interval arithmetic operations are well-defined, we have the inclusion

$$\mathrm{range}(f;\boldsymbol{x}) \subseteq F(\boldsymbol{x}), \tag{5.23}$$

by property (5.22). In fact, this type of range enclosure can be obtained for any elementary function.

A higher-dimensional function $f\colon \mathbb{R}^n \to \mathbb{R}$ can be extended to an interval function $F\colon \mathbb{IR}^n \to \mathbb{IR}$ in a similar manner. The function argument is then an interval-vector $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$, which we also refer to as a *box*. There exist several open source programming packages for interval analysis [16,26,40], as well as commercial products such as [13].

We will now illustrate the use of interval techniques, with a special emphasis on non-linear equation solving. Given a real-valued continuous function f together with an interval domain $\boldsymbol{x}$, we want to locate all zeros of f restricted to $\boldsymbol{x}$. We will do this by subdividing the domain into smaller intervals:

$$\boldsymbol{x} = \bigcup_{i=1}^{N} \boldsymbol{x}_i. \tag{5.24}$$

The contrapositive version of (5.23) gives

$$0 \notin F(\boldsymbol{x}_i) \Longrightarrow \forall x \in \boldsymbol{x}_i, f(x) \neq 0. \tag{5.25}$$

This is an effective criterion for discarding subsets of the domain that provably do not contain zeros of f. By continuity, the intermediate value theorem provides a simple check for a subinterval to enclose (at least) one zero of f: if $f(\underline{\boldsymbol{x}}_i)$ and $f(\overline{\boldsymbol{x}}_i)$ have opposite signs, then $f(x) = 0$ for some $x \in \boldsymbol{x}_i$. If f is continously differentiable, and we also have $0 \notin F'(\boldsymbol{x}_i)$, then we know that $\boldsymbol{x}_i$ encloses a unique zero of f. In higher dimensions, the intermediate value theorem is replaced by more general statements such as Miranda's theorem [32].

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Received: November 28, 2016.
Revised: July 31, 2017.