Combinatorial intersection cohomology: a survey

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Projective toric varieties and lattice polytopes may be considered as two faces of the same coin. Accordingly, in the last 25 years, investigations related with toric varieties and their cohomology have played an increasingly important role in studying the combinatorics of convex polytopes. This started around 1980 with Stanley's spectacular proof of the necessity of McMullen's conditions (characterizing the face numbers of simple polytopes) using the cohomology of "rationally smooth" projective toric varieties. It continued with his introduction of a generalized h-vector for non-simple polytopes, modeled on the properties of the intersection cohomology Betti numbers of general projective toric varieties. In the last five years, attempts to prove the conjectured properties of this generalized h-vector led to the introduction of a purely combinatorial "virtual" intersection cohomology for polytopes, inspired by equivariant intersection cohomology of projective toric varieties. This work culminated in the recent proof of a "combinatorial Hard Lefschetz" theorem, which provides the keystone to proving Stanley's conjectures. – The aim of the present talk is to survey these developments.

The most basic combinatorial data of a convex polytope in $\mathbf{R}^n$ are the numbers f_i of i-dimensional faces, collected in the f(ace)-vector $(f_0, \dots, f_n)$ or, equivalently, encoded in the f(ace)-polynomial $f(t) := \sum_{i=0}^n f_i t^i$. For *simple* polytopes, i.e., where each vertex lies on exactly n edges, the possible f-polynomials are characterized by *McMullen's conditions*. These are most conveniently stated in terms of the *h-vector* $(h_0, \dots, h_n)$, i.e., the coefficient vector of the "*h-polynomial*" $h(t) := f(t-1) =: \sum_{i=0}^n h_i t^i$: The integers h_i are strictly *positive*, they satisfy the *symmetry* relation $h_i = h_{n-i}$, the "*unimodality* property" $h_i \leq h_{i+1}$ holds for $i \leq n/2 - 1$, and there are specific estimates for the growth of the differences $h_{i+1} - h_i$. By duality, there is a corresponding characterization for the class of simplicial polytopes.

The h-polynomial occurs in quite a different context if a simple polytope P is *rational*: The outer normal fan $\Delta(P)$ determines a projective toric variety $X_{\Delta(P)}$. Since the fan is simplicial, this variety is a rational homology manifold. It turns out that its Poincaré polynomial agrees with $h(t^2)$. This yields Stanley's "topological" proof for the necessity of McMullen's conditions: Symmetry corresponds to Poincaré duality, positivity and unimodality come from the Hard Lefschetz theorem, and the growth conditions follow from the fact that the cohomology algebra $H^*(X_{\Delta(P)})$ – and hence also its factor algebra $H^*(X_{\Delta(P)})/(\omega)$ with the hyperplane class ω – is generated by elements of degree 2.

On the other hand, if the simple polytope P is *non-rational*, then there is no longer an associated toric variety and thus, no cohomology algebra. Nevertheless, the above argument for the h-vector still can be used: Regarding P as an intersection of half-spaces, any nearby rational polytope has the same combinatorial type. But there is a more systematic approach, namely, to associate to P itself – or rather to the simplicial fan $\Delta(P)$ – a "virtual" cohomology algebra $H^*(\Delta(P))$ with Hilbert polynomial $h(t^2)$ as follows: Let V denote the ambient vector space of $\Delta(P)$, so P "lives" in V^*. Let us consider $A := S(V^*)$, the algebra of polynomial functions on V, graded by $V^* =: A^{(2)}$, and the homogeneous maximal ideal $\mathfrak{m}$ of all polynomials vanishing at 0. For a graded A-module M, we denote with $\overline{M} := (A/\mathfrak{m}) \otimes_A M$ the graded real vector space obtained by reduction modulo $\mathfrak{m}$. In this setting, we associate to $\Delta(P)$ the graded A-module $A_{\Delta(P)}$ of all cone-wise polynomial functions, and then define $H^*(\Delta(P)) := \overline{A}_{\Delta(P)}$. This approach is motivated by the equivariant cohomology of the toric variety $X_{\Delta(P)}$ associated to P in the rational case: There is a natural action of an algebraic torus T. If P is simple, the variety $X_{\Delta(P)}$ is T-equivariantly formal, i.e., the ordinary cohomology $H^*(X_{\Delta(P)})$ is obtained from $H^*_T(X_{\Delta(P)}) \cong A_{\Delta(P)}$ by reduction modulo the homogeneous maximal ideal $\mathfrak{m}$ in $H^*(BT) \cong A$.

We now consider polytopes that are *not simple*, so their outer normal fan fails to be simplicial. If such a polytope P is *rational*, the associated projective toric variety $X_{\Delta(P)}$ is never a rational homology manifold. Neither its Betti numbers nor the h-vector of P in general enjoy the properties mentioned above. Considering intersection cohomology instead of the "usual" theory, however, yields an even Poincaré polynomial $\mathcal{Q}$ with "good" properties since both, Poincaré duality and the Hard Lefschetz theorem, hold for $IH^*(X_{\Delta(P)})$. One may thus assign the polynomial h with $\mathcal{Q}(t) = h(t^2)$ to the polytope P as *generalized* h-polynomial. The corresponding generalized h-vector then enjoys three of the properties that hold for simple polytopes, namely, *positivity, symmetry, and unimodality.* In contrast to the simple case, however, there is no natural algebra structure on $IH^*(X_{\Delta(P)})$, so the proof of the growth estimates does not carry over; furthermore, there is no immediate connection between this new h-polynomial and the face polynomial. On the other hand, there is a recursion method to compute h from combinatorial data of P, so the same recursion allows to assign a generalized h-polynomial also to *non-rational* polytopes, cf. [St].

In contrast to the situation for simple polytopes, nearby polytopes now do not necessarily have the same combinatorial type. So the following question is natural: In the non-rational case, does the new h-vector still have the same three properties: positivity, symmetry, and unimodality? It motivated the search for a "virtual" intersection cohomology theory $IH^*(\Delta(P))$, as in the case of simple polytopes. In fact, the investigation of the "sheafified" equivariant intersection cohomology of toric varieties leads to the following construction entirely in terms of the fan Δ: To apply sheaf theory, the fan is endowed with the "fan topology", where the subfans $\Lambda \subset \Delta$ are the open subsets. On that fan space, there is a natural structure sheaf $\mathcal{A}$ of graded rings given by the assignment $\Lambda \mapsto A_\Lambda$, so in particular $A_\sigma := \mathcal{A}(\sigma) = S(V_\sigma^*)$ with $V_\sigma := \mathrm{span}(\sigma)$. A sheaf $\mathcal{F}$ of graded $\mathcal{A}$-modules is called *pure* if it is flabby and satisfies the following condition:

(*) For each $\sigma \in \Delta$, the A_σ-module $F_\sigma := \mathcal{F}(\sigma)$ is finitely generated and free.

A sheaf $\mathcal{F}$ on Δ is flabby iff for each cone σ, the restriction map $F_\sigma \to F_{\partial\sigma}$ is surjective; if $\mathcal{F}$ even satisfies (*), then this surjectivity is equivalent to that of $\overline{F}_\sigma \to \overline{F}_{\partial\sigma}$. The structure sheaf $\mathcal{A}$ clearly satisfies condition (*); it is flabby iff Δ is simplicial, which holds for a polytopal fan $\Delta(P)$ iff P is simple. Up to isomorphism, among the pure sheaves $\mathcal{F}$ on Δ with $F_o = \mathbf{R}$, there is a unique minimal object $\mathcal{E}$ determined by the condition that $\overline{E}_\sigma \xrightarrow{\cong} \overline{E}_{\partial\sigma}$ even is an isomorphism for each $\sigma \neq o$. It is called the "equivariant intersection cohomology sheaf" of Δ, and $IH^*(\Delta) := \overline{E}_\Delta$ is the "virtual" intersection cohomology sought after, cf. [BBFK$_2$, BreLu$_1$].

We now have to analyse how far Poincaré duality and, in the case of polytopal fans $\Delta = \Delta(P)$, the Hard Lefschetz theorem continue to hold. As to Poincaré duality, we note that for any oriented fan Δ, the category of pure sheaves admits an involutive duality functor $\mathcal{F} \mapsto \mathcal{D}\mathcal{F}$. After fixing a volume form on V, that provides a natural isomorphism $\mathcal{D}\mathcal{E} \cong \mathcal{E}$. In fact, the naturality is not immediate since it relies on the Hard Lefschetz theorem for polytopal fans of lower dimensions. This duality isomorphism provides a natural intersection product "$\cap$" on $IH^*(\Delta)$. In particular, this yields Poincaré duality between homogeneous subspaces of complementary dimensions, cf. [BBFK$_3$, BreLu$_2$].

As to the Hard Lefschetz Theorem, one assigns to a polytope P a natural strictly convex conewise linear function ψ on its outer normal fan $\Delta := \Delta(P)$ as follows: For each n-dimensional cone σ, the restriction $\psi|_\sigma \in V^*$ is precisely the corresponding vertex of the polytope $P \subset V^*$. The multiplication endomorphism $\mu_\psi \colon E_\Delta \to E_\Delta$ induces the "Lefschetz operator" $L := \overline{\mu}_\psi : IH^*(\Delta) \to IH^{*+2}(\Delta)$, and the Hard Lefschetz Theorem states that each $L^k\colon IH^{n-k}(\Delta) \to IH^{n+k}(\Delta)$ is an isomorphism for $k \geq 0$. Its proof follows easily from the "Hodge-Riemann bilinear relations" (HRR), according to which the pairing

$$IH^{n-k}(\Delta) \times IH^{n-k}(\Delta) \longrightarrow \mathbf{R}\ , \quad (\xi, \eta) \mapsto \xi \cap L^k(\eta)$$

is $(-1)^{(n-k)/2}$-definite on the "primitive" subspace $IP^{n-k}(\Delta) := \ker(L^{k+1})$. For a simple polytope P, these relations have been proved in [Mc], to which the general case can be reduced according to [Ka].

Let us sketch a geometric idea for such a reduction: We successively cut off "bad" faces from the polytope P, lowering their number in each step. Since a polytope without bad faces is simple, this procedure eventually yields the starting point for an induction. We describe the typical step: We call a face $F \subset P$ "bad" if its link is not a cone $C(Q)$ over some polytope Q. A bad face F of minimal dimension is itself a simple polytope and admits a "tubular neighbourhood" in P. To cut off F, we write $F = P \cap H$ with a hyperplane H and move H slightly towards the interior of P. Intersecting P with the two corresponding half-spaces yields a decomposition $P = P_1 \cup P_2$ into polytopes, with P_2 containing F and P_1 on the other side of the hyperplane. By induction hypothesis, HRR holds for P_1 since it has less bad faces than P. The fact that HRR also holds for P_2 can be derived from the lower-dimensional case: The polytope P_2 is "hip-roofed" with ridge F, and a transversal cross-section is a cone $C(Q)$ over a polytope Q of dimension $n-1-\dim F$. Now HRR for Q implies HRR for $C(Q)$, and for $\dim F > 0$ the polytope P_2 is "trivialized" by moving the ridge to infinity. Patching together the HRR for P_1 and P_2 by a Mayer-Vietoris argument yields the result for P.

Hence, even for non-rational polytopes, the generalized h-vector satisfies the three properties: positivity, symmetry, and unimodality.

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