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Abstract

It is natural to expect that the performance of a sensor network improves if sensor nodes are allowed to move, assuming that messages are not time-critical and longer propagation delays are permitted. To understand such delays we analyze a one-dimensional version of a sensor network, where the initial positions of the sensors are determined by a Poisson process on the line with constant intensity. Initially a message reaches one of the sensors which is activated and starts to move as a Brownian particle. The message is passed on successively to any dormant sensor, which then itself turns into such a Brownian sensor, at the first instant it is hit by a previously activated one. The system of active sensors grows in a tree like fashion with branch points corresponding to message transmission times. It is shown that the expected times between subsequent activations are all equal, independently of the number of active sensors, and subject to a simple scaling relation of the Poisson intensity and the variance of the Brownian motion. Furthermore, the sequence of transmission times satisfies a strong law of large numbers.

1 Introduction

A random sensor network is a wireless communication network where the nodes are randomly placed sensor units. In addition to monitoring the surrounding environment, such as measuring temperature, moisture etc., the nodes are able to generate and forward messages. They are equipped with a low functionality radio mechanism that allows for messages to be transmitted between nodes which are sufficiently close to each other. A message present in a given node is instantly transmitted to any other node within transmission range. The capability of a network to forward messages over longer distances should increase with node density, at least if we ignore harmful effects such as interference between nodes. As an alternative method for improving network capacity it is natural to consider random sensor mobility. With mobile sensors and in a time dynamical setting, we expect the message propagation properties of the network to improve at the cost of the delay imposed by the waiting time of sensors to reach within transmission range.

Our goal is to understand the trade-off between node density and node mobility and to study the message propagation properties in a mobile sensor network. In

this work, we consider a simplified one dimensional sensor network, where the sensor units have zero transmission range and move according to independent Brownian motions. This corresponds to a non-radio system where sensors move randomly on the real line and an active sensor that carries a message must reach, and physically activate, a neighbor node in order for the message to be passed on. Our measure of performance will be the rate of growth at which the sensors are activated. Specifically, our main object of study is the random time required for a single message to reach a fixed number of nodes, or, asymptotically, a large number of nodes.

Mathematical aspects of sensor networks have been treated in [10] where coverage properties are considered when sensors move as independent Brownian motions. A preliminary presentation of the results in this paper appears in [7], which also includes simulation studies for higher dimensions. A dynamical network where active sensors move as independent Brownian motions on adjacent line segments with reflecting barriers is studied in [6]. Another type of dynamics in a sensor network is treated in [2], where sensors switches on and off to save power. The trade-off between node density and interference mentioned above is examined in [1].

2 Model and results

We begin by describing informally our model for the spread of a message among random sensors in one spatial dimension. The points of a Poisson point process with constant intensity on the real line are the locations of dormant sensor units which form nodes of a sensor network. The evolution of the system begins as a message is initiated by one of the nodes. This activates the mobility function of the node which starts to perform a Brownian motion on the line, hence effectively changing from a dormant sensor into a Brownian sensor. At the first instant the Brownian sensor reaches one of its two nearest neighbors the message is passed on, and the receiving node starts to perform a Brownian motion which is independent of the motion of the initial node. As of this time we have two active Brownian sensors generated by two independent Brownian motions. We can imagine they search off the space around them until either hits a third node. In this way the number of active sensors grows over time as the message is passed on to further dormant nodes, which successively turn into Brownian sensors that help the network to spread the message effectively. We are interested in the rate of growth of the system, and the dynamics and distributional properties of such a collection of Brownian sensors.

To set up the model properly suppose that on a probability space with probability measure P we are given a Poisson point measure on the real line with intensity $\lambda > 0$. These points correspond to the initial locations of dormant network sensor units. The distances between sensors are i.i.d. exponential random variables (V_i) with parameter $\lambda > 0$, where the ordering is going to be determined by the succes-

sive order at which the sensors are activated. On the same probability space are given furthermore a family of independent standard zero mean Brownian motions (B_i) , $B_i = \{B_i(t), t \geq 0\}$; to each initial Poisson point there is a Brownian motion associated. These provide mobility of the sensors in such a way that active sensor units move independently of each other with motions corresponding to Brownian particles σB_i , where $\sigma^2 > 0$ is a variance parameter. Thus, λ and σ are parameters for node density and node mobility respectively.

2.1 Activation times

We will construct a sequence of random variables (τ_k) which give the successive times between sensor activations. Fix one of the Poisson points and start there at time $t = 0$ a Brownian motion with no drift and variance parameter σ^2 . This is the initially activated Brownian sensor from which the message is passed on as the active nodes sweep out larger portions of space and step by step reaches more distant dormant nodes on either side.

Seen from the initial active node, the two nearest Poisson points, one on each side, are independent exponential distances away, say V_1 and V_2 with mean $1/\lambda$. Thus, at $t = 0$ the active sensor starts to perform a Brownian particle motion initially located on an interval of gamma distributed length $R_2(\lambda) = V_1 + V_2 \in \Gamma(2, \lambda)$, at the relative position $X_{1,1} = V_1/(V_1 + V_2)$ which has the uniform distribution $U[0, 1]$ on the corresponding normalized interval. Here, $X_{1,1}$ and $R_2(\lambda)$ are independent. Define τ_1 to be the first exit time of the Brownian motion from the interval of length $R_2(\lambda)$. Using Brownian scaling we have the representations

$$\tau_1 = \inf\{t : V_1 + \sigma B_1(t) \notin (0, R_2(\lambda))\} \stackrel{d}{=} \frac{1}{\sigma^2} R_2^2(\lambda) \inf\{t : X_{1,1} + \tilde{B}_1(t) \notin (0, 1)\},$$

where $\tilde{B}_1(t) = B_1(R_2^2(\lambda)t/\sigma^2)\sigma/R_2(\lambda)$ is again a standard Brownian motion, independent of the Poisson process. It is convenient to use yet another distributional representation where we introduce the notation $R_2 = \lambda R_2(\lambda) \in \Gamma(2, 1)$ in order to separate the parameter λ , namely

$$\tau_1 \stackrel{d}{=} \frac{1}{\lambda^2 \sigma^2} R_2^2 \inf\{t : X_{1,1} + \tilde{B}_1(t) \notin (0, 1)\}.$$

We interpret the exit event at τ_1 such that the initial Brownian sensor has activated one of its neighbor nodes, which itself becomes a Brownian sensor independent of the first.

In the beginning of the next phase of the process, the two nearest dormant sensors are distance $R_3(\lambda) = R_2(\lambda) + V_3$ apart and we have two active Brownian sensors; one continuing and another starting its Brownian motion. At time τ_1 they are located at the same position on their interval of length $R_3(\lambda)$. The relative position $X_{2,1} = X_{2,2}$ of the two sensors is given by a mixture of beta distributions due to the properties of gamma distributed interval partitions. The distribution is either that of $V_3/R_3(\lambda) \in \beta(1, 2)$ or of $(V_1 + V_2)/R_3(\lambda) \in \beta(2, 1)$, with probability

one half for each case. The subsequent time until the two sensors trigger activation of a third is now obtained as

$$\begin{aligned}\tau_2 &= \min_{i=1,2} \inf\{t : R_3(\lambda)X_{2,i} + \sigma B_{2,i}(t) \notin (0, R_3(\lambda))\} \\ &\stackrel{d}{=} \frac{1}{\lambda^2 \sigma^2} R_3^2 \min_{i=1,2} \inf\{t : X_{2,i} + \tilde{B}_{2,i}(t) \notin (0, 1)\} < \infty, \text{ a.s.}\end{aligned}$$

reindexing the Brownian motions for convenience. Here, $R_3 \in \Gamma(3, 1)$. Moreover, $X_{2,i}$, $i = 1, 2$, and R_3 are independent.

In general let τ_k , denote the length of the time interval during which k Brownian sensors are active, $k \geq 1$. These are stopping times with respect to the corresponding k Brownian motions, and are defined recursively as follows. For $k \geq 2$, at time $\tau_1 + \dots + \tau_{k-1}$ the interval between the nearest dormant sensors extends from $R_k(\lambda)$ to $R_{k+1}(\lambda) = R_k(\lambda) + V_{k+1} \in \Gamma(k+1, \lambda)$. At this time the number of active sensors increases from $k-1$ to k , and the ordered locations of active sensors is given by a vector of relative positions $(X_{k,1}, \dots, X_{k,k})$ on the unit interval, such that either $X_{k,1} = X_{k,2}$ or $X_{k,k-1} = X_{k,k}$. This yields

$$\tau_k \stackrel{d}{=} \frac{1}{\lambda^2 \sigma^2} R_{k+1}^2 \eta_k, \quad \eta_k = \min_{1 \leq i \leq k} \inf\{t : X_{k,i} + B_{k,i}(t) \notin (0, 1)\}, \quad k \geq 1, \quad (1)$$

where $R_{k+1} = \lambda \sum_{i=1}^{k+1} V_i \in \Gamma(k+1, 1)$, $k \geq 1$ and $B_{k,i}$, $1 \leq i \leq k$, $k \geq 1$, are independent standard Brownian motions (suppressing the tilde-notation). For each fixed k , R_{k+1} and $\{B_{k,i}\}$ are independent. Since the exit time of a Brownian motion in the unit cube of \mathbb{R}^k is finite, we have $\tau_k < \infty$ a.s. for all $k \geq 1$.

2.2 Growth of the sensor network process

Let S_n denote the time until the network has managed to propagate the initial message to n sensor nodes (in addition to the initial node), so that

$$S_n = \sum_{k=1}^n \tau_k \stackrel{d}{=} \frac{1}{\lambda^2 \sigma^2} \sum_{k=1}^n R_{k+1}^2 \eta_k. \quad (2)$$

We are going to derive the distribution of the relative positions $(X_{k,1}, \dots, X_{k,k})$ at the time of the activation of the k th Brownian sensor and from this find the expected growth times ES_n for the one-dimensional sensor network.

The sequence S_n measures the time instances until the network of active sensors is about to change from size n to size $n+1$. To obtain additional information on the symmetric aspects of the growing tree of active Brownian sensors we study also the one-sided rate of growth in the system. Enumerate the dormant sensors arbitrarily $\dots, m-1, m, m+1, \dots$ in order of their initial Poisson positions on the real line. Define for arbitrary integers $m \neq n$,

$$T_{m,n} = \text{the activation time of sensor } n \text{ if initially a message is initiated by sensor } m,$$

and let $T_{m,m} = 0$. We may think of these variables modeling the required time for a message to be carried from one node to a distant other node with the help of Brownian sensors relaying the message step by step. In particular, $T_{m,n}$ is the time to transport a message a random distance with distribution $\Gamma(|n - m|, \lambda)$.

We provide strong laws of large numbers for S_n and $T_{0,n}$. Our analysis relies on the basic subadditivity property for the collection of random variables $\{T_{m,n}\}$,

$$T_{0,n} \leq T_{0,m} + T_{m,n} \quad 0 \leq m \leq n. \quad (3)$$

2.3 Fixed number of sensors

We are now prepared to state our results for the distribution of the location of the Brownian sensors at the message transmission times, and for the expected time duration between such transmissions.

Theorem 1. *For each $k \geq 1$, the locations of sensors at the time when the k th sensor is activated is distributed on an interval of length $R_{k+1}(\lambda) \in \Gamma(k + 1, \lambda)$, such that the relative positions of the sensors are independent of $R_{k+1}(\lambda)$ and given for $k = 1$ by the uniform distribution $X_{1,1} \in U(0, 1)$, and for $k \geq 2$ in increasing order by the vector*

$$(X_{k,1}, \dots, X_{k,k}) \stackrel{d}{=} \begin{cases} (U_{(1)}, U_{(1)}, \dots, U_{(k-1)}), & \text{pr. } 1/2 \\ (U_{(2)}, \dots, U_{(k)}, U_{(k)}), & \text{pr. } 1/2, \end{cases}$$

where $(U_{(1)}, \dots, U_{(k)})$ is the order statistic of k independent and uniformly distributed points on the unit interval. Moreover, the expected time during which there are k active sensors is given by

$$E\tau_k = \frac{1}{\lambda^2 \sigma^2}, \quad k \geq 1.$$

2.4 Asymptotic results

It follows immediately from Theorem 1 that the expected activation time of the $(n + 1)$ th sensor is $ES_n = n/(\lambda\sigma)^2$. We will show in addition that S_n satisfies a strong law of large numbers. Indeed, S_n/n converges to the expected value $1/(\lambda\sigma)^2$ almost surely as $n \rightarrow \infty$

Theorem 2. *Given the intensity λ of the Poisson process and the variance parameter σ^2 for the Brownian motions it holds that,*

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} T_{0,n} &= \frac{2}{\lambda^2 \sigma^2} \quad \text{a.s. and in } L_1 \\ \lim_{n \rightarrow \infty} \frac{1}{n} S_n &= \frac{1}{\lambda^2 \sigma^2} \quad \text{a.s. and in } L_1. \end{aligned}$$

Simulations indicate that the convergence is very slow.

3 Properties of the sensor network model

3.1 Growth tree levels

It is convenient to consider separately the dynamics of the Brownian sensors between each pair of successive activation times. The collection of active sensors that evolve over time form a spatial tree of branching Brownian motions. During the time interval $(S_{k-1}, S_k]$, $S_0 = 0$, there are exactly k branches in the tree, $k \geq 1$, which yields a natural partitioning of the tree into levels. The motion of the corresponding k sensors form a level k process which we analyze in the following framework. Place k particles at the ordered positions $u = (u_1, \dots, u_k)$ on the unit interval. Let the particles perform Brownian motions $\xi_t = (\xi_t^1, \dots, \xi_t^k)$, where $\xi_t^i = u_i + B_i(t)$ and $B_i(t)$ is a standard Brownian motion, until the first exit time

$$\eta_k^u = \min_{1 \leq i \leq k} \inf\{t : \xi_t^i \notin (0, 1)\}.$$

The quantity η_k introduced in (1) is η_k^u further randomized by choosing initial distribution $u = \xi_0$ according to $(X_{k,1}, \dots, X_{k,k})$. Multiplication of η_k with the square of an independent gamma distributed random variable and normalization with the relevant model parameters $\lambda^2 \sigma^2$ yield a representation of the sensor activation time, as $\tau_k = R_{k+1}^2 \eta_k / (\lambda \sigma)^2$.

3.2 Distribution at growth times

Recall that the successive distances between nearest dormant sensors are given by sums of exponential random variables $R_k(\lambda) = \sum_{i=1}^k V_i \sim \Gamma(k, \lambda)$. On level k the ratios $\tilde{U}_i = R_i(\lambda) / R_{k+1}(\lambda)$, $1 \leq i \leq k$, are $\beta(i, k+1-i)$ -distributed and independent of $R_{k+1}(\lambda)$. Letting U_1, \dots, U_k be independent and uniform on $[0, 1]$, the corresponding ordered sample $U_{(1)}, \dots, U_{(k)}$ has the same distribution as $(\tilde{U}_1, \dots, \tilde{U}_k)$. The relevant density functions are

$$\begin{aligned} f_{U_{(1)}, \dots, U_{(k)}}(u_1, \dots, u_k) &= k! \mathbf{1}_{\{0 < u_1 < \dots < u_k < 1\}}, \\ f_{U_{(2)}, \dots, U_{(k)}}(u_2, \dots, u_k) &= k! u_2 \mathbf{1}_{\{0 < u_2 < \dots < u_k < 1\}}, \\ f_{U_{(1)}, \dots, U_{(k-1)}}(u_1, \dots, u_{k-1}) &= k! (1 - u_{k-1}) \mathbf{1}_{\{0 < u_1 < \dots < u_{k-1} < 1\}}. \end{aligned}$$

It is claimed in Theorem 1, and remains to be proved, that the normalized distribution $(X_{k,1}, \dots, X_{k,k})$ is with equal probabilities either $(U_{(2)}, U_{(3)}, \dots, U_{(k)}, U_{(k)})$ or $(U_{(1)}, U_{(1)}, U_{(2)}, \dots, U_{(k-1)})$. In the equivalent setting of a probability distribution defined on the unit cube $[0, 1]^k$ this yields for $x = (x_1, \dots, x_k) \in [0, 1]^k$ the (generalized) density function

$$\begin{aligned} h_k(x_1, \dots, x_k) &= \sum_{a \in I} \frac{1}{2} \left(x_{a_1} \delta(x_{a_{k-1}} - x_{a_k}) \mathbf{1}_{\{0 < x_{a_1} < \dots < x_{a_{k-1}} < 1\}} \right. \\ &\quad \left. + (1 - x_{a_k}) \delta(x_{a_1} - x_{a_2}) \mathbf{1}_{\{0 < x_{a_2} < \dots < x_{a_k} < 1\}} \right), \quad (4) \end{aligned}$$

where the summation is over the set I of all $k!$ distinct permutations $a = (a_1, \dots, a_k)$ of $(1, \dots, k)$. For continuous functions f defined on $[0, 1]^k$ we will write

$$\begin{aligned} Ef(X_{k,1}, \dots, X_{k,k}) &= \int_{[0,1]^k} f(x) h_k(dx) \\ &= \int_{[0,1]^{k-1}} \left[\sum_{a \in I} \frac{1}{2} \left(f(x_{a_1}, \dots, x_{a_{k-1}}, x_{a_k}) x_{a_1} 1_{\{0 < x_{a_1} < \dots < x_{a_{k-1}} < 1\}} dx_{a_1} \dots dx_{a_{k-1}} \right. \right. \\ &\quad \left. \left. + f(x_{a_2}, x_{a_2}, \dots, x_{a_k})(1 - x_{a_k}) 1_{\{0 < x_{a_2} < \dots < x_{a_k} < 1\}} dx_{a_2} \dots dx_{a_k} \right) \right]. \end{aligned} \quad (5)$$

3.3 The level 1 and level 2 processes

For $k = 1$ we have $X_{1,1} \in U[0, 1]$ and $E\tau_1 = E(R_2^2)E(\eta_1)/(\lambda\sigma)^2$ with $ER_2^2 = 6$ and

$$\begin{aligned} E\eta_1 &= E(X_{1,1} + B_{1,1}(\eta_1))^2 - EX_{1,1}^2 \\ &= E(0 \cdot (1 - X_{1,1}) + 1 \cdot X_{1,1}) - EX_{1,1}^2 = 1/6, \end{aligned}$$

using optional stopping of the martingale $\xi_t^2 - \xi_0^2 - t$, $t \geq 0$, at η_1 . Thus, $E\tau_1 = 1/(\lambda\sigma)^2$.

We turn to level $k = 2$. At time τ_1 the distance between the two possible candidates for being the next, third, Brownian sensor extends from length $R_2(\lambda)$ to length $R_3(\lambda)$. The two already activated Brownian sensors are at this time both located in the same point, the relative positions being with equal probabilities either $V_3/R_3(\lambda)$ or $(V_1 + V_2)/R_3(\lambda)$. These variables have the same beta distributions, $\beta(1, 2)$ and $\beta(2, 1)$, respectively, as the order variables $U_{(1)}$, $U_{(2)}$ generated by 2 independent uniform points in $(0, 1)$. Thus, $(X_{2,1}, X_{2,2})$ is equal to $(U_{(1)}, U_{(1)})$ or $(U_{(2)}, U_{(2)})$ with equal probability. By analyzing the two Brownian motions with some care it is possible to show that the expected time until the next branch point, where the third sensor is activated, is $E(\tau_2) = 1/(\lambda\sigma)^2$ and the resulting relative positions are with equal probabilities either of the form $(U_{(1)}, U_{(1)}, U_{(2)})$ or $(U_{(2)}, U_{(3)}, U_{(3)})$.

We are going to device for levels $k \geq 2$ a general martingale technique based on which the statement in Theorem 1 follows by an induction argument.

3.4 Martingale technique

Let $C([0, 1]^k)$ denote the class of continuous real-valued functions on the unit cube in \mathbb{R}^k and let $C^2([0, 1]^k)$ denote the subset of functions which are twice differentiable with continuous second derivatives. Furthermore, let $C_0^2([0, 1]^k)$ be the subset of functions ϕ in $C^2([0, 1]^k)$ with the additional boundary condition

$$\phi(x) = 0, \quad \text{for all } x \in \partial[0, 1]^k \quad (6)$$

Consider the σ -algebra $\mathcal{F}_t^k = \sigma(\xi_s, 0 \leq s \leq t)$. By Ito's formula, for $f \in C^2([0, 1]^k)$,

$$f(\xi_t) = f(\xi_0) + \frac{1}{2} \int_0^t \Delta f(\xi_s) ds + M_t,$$

where M_t is an (\mathcal{F}_t^k) -martingale. Since η_k is an a.s. finite (\mathcal{F}_t^k) -stopping time and $|M_{t \wedge \eta_k}|$ is bounded by a constant, the optional sampling theorem implies

$$Ef(\xi_{\eta_k}) = Ef(\xi_0) + \frac{1}{2}E \int_0^{\eta_k} \Delta f(\xi_s) ds, \quad f \in C^2([0, 1]^k), \quad (7)$$

where

$$\begin{aligned} E \int_0^{\eta_k} |\Delta f(\xi_s)| ds &\leq E \int_0^{\eta_k} \sup_{y \in [0, 1]^k} |\Delta f(y)| ds \\ &= \sup_{y \in [0, 1]^k} |\Delta f(y)| E\eta_k < \infty. \end{aligned}$$

Hence, by Fubini's theorem,

$$E \int_0^{\eta_k} \Delta f(\xi_s) ds = \int_0^\infty E(\Delta f(\xi_s); \eta_k > s) ds. \quad (8)$$

It is well known [12] that the transient probability density of Brownian motion, with initial state $\xi_0 = u \in (0, 1)$ and forced to remain in the unit interval, equals

$$P_u(\xi_t^i \in dx, 0 < \xi_s^i < 1, 0 \leq s \leq t) = 2 \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t / 2} \sin n \pi u_i \sin n \pi x dx.$$

Hence, writing $|n|^2 = n_1^2 + \dots + n_k^2$,

$$\begin{aligned} P_u(\xi_t^1 \in dx_1, \dots, \xi_t^k \in dx_k, \eta_k > t) \\ = 2^k \sum_{n_1=1}^{\infty} \dots \sum_{n_k=1}^{\infty} e^{-|n|^2 \pi^2 t / 2} \prod_{i=1}^k \sin n_i \pi u_i \prod_{j=1}^k \sin n_j \pi x_j dx_j. \end{aligned}$$

It follows by averaging over a general initial distribution ξ_0 that

$$\begin{aligned} P(\xi_t^1 \in dx_1, \dots, \xi_t^k \in dx_k, \eta_k > t) \\ = 2^k \sum_{n_1=1}^{\infty} \dots \sum_{n_k=1}^{\infty} e^{-|n|^2 \pi^2 t / 2} E \left(\prod_{i=1}^k \sin n_i \pi \xi_0^i \right) \prod_{j=1}^k \sin n_j \pi x_j dx_j \end{aligned}$$

This shows that the integrand in (8) is given by

$$\begin{aligned} E(\Delta f(\xi_t), \eta_k > t) \\ = 2^k \int_0^1 \dots \int_0^1 \Delta f(x) \sum_{n_1=1}^{\infty} \dots \sum_{n_k=1}^{\infty} e^{-|n|^2 \pi^2 t / 2} E \left(\prod_{i=1}^k \sin n_i \pi \xi_0^i \right) \prod_{j=1}^k \sin n_j \pi x_j dx_j. \end{aligned} \quad (9)$$

A further application of Fubini's theorem yields

$$\begin{aligned} E \int_0^{\eta_k} \Delta f(\xi_s) ds &= 2^k \int_0^1 \dots \int_0^1 \Delta f(x) \int_0^\infty \sum_{n_1=1}^{\infty} \dots \sum_{n_k=1}^{\infty} e^{-|n|^2 \pi^2 t / 2} \\ &\quad \times E \left(\prod_{i=1}^k \sin n_i \pi \xi_0^i \right) \prod_{j=1}^k \sin n_j \pi x_j dt dx_j. \end{aligned} \quad (10)$$

Summing up, relation (7) takes the form

$$Ef(\xi_{\eta_k}) = Ef(\xi_0) + \frac{1}{2} \int_{[0,1]^k} \Delta f(x) g_k(x) dx, \quad f \in C^2([0,1]^k), \quad (11)$$

where the density function $g_k \geq 0$ in

$$g_k(x) dx = \int_0^\infty P(\xi_t^1 \in dx_1, \dots, \xi_t^k \in dx_k, \eta_k > t) dt$$

is

$$g_k(x) = 2^k \int_0^\infty \sum_{n_1=1}^\infty \dots \sum_{n_k=1}^\infty e^{-|n|^2 \pi^2 t/2} E\left(\prod_{i=1}^k \sin n_i \pi \xi_0^i\right) \prod_{j=1}^k \sin n_j \pi x_j dt \quad (12)$$

Restricting further to functions ϕ which satisfy the boundary condition (6), we obtain

$$E\phi(\xi_0) + \frac{1}{2} \int_{[0,1]^k} \Delta \phi(x) g_k(x) dx = 0, \quad \phi \in C_0^2([0,1]^k). \quad (13)$$

3.5 Three technical lemmas

The following three technical lemmas form the basis for the proof of Theorem 1. The first lemma considers the expected value $E\phi(\xi_0)$ with the initial distribution of ξ_0 given by the generalized density function h_k defined in (4). In the second lemma, we apply the representation of $E\phi(\xi_0)$ found in Lemma 1 to determine the function $g_k(x)$ which appeared in (12) in a surprisingly simple and convenient form. Finally, the third lemma gives the distribution of the Brownian sensor locations at the time S_k (given the particular choice of initial distribution).

Lemma 1. *Assume that ξ_0 has the distribution given by the generalized density function h_k in (4). Define a function \tilde{g}_k on $[0,1]^k$ by*

$$\tilde{g}_k(x) = \min_{1 \leq i \leq k} x_i (1 - \max_{1 \leq i \leq k} x_i) = \sum_{a \in I} x_{a_1} (1 - x_{a_k}) 1_{\{0 < x_{a_1} < \dots < x_{a_k} < 1\}}, \quad (14)$$

where $a = (a_1, \dots, a_k)$ and I is the set consisting of the $k!$ different permutations of $(1, \dots, k)$. Then

$$\int_{[0,1]^k} \phi(x) h_k(dx) + \frac{1}{2} \int_{[0,1]^k} \Delta \phi(x) \tilde{g}_k(x) dx = 0, \quad \phi \in C_0^2([0,1]^k).$$

Proof. Let

$$\tilde{g}_k^a(x) = x_{a_1} (1 - x_{a_k}) 1_{\{0 < x_{a_1} < \dots < x_{a_k} < 1\}}$$

and

$$h_k^a(x) = \frac{1}{2} \left(x_{a_1} \delta(x_{a_{k-1}} - x_{a_k}) 1_{\{0 < x_{a_1} < \dots < x_{a_{k-1}} < 1\}} \right. \\ \left. + (1 - x_{a_k}) \delta(x_{a_1} - x_{a_2}) 1_{\{0 < x_{a_2} < \dots < x_{a_k} < 1\}} \right),$$

so

$$\tilde{g}_k(x) = \sum_{a \in I} \tilde{g}_k^a(x) \quad \text{and} \quad h_k(x) = \sum_{a \in I} h_k^a(x).$$

Now,

$$\int_{[0,1]^k} \Delta\phi(x) \tilde{g}_k(x) dx = \sum_{a \in I} \int_{[0,1]^k} \sum_{i=1}^k \phi_i''(x) \tilde{g}_k^a(x) dx, \quad \phi_i''(x) = \frac{\partial^2 \phi(x)}{\partial x_i^2}.$$

Here, for each fixed permutation a

$$\int_{[0,1]^k} \sum_{i=1}^k \phi_i''(x) \tilde{g}_k^a(x) dx = \sum_{i=1}^k \int_0^1 dx_{a_k} \cdots \int_0^{x_{a_2}} dx_{a_1} \phi_{a_i}''(x) x_{a_1} (1 - x_{a_k}) =: RS.$$

Change the order of integration on the right hand side to continue with

$$\begin{aligned} RS &= \sum_{i=2}^{k-1} \int_0^1 dx_{a_k} \cdots \int_0^{x_{a_{i+1}}} dx_{a_{i-1}} \cdots \int_0^{x_{a_2}} dx_{a_1} \\ &\quad \times x_{a_1} (1 - x_{a_k}) \int_{x_{a_{i-1}}}^{x_{a_{i+1}}} dx_{a_i} \phi_{a_i}''(x) \\ &\quad + \int_0^1 dx_{a_k} \cdots \int_0^{x_{a_3}} dx_{a_2} (1 - x_{a_k}) \int_0^{x_{a_2}} dx_{a_1} \phi_{a_1}''(x) x_{a_1} \\ &\quad + \int_0^1 dx_{a_{k-1}} \cdots \int_0^{x_{a_2}} dx_{a_1} x_{a_1} \int_{x_{a_{k-1}}}^1 dx_{a_k} \phi_{a_k}''(x) (1 - x_{a_k}) \end{aligned}$$

and carry out the partial integrations to obtain

$$\begin{aligned} RS &= \sum_{i=2}^{k-1} \int_0^1 dx_{a_k} \cdots \int_0^{x_{a_{i+1}}} dx_{a_{i-1}} \cdots \int_0^{x_{a_2}} dx_{a_1} x_{a_1} (1 - x_{a_k}) \\ &\quad \times (\phi_{a_i}'(x)|_{x_{a_i}=x_{a_{i+1}}} - \phi_{a_i}'(x)|_{x_{a_i}=x_{a_{i-1}}}) \\ &\quad + \int_0^1 dx_{a_k} \cdots \int_0^{x_{a_3}} dx_{a_2} (1 - x_{a_k}) \\ &\quad \times \left([\phi_{a_1}'(x) x_{a_1}]_0^{x_{a_2}} - \int_0^{x_{a_2}} \phi_{a_1}'(x) dx_{a_1} \right) \\ &\quad + \int_0^1 dx_{a_{k-1}} \cdots \int_0^{x_{a_2}} dx_{a_1} x_{a_1} \\ &\quad \times \left([\phi_{a_k}'(x) (1 - x_{a_k})]_{x_{a_{k-1}}}^1 - \int_{x_{a_{k-1}}}^1 \phi_{a_k}'(x) (-1) dx_{a_k} \right). \end{aligned}$$

To reveal the structure in these expressions we denote, for $2 \leq i \leq k-1$,

$$\begin{aligned} A_i^a &= \int_0^1 dx_{a_k} \cdots \int_0^{x_{a_{i+1}}} dx_{a_{i-1}} \cdots \int_0^{x_{a_2}} dx_{a_1} x_{a_1} (1 - x_{a_k}) \phi_{a_i}'(x)|_{x_{a_i}=x_{a_{i+1}}}, \\ B_i^a &= \int_0^1 dx_{a_k} \cdots \int_0^{x_{a_{i+1}}} dx_{a_{i-1}} \cdots \int_0^{x_{a_2}} dx_{a_1} x_{a_1} (1 - x_{a_k}) \phi_{a_i}'(x)|_{x_{a_i}=x_{a_{i-1}}}, \end{aligned}$$

and, additionally,

$$\begin{aligned} A_1^a &= \int_0^1 dx_{a_k} \cdots \int_0^{x_{a_3}} dx_{a_2} x_{a_2} (1 - x_{a_k}) \phi'_{a_1}(x) \Big|_{x_{a_1}=x_{a_2}}, \\ B_k^a &= \int_0^1 dx_{a_{k-1}} \cdots \int_0^{x_{a_2}} dx_{a_1} x_{a_1} (1 - x_{a_{k-1}}) \phi'_{a_k}(x) \Big|_{x_{a_k}=x_{a_{k-1}}}. \end{aligned}$$

Since $\phi_{a_k}(x) \Big|_{x_{a_k}=1} = 0$ and $\phi(x) \Big|_{x_{a_1}=0} = 0$, we can write first

$$\begin{aligned} RS &= \sum_{i=2}^{k-1} (A_i^a - B_i^a) \\ &+ \int_0^1 dx_{a_k} \cdots \int_0^{x_{a_3}} dx_{a_2} (1 - x_{a_k}) \left(\phi'_{a_1}(x) \Big|_{x_{a_1}=x_{a_2}} x_{a_2} - \phi(x) \Big|_{x_{a_1}=x_{a_2}} \right) \\ &+ \int_0^1 dx_{a_{k-1}} \cdots \int_0^{x_{a_2}} dx_{a_1} x_{a_1} \\ &\quad \times \left(-\phi'_{a_k}(x) \Big|_{x_{a_k}=x_{a_{k-1}}} (1 - x_{a_{k-1}}) - \phi(x) \Big|_{x_{a_k}=x_{a_{k-1}}} \right) \end{aligned}$$

and then, splitting the last two terms, we get

$$\begin{aligned} RS &= \sum_{i=1}^{k-1} A_i^a - \sum_{i=2}^k B_i^a \\ &- \int_0^1 dx_{a_k} \cdots \int_0^{x_{a_3}} dx_{a_2} (1 - x_{a_k}) \phi(x_{a_2}, x_{a_2}, \dots, x_{a_k}) \\ &- \int_0^1 dx_{a_{k-1}} \cdots \int_0^{x_{a_2}} dx_{a_1} x_{a_1} \phi(x_{a_1}, \dots, x_{a_{k-1}}, x_{a_{k-1}}). \end{aligned}$$

By identifying the two remaining integral terms in the previous expression as an integral with respect to $h_k^a(dx)$, we obtain

$$RS = \sum_{i=1}^{k-1} A_i^a - \sum_{i=2}^k B_i^a - 2 \int_{[0,1]^k} \phi(x) h_k^a(dx).$$

Thus

$$\begin{aligned} \int_{[0,1]^k} \Delta \phi(x) \tilde{g}_k(x) dx &= \sum_{a \in I} \left(\sum_{i=1}^{k-1} A_i^a - \sum_{i=2}^k B_i^a - 2 \int_{[0,1]^k} \phi(x) h_k^a(dx) \right) \\ &= \sum_{i=1}^{k-1} \sum_{a \in I} (A_i^a - B_{i+1}^a) - 2 \int_{[0,1]^k} \phi(x) h_k(dx), \end{aligned}$$

and it remains to show that

$$\sum_{i=1}^{k-1} \sum_{a \in I} (A_i^a - B_{i+1}^a) = 0. \quad (15)$$

Let

$$I_i^+ = \{a \in I : a_i < a_{i+1}\} \quad I_i^- = \{a \in I : a_i > a_{i+1}\}, \quad 1 \leq i \leq k-1. \quad (16)$$

Then $I_i^+ \cap I_i^- = \emptyset$ and $|I_i^\pm| = k!/2$. For each $1 \leq i \leq k-1$, this yields a partition $I = I_i^+ \cup I_i^-$ of I into disjoint sets where each permutation $a^+ \in I_i^+$ has a dual element $a^- \in I_i^-$ such that $a_i^+ = a_{i+1}^- < a_{i+1}^+ = a_i^-$. Thus, a^+ and a^- coincide at all elements except i and $i+1$, where a^+ increases and a^- is the reverse permutation with decreasing indices from i to $i+1$. By listing all such pairs of permutations it follows that

$$\begin{aligned} \sum_{a \in I} (A_i^{a^+} - B_{i+1}^{a^+}) &= \sum_{a^+ \in I_i^+} (A_i^{a^+} + A_i^{a^-} - B_{i+1}^{a^+} - B_{i+1}^{a^-}) \\ &= \sum_{a^+ \in I_i^+} (A_i^{a^+} - B_{i+1}^{a^-}) + (A_i^{a^-} - B_{i+1}^{a^+}), \quad 1 \leq i \leq k-1. \end{aligned}$$

Since $A_i^{a^+} - B_{i+1}^{a^-} = 0$ and $A_i^{a^-} - B_{i+1}^{a^+} = 0$, all terms in the last sum vanishes. Hence the sum in (15) vanishes, which concludes the proof of the lemma. \square

Lemma 2. *Assume that ξ_0 has the distribution given by the generalized density function h_k . Then the functions g_k and \tilde{g}_k , defined in (12) and (14), are equal, i.e.,*

$$g_k(x) = \sum_{a \in I} x_{a_1} (1 - x_{a_k}) 1_{\{0 < x_{a_1} < \dots < x_{a_k} < 1\}}, \quad x \in [0, 1]^k,$$

where $a = (a_1, \dots, a_k)$ and I are as in Lemma 1.

Proof. Let $C_c^\infty((0, 1)^k)$ denote the class of functions which have continuous derivatives of all orders and compact support in the open unit cube $(0, 1)^k$. Let $\mathcal{D}'((0, 1)^k)$ be the dual vector space of distributions on $(0, 1)^k$. We write $\langle u, \phi \rangle$ for the action of an element $u \in \mathcal{D}'((0, 1)^k)$ on $\phi \in C_c^\infty((0, 1)^k)$.

The nonnegative functions g_k and \tilde{g}_k are integrable in $(0, 1)^k$. Hence we may associate with $g_k - \tilde{g}_k$ the distribution $u_g \in \mathcal{D}'((0, 1)^k)$ defined by

$$\langle u_g, \phi \rangle = \int_{(0,1)^k} (g_k(x) - \tilde{g}_k(x)) \phi(x) dx, \quad \phi \in C_c^\infty((0, 1)^k).$$

The distribution $\Delta u_g = \sum_{i=1}^k \frac{\partial^2 u_g}{\partial x_i^2}$ is then defined by

$$\langle \Delta u_g, \phi \rangle = \int_{(0,1)^k} (g_k(x) - \tilde{g}_k(x)) \Delta \phi(x) dx, \quad \phi \in C_c^\infty((0, 1)^k).$$

By combining (13) and Lemma 1,

$$\int_{[0,1]^k} \Delta \phi(x) g_k(x) dx = \int_{[0,1]^k} \Delta \phi(x) \tilde{g}_k(x) dx, \quad \phi \in C_c^2([0, 1]^k).$$

In particular,

$$\langle \Delta u_g, \phi \rangle = \int_{[0,1]^k} (g_k(x) - \tilde{g}_k(x)) \Delta \phi(x) dx = 0, \quad \phi \in C_c^\infty((0,1)^k).$$

Thus, u_g is a distribution that satisfies $\Delta u_g = 0$. However, every solution $u \in \mathcal{D}'((0,1)^k)$ of $\Delta u = 0$ belongs to $C_c^\infty((0,1)^k)$ by the elliptic regularity theorem, see e.g. [5, Theorem 8.6.1] or [13, Theorem 8.12, Corollary]. In particular $u_g \in C_c^\infty((0,1)^k)$. But this is to say that $g_k - \tilde{g}_k \in C_c^\infty((0,1)^k)$. Hence

$$\langle \Delta u_g, \phi \rangle = \int_{[0,1]^k} \Delta(g_k(x) - \tilde{g}_k(x)) \phi(x) dx = 0, \quad \phi \in C_c^\infty((0,1)^k)$$

and so $\Delta(g_k - \tilde{g}_k) = 0$, [8, Theorem 1.2.4]. Since $g_k(x) = \tilde{g}_k(x) = 0$ on the boundary $x \in \partial(0,1)^k$, it follows from the weak maximum principle, see e.g. [3, Section 6.4], that $g_k(x) - \tilde{g}_k(x) = 0$ for every $x \in [0,1]^k$. □

Lemma 3. *For the normalized growth model on level $k \geq 2$ with ordered initial distribution that of $(X_{k,1}, \dots, X_{k,k})$ in Theorem 1, the k particles at the time of growth η_k are distributed with equal probabilities either as $(0, U_{(1)}, \dots, U_{(k-1)})$ or as $(U_{(2)}, \dots, U_{(k)}, 1)$, with $U_{(1)}, \dots, U_{(k)}$ as in Theorem 1. Equivalently, if the initial distribution for ξ_0 on $[0,1]^k$ is h_k , then the density function for ξ_{η_k} equals*

$$\begin{aligned} q_k(x) &= \frac{1}{2} \sum_{a \in I} (x_{a_1} \delta(x_{a_k} - 1) 1_{\{0 < x_{a_1} < \dots < x_{a_{k-1}} < 1\}} \\ &\quad + (1 - x_{a_k}) \delta(x_{a_1}) 1_{\{0 < x_{a_2} < \dots < x_{a_k} < 1\}}), \end{aligned} \quad (17)$$

where $a = (a_1, \dots, a_k)$ and I are as in Lemma 1.

Proof. Applying (11) with $f \in C^2([0,1]^k)$, initial distribution h_k according to (4), and g_k as in Lemma 2,

$$\begin{aligned} Ef(\xi_{\eta_k}) &= \int_{[0,1]^k} f(x) h_k(dx) + \frac{1}{2} \int_{[0,1]^k} \Delta f(x) g_k(x) dx \\ &= \sum_{a \in I} \int_{S^a} f(x) h_k^a(dx) + \frac{1}{2} \sum_{a \in I} \int_{S^a} \Delta f(x) g_k^a(x) dx, \end{aligned}$$

where $S^a = \{0 \leq x_{a_1} \leq \dots \leq x_{a_k} \leq 1\}$. Let ∂S^a denote the boundary of S^a . Since $\Delta g_k = 0$ almost surely on each S^a , applying Green's identity to each term in the last sum gives

$$\begin{aligned} Ef(\xi_{\eta_k}) &= \sum_{a \in I} \int_{S^a} f(x) h_k^a(dx) + \frac{1}{2} \sum_{a \in I} \int_{S^a} f(x) \Delta g_k(x) dx \\ &\quad + \frac{1}{2} \sum_{a \in I} \int_{\partial S^a} \left(g_k^a \frac{df}{dn} - f \frac{dg_k^a}{dn} \right) d\sigma \\ &= \sum_{a \in I} \int_{S^a} f(x) h_k^a(dx) + \frac{1}{2} \sum_{a \in I} \int_{\partial S^a} g_k^a \frac{df}{dn} d\sigma - \frac{1}{2} \sum_{a \in I} \int_{\partial S^a} f \frac{dg_k^a}{dn} d\sigma, \end{aligned} \quad (18)$$

where $\frac{d}{dn}$ indicates differentiation in the direction of the unit normal pointing out from ∂S^a and $d\sigma$ denotes integration of the surface elements. To analyze the boundary integral terms in the previous expression, we introduce the hyperplanes

$$\begin{aligned}\partial S_1^a &= \{x \in \partial S^a : 0 = x_{a_1} \leq \cdots \leq x_{a_k} \leq 1\}, \\ \partial S_i^a &= \{x \in \partial S^a : 0 \leq x_{a_1} \leq \cdots \leq x_{a_{i-1}} = x_{a_i} \leq \cdots \leq x_{a_k} \leq 1\}, \quad 2 \leq i \leq k, \\ \partial S_{k+1}^a &= \{x \in \partial S^a : 0 \leq x_{a_1} \leq \cdots \leq x_{a_k} = 1\},\end{aligned}$$

which together yield ∂S^a . Let e_i , $i = 1, \dots, k$, denote the standard base vectors in \mathbb{R}^k and let N_i^a be the unit normal vectors on ∂S_i^a pointing outwards from S^a , that is,

$$N_1^a = -e_{a_1}, \quad N_i^a = \frac{1}{\sqrt{2}}(e_{i-1} - e_i), \quad 2 \leq i \leq k, \quad N_{k+1}^a = e_k.$$

Then d/dn equals $\nabla \cdot N_i^a$ on ∂S_i^a , $1 \leq i \leq k+1$. Clearly, g_k^a vanishes on both ∂S_1^a and ∂S_{k+1}^a . Hence

$$\sum_{i=1}^{k+1} \int_{\partial S_i^a} g_k^a \frac{df}{dn} d\sigma = \frac{1}{\sqrt{2}} \sum_{i=2}^k \int_{\partial S_i^a} x_{a_1}(1 - x_{a_k}) \left(\frac{\partial f}{\partial x_{a_{i-1}}} - \frac{\partial f}{\partial x_{a_i}} \right) d\sigma$$

so that

$$\int_{\partial S^a} g_k^a \frac{df}{dn} d\sigma = \frac{1}{\sqrt{2}} \sum_{i=1}^{k-1} (A_i^a - B_i^a),$$

where

$$\begin{aligned}A_i^a &= \int_{\partial S_{i+1}^a} x_{a_1}(1 - x_{a_k}) \frac{\partial f}{\partial x_{a_i}} d\sigma, \\ B_i^a &= \int_{\partial S_{i+1}^a} x_{a_1}(1 - x_{a_k}) \frac{\partial f}{\partial x_{a_{i+1}}} d\sigma, \quad 1 \leq i \leq k-1.\end{aligned}$$

To compute the sum $\sum_a \int_{\partial S^a} g_k^a \frac{df}{dn} d\sigma$ over permutations a , we proceed similarly as in the proof of Lemma 1. Recall from (16) that I_i^+ , $1 \leq i \leq k-1$, denotes the set of permutations which increases from element i to $i+1$. If $a^+ \in I_i^+$, then a^- is the corresponding permutation where elements i and $i+1$ have switched. It follows

$$\begin{aligned}\sum_{a \in I} \int_{\partial S^a} g_k^a \frac{df}{dn} d\sigma &= \frac{1}{\sqrt{2}} \sum_{i=1}^{k-1} \sum_{a \in I} (A_i^a - B_i^a) \\ &= \frac{1}{\sqrt{2}} \sum_{i=1}^{k-1} \sum_{a^+ \in I_i^+} (A_i^{a^+} + A_i^{a^-} - B_i^{a^+} - B_i^{a^-}) = 0, \quad (19)\end{aligned}$$

since the terms cancel pairwise according to $A_i^{a^+} = B_i^{a^-}$ and $A_i^{a^-} = B_i^{a^+}$ for each $1 \leq i \leq k-1$ and $a \in I$. Observe that this also holds for $i = 1$ and $i = k-1$ because on S_2^a we have $a_1 = a_2$ and on S_k^a it holds $a_{k-1} = a_k$.

We turn to the last term in (18) by considering

$$\begin{aligned} \int_{\partial S^a} f \frac{dg_k^a}{dn} d\sigma &= \int_{\partial S_1^a} f(x) \left(-\frac{\partial g}{\partial x_{a_1}} \right) d\sigma + \frac{1}{\sqrt{2}} \int_{\partial S_2^a} f(x) \left(\frac{\partial g_k^a}{\partial x_{a_1}} - \frac{\partial g_k^a}{\partial x_{a_2}} \right) d\sigma \\ &+ \frac{1}{\sqrt{2}} \sum_{i=3}^{k-1} \int_{\partial S_i^a} f(x) \left(\frac{\partial g_k^a}{\partial x_{a_{i-1}}} - \frac{\partial g_k^a}{\partial x_{a_i}} \right) d\sigma \\ &+ \frac{1}{\sqrt{2}} \int_{\partial S_k^a} f(x) \left(\frac{\partial g_k^a}{\partial x_{a_{k-1}}} - \frac{\partial g_k^a}{\partial x_{a_k}} \right) d\sigma + \int_{\partial S_{k+1}^a} f(x) \frac{\partial g_k^a}{\partial x_{a_k}} d\sigma, \end{aligned}$$

where

$$\frac{\partial g_k^a}{\partial x_{a_i}} = 0, \quad 2 \leq i \leq k-1.$$

Thus, we have

$$\begin{aligned} \int_{\partial S^a} f \frac{dg_k^a}{dn} d\sigma &= - \int_{\partial S_1^a} f(x)(1-x_{a_k}) d\sigma + \frac{1}{\sqrt{2}} \int_{\partial S_2^a} f(x)(1-x_{a_k}) d\sigma \\ &+ \frac{1}{\sqrt{2}} \int_{\partial S_k^a} f(x)x_{a_1} d\sigma - \int_{\partial S_{k+1}^a} f(x)x_{a_1} d\sigma \\ &= - \int_{\partial S_1^a} f(x)(1-x_{a_k}) d\sigma - \int_{\partial S_{k+1}^a} f(x)x_{a_1} d\sigma \\ &+ 2 \int_{S^a} f(x) h_k^a(dx). \end{aligned}$$

Inserting this expression together with (19) in (18) yields

$$\begin{aligned} Ef(\xi_{\eta_k}) &= \frac{1}{2} \sum_{a \in I} \int_{\partial S_1^a} f(x)(1-x_{a_k}) d\sigma + \frac{1}{2} \sum_{a \in I} \int_{\partial S_{k+1}^a} f(x)x_{a_1} d\sigma \\ &= \frac{1}{2} \sum_{a \in I} \int_{[0,1]^{k-1}} f(0, x_{a_2}, \dots, x_{a_k})(1-x_{a_k}) 1_{\{0 \leq x_{a_2} \leq \dots \leq x_{a_k} \leq 1\}} dx \\ &+ \frac{1}{2} \sum_{a \in I} \int_{[0,1]^{k-1}} f(x_{a_1}, \dots, x_{a_{k-1}}, 1)x_{a_1} 1_{\{0 \leq x_{a_1} \leq \dots \leq x_{a_{k-1}} \leq 1\}} dx. \end{aligned}$$

If we interpret integration with respect to the generalized density function $q_k(x)$ (defined in (17)) in the same way as in (5), we see that,

$$Ef(\xi_{\eta_k}) = \int_{[0,1]} f(x) q_k(dx), \quad \phi \in C^2([0,1]^k).$$

Since $C^2([0,1]^k)$ is dense in $C([0,1]^k)$, to each function f in $C([0,1]^k)$ there exists a sequence f_1, f_2, \dots of functions in $C^2([0,1]^k)$ such that $f_n \rightarrow f$ uniformly. From

the continuity of the functions and the compactness of the domain it follows that there exists a constant $M < \infty$ such that $|f|, |f_1|, |f_2|, \dots$ are all bounded by M . Using dominated convergence twice,

$$\begin{aligned} Ef(\xi_{\eta_k}) &= E \lim_{n \rightarrow \infty} f_n(\xi_{\eta_k}) = \lim_{n \rightarrow \infty} Ef_n(\xi_{\eta_k}) = \lim_{n \rightarrow \infty} \int_{[0,1]^k} f_n(x) q_k(dx) \\ &= \int_{[0,1]^k} \lim_{n \rightarrow \infty} f_n(x) q_k(dx) = \int_{[0,1]^k} f(x) q_k(dx), \quad f \in C([0,1]^k). \end{aligned}$$

Since $q_k(dx)$ has no support outside $[0,1]^k$, this relation extends to all continuous functions $C(\mathbb{R}^k)$ on \mathbb{R}^k , that is

$$\int_{\mathbb{R}^k} f(x) q_k(dx) = Ef(\xi_{\eta_k}), \quad f \in C(\mathbb{R}^k).$$

Hence we know the value of $Ef(\xi_{\eta_k})$ for each $f \in C(\mathbb{R}^k)$ which uniquely determines the distribution of ξ_{η_k} . In particular, $P(\xi_{\eta_k} \in A) = \int_A q_k(dx)$ for each measurable set $A \subset [0,1]^k$, see e.g. [4, Lemma 2, section V.1]. \square

3.6 Proof of Theorem 1

For $k = 1$ we have $\xi_0 \in \text{Re}(0,1)$ and by optional stopping,

$$E\tau_1 = \frac{1}{\lambda^2 \sigma^2} ER_2^2 E\eta_1 = \frac{1}{\lambda^2 \sigma^2} ER_2^2 (E\xi_\eta^2 - E\xi_0^2) = \frac{6}{\lambda^2 \sigma^2} \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{1}{\lambda^2 \sigma^2}.$$

The subsequent initial state for the level 2 process is that of two Brownian particles located on an interval of gamma distributed length $R_3(\lambda) \in \Gamma(3, \lambda)$, both particles in the same point chosen as a fraction of the interval according to either $\beta(1,2)$ or $\beta(2,1)$ with equal probabilities. Thus, the level 2 process starts off with initial locations $(X_{2,1}, X_{2,2})$ governed by the density $h_2(x_1, x_2)$ and, by construction, independent of $R_3(\lambda)$.

Consider now level k and assume that the initial distribution has the desired distribution $(X_{k,1}, \dots, X_{k,k})$ stated in the theorem for relative sensor locations on the interval of length $R_{k+1}(\lambda)$. By Lemma 3, the corresponding distribution at the activation time $\tau_k = R_{k+1}^2 \eta_k / (\lambda \sigma)^2$ of the $(k+1)$ th sensor is given by (17), that is

$$(0, U_{(1)}, U_{(2)}, \dots, U_{(k-1)}) \text{ or } (U_{(2)}, U_{(3)}, \dots, U_{(k)}, 1),$$

with equal probability. Let $T_1, T_2, \dots, T_{k+1}, T_{k+2}$ be the times of the $k+2$ first occurrences in a Poisson process with intensity λ , then,

$$(U_{(2)}, U_{(3)}, \dots, U_{(k)}, 1) \stackrel{d}{=} \left(\frac{T_2}{T_{k+1}}, \frac{T_3}{T_{k+1}}, \dots, \frac{T_k}{T_{k+1}}, 1 \right).$$

After the new sensor is activated we extend the interval R_{k+1} to R_{k+2} by adding the exponentially distributed interval $V_{k+2} \stackrel{d}{=} T_{k+2} - T_{k+1}$. Thus the relative

positions of the sensors in the interval of length R_{k+2} have the same distribution as the relative positions of the occurrence times T_2, \dots, T_{k+1} on the interval of length T_{k+2} . This distribution is determined by

$$(U_{(2)}, U_{(3)}, \dots, U_{(k+1)}),$$

where $(U_{(1)}, U_{(2)}, \dots, U_{(k+1)})$ is the order statistic of $k+1$ independent uniformly distributed points on the unit interval. By symmetry it's seen that the relative positions of the k sensors on the interval R_{k+2} are

$$(U_{(1)}, U_{(2)}, \dots, U_{(k)}) \text{ or } (U_{(2)}, U_{(3)}, \dots, U_{(k+1)})$$

with equal probability.

The initial distribution at the next level $k+1$ is therefore composed of two states each with probability one half, one with two particles at the minimum position and one with two particles at the maximum, all in all described by a vector $(X_{k+1,1}, \dots, X_{k+1,k+1})$ with density $h_{k+1}(x_1, \dots, x_{k+1})$. These positions are independent of the new interval width $R_{k+2}(\lambda)$. It follows by induction that for each $k \geq 1$, the sensor locations are distributed as stated in the theorem.

In particular, for each k R_{k+1} and η_k are independent. Thus,

$$E\tau_k = \frac{1}{\lambda^2 \sigma^2} ER_{k+1}^2 E\eta_k.$$

It remains to compute the expected time $E\eta_k$. Note that the function $x \rightarrow |x|^2 = x_1^2 + \dots + x_k^2$ belongs to $C^2([0, 1]^k)$ and satisfies $\Delta(|x|^2) = 2k$. By (7),

$$E|\xi_{\eta_k}|^2 = E|\xi_0|^2 + kE\eta_k \tag{20}$$

where,

$$E|\xi_{\eta_k}|^2 = \int_{[0,1]^k} |x|^2 q_k(dx) \quad \text{and} \quad E|\xi_0|^2 = \int_{[0,1]^k} |x|^2 h_k(dx).$$

Hence,

$$\begin{aligned} kE\eta_k &= \int_{[0,1]^k} |x|^2 q_k(dx) - \int_{[0,1]^k} |x|^2 h_k(dx) \\ &= \sum_{a \in I} \frac{1}{2} \left(\int_0^1 dx_{a_{k-1}} \int_0^{x_{a_{k-1}}} dx_{a_{k-2}} \dots \int_0^{x_{a_2}} dx_{a_1} x_{a_1} (x_{a_1}^2 + \dots + x_{a_{k-1}}^2 + 1) \right. \\ &\quad + \int_0^1 dx_{a_k} \int_0^{x_{a_k}} dx_{a_{k-1}} \dots \int_0^{x_{a_3}} dx_{a_2} (1 - x_{a_k})(0 + x_{a_2}^2 + \dots + x_{a_k}^2) \\ &\quad - \int_0^1 dx_{a_{k-1}} \int_0^{x_{a_{k-1}}} dx_{a_{k-2}} \dots \int_0^{x_{a_2}} dx_{a_1} x_{a_1} (x_{a_1}^2 + \dots + x_{a_{k-2}}^2 + 2x_{a_{k-1}}^2) \\ &\quad \left. - \int_0^1 dx_{a_k} \int_0^{x_{a_k}} dx_{a_{k-1}} \dots \int_0^{x_{a_3}} dx_{a_2} (1 - x_{a_k})(2x_{a_2}^2 + x_{a_3}^2 + \dots + x_{a_k}^2) \right). \end{aligned}$$

By observing that the summand gives the same value for each $a \in I$, this yields,

$$\begin{aligned} kE\eta_k &= k! \frac{1}{2} \left(\int_0^1 dx_{a_{k-1}} \int_0^{x_{a_{k-1}}} dx_{a_{k-2}} \cdots \int_0^{x_{a_2}} dx_{a_1} x_{a_1} (1 - x_{a_{k-1}}^2) \right. \\ &\quad \left. + \int_0^1 dx_{a_k} \int_0^{x_{a_k}} dx_{a_{k-1}} \cdots \int_0^{x_{a_3}} dx_{a_2} (1 - x_{a_k})(-x_{a_2}^2) \right) \\ &= \frac{k}{(k+1)(k+2)}. \end{aligned}$$

Thus,

$$E\tau_k = \frac{1}{\lambda^2 \sigma^2} (k+1)(k+2) \cdot \frac{1}{(k+1)(k+2)} = \frac{1}{\lambda^2 \sigma^2}.$$

□

3.7 One-sided rate of growth

Enumerate the sensors according to their initial ordering $\dots, -2, -1, 0, 1, 2, \dots$ by setting index 0 to an arbitrary node. Denoting the initial position of sensor i by Y_i , we have $Y_i < Y_j$ whenever $i < j$. By interpreting the growth process as a branching Brownian motion where ancestors live forever and each branching event is a birth of an independent Brownian motion, shifted to the branch location, we can operate on the probability space $\Omega = \cdots \times \Omega_{-1} \times \Omega_0 \times \Omega_1 \times \cdots$, where

$$\Omega_i = \mathbb{R}_+ \times \{f : f \text{ continuous, } f(0) = 0\}.$$

On Ω we define a product measure $P = \cdots \times P_{-1} \times P_0 \times P_1 \times \cdots$ where $P_i = \mu_{\text{exp}} \times \mu_{\text{Gauss}}$, where μ_{exp} and μ_{Gauss} correspond to the exponential and Gaussian measures, respectively.

For each $\omega \in \Omega$ we define $R_t^i(\omega)$ and $L_t^i(\omega)$, respectively, to be the locations of the Brownian sensor furthest to the right and to the left at time t , if the original message was injected in sensor i at time $t = 0$. The time to activate the sensor in position Y_n if the branching started in Y_m is then given by

$$T_{m,n} = \begin{cases} \inf\{t : R_t^m \geq Y_n\}, & m \leq n, \\ \inf\{t : L_t^m \leq Y_n\}, & m \geq n. \end{cases}$$

Then $T_{k,m,n} = T_{k,n} - T_{k,m}$ determines the time difference between the activation of the sensors in Y_m and Y_n if the system is started in Y_k . Note that $T_{k,m,n}$ can be negative, unless $k < m < n$ or $n < m < k$.

The proof of Theorem 2 will be based on the subadditive ergodic theorem, see e.g. [11, Theorem 1.10] or [9, Theorem 10.22], which we quote for easy reference.

Theorem. *Let $\{T_{m,n}\}$ be a collection of random variables indexed by integers satisfying $0 \leq m < n$. Suppose $\{T_{m,n}\}$ has the following properties:*

(i) $T_{0,n} \leq T_{0,m} + T_{m,n}$.

(ii) For each n , $E|T_{0,n}| < \infty$ and $ET_{0,n} \geq cn$ for some constant $c > -\infty$.

(iii) The distribution of $\{T_{m,m+k} : k \geq 1\}$ does not depend on m .

(iv) For each $k \geq 1$, $\{T_{nk,(n+1)k} : n \geq 0\}$ is a stationary sequence.

Then:

(a) $\eta := \lim_{n \rightarrow \infty} ET_{0,n}/n = \inf_{n \geq 1} E(T_{0,n})/n$.

(b) $T := \lim_{n \rightarrow \infty} T_{0,n}/n$ exists a.s. and in L_1 .

(c) $ET = \eta$

Furthermore, if

(v) for each $k \geq 1$, $\{T_{nk,(n+1)k} : n \geq 0\}$ is ergodic,

then

(d) $T = \eta$ a.s.

To check property (ii) above we will use the following lemma.

Lemma 4. Consider k independent standard Brownian motions starting at the origin. Denote by $\tau_k(u, v)$ the first exit time from the interval $(-u, v)$ and by $\hat{\tau}_k(v)$ the first exit time from the interval $(-\infty, v)$, where u and v are positive. Then $E\tau_k(u, v) \leq \max(u, v)^2$ for all $k \geq 1$ and there exists a constant C such that $E\hat{\tau}_k(v) \leq Cv^2$ for all $k \geq 3$.

Proof. Let $\{\xi_t^i\}_{i=1}^k$ denote independent standard Brownian motions. The inequality $E\tau_k(u, v) \leq \max(u, v)^2$ follows from

$$\begin{aligned} E\tau_k(u, v) &= E \min_{1 \leq i \leq k} \inf\{t \geq 0 : \xi_t^i \notin (-u, v)\} \leq E \inf\{t \geq 0 : \xi_t^1 \notin (-u, v)\} \\ &\leq E \inf\{t \geq 0 : \xi_t^1 \notin (-\max(u, v), \max(u, v))\} = \max(u, v)^2, \end{aligned}$$

where the last equality follows from Dynkin's formula. To show that $E\hat{\tau}_k(v) \leq Cv^2$ for $k \geq 3$, define $\sigma_i(v) = \inf\{t \geq 0 : \xi_t^i \geq v\}$. Then $\{\sigma_i(v)\}_{i=1}^k$ is an i.i.d. sequence of random variables and the density for $\sigma_i(v)$ is,

$$f_{\sigma_i(v)}(t) = \frac{v}{\sqrt{2\pi}t^{3/2}} \exp\left(-\frac{v^2}{2t}\right),$$

which is a stable $1/2$ Lévy distribution. Hence $\sigma_i^p(v)$ has finite mean if $0 < p < 1/2$. The density of $\hat{\tau}_k = \min_{1 \leq i \leq k} \sigma_i(v)$ is

$$f_{\hat{\tau}_k}(t) = k[P(\sigma_1(v) \geq t)]^{k-1} f_{\sigma_1(v)}(t).$$

Markov's inequality applied to $P(\sigma_1(v) \geq t)$ gives, for $p \geq 0$,

$$\begin{aligned}
E\hat{\tau}_k(v) &\leq \int_0^\infty tk[E\sigma_1^p(v)t^{-p}]^{k-1}f_{\sigma_1(v)}(t)dt \\
&= k[E\sigma_1^p(v)]^{k-1}\int_0^\infty t^{1-p(k-1)}f_{\sigma_1(v)}(t)dt \\
&= k[E\sigma_1^p(v)]^{k-1}E\sigma_1^{1-p(k-1)}(v).
\end{aligned} \tag{21}$$

For any $q \in [0, 1/2)$ it holds that

$$\begin{aligned}
E\sigma_1^q(v) &= \int_0^\infty t^q \frac{v}{\sqrt{2\pi t^{3/2}}} \exp\left(-\frac{v^2}{2t}\right) dt \\
&= v^{2q} \int_0^\infty x^q \frac{1}{\sqrt{2\pi x^{3/2}}} \exp\left(-\frac{1}{2x}\right) dx \\
&= v^{2q} E\sigma_1^q(1) = C_q v^{2q},
\end{aligned}$$

where C_q is a finite constant. Inserting this expression in (21) yields,

$$E\hat{\tau}_k(v) \leq k[C_p v^{2p}]^{k-1} C_{1-p(k-1)} v^{2(1-p(k-1))} = C v^2,$$

where $C = k C_p^{k-1} C_{1-p(k-1)}$. Thus, if $p < 1/2$ and $1 - p(k - 1) < 1/2$, C is a finite constant. By choosing $p \in (1/4, 1/2)$, we conclude that C is finite for $k \geq 3$. \square

3.8 Proof of Theorem 2

First we show that the assumptions of the subadditive ergodic theorem hold true for the collection $\{T_{m,n}\}$ of random variables.

Let $m, n \in \mathbb{Z}_+$ with $0 < m < n$.

(i) For a given $\omega \in \Omega$, the Brownian motions and sensor locations are fixed, i.e., independently of the initial point and which Brownian motion that crossed first, the motion is the same (except the time shift). Thus for all $t \geq 0$,

$$\min_{0 \leq s \leq T_{0,m}+t} L_s^0 \leq \min_{0 \leq s \leq t} L_s^m \leq \max_{0 \leq s \leq t} R_s^m \leq \max_{0 \leq s \leq T_{0,m}+t} R_s^0$$

which implies that $T_{0,m,n} \leq T_{m,n}$. By definition, we have $T_{0,n} = T_{0,m} + T_{0,m,n}$, hence, $T_{0,n} \leq T_{0,m} + T_{m,n}$.

(ii) Consider the growth tree starting in Y_0 . In order to reach Y_1 , we have to first exit from $[Y_{-1}, Y_1]$, which takes time $\tau_1(|Y_0 - Y_{-1}|, |Y_1 - Y_0|)$. If we first reached Y_{-1} , then the next branching occurs when one of the two Brownian motions exits from $[Y_{-2}, Y_1]$. The time between $T_{0,-1}$ and the exit time from $[Y_{-2}, Y_1]$ is equal in distribution to $\tau_2(|Y_{-1} - Y_{-2}|, |Y_1 - Y_{-1}|)$. If the next sensor that activates is -2 , at time $T_{0,-2}$ we have three Brownian sensors; two at Y_{-2} and one somewhere in (Y_{-2}, Y_1) . Now if the latter Brownian sensor is moved to Y_{-2} , the time for any of these three sensors to exit from $(-\infty, Y_1]$ increases. This time is equal in

distribution to $\hat{\tau}_3(|Y_1 - Y_{-2}|)$. Since, $|Y_0 - Y_{-1}|, |Y_1 - Y_0|, |Y_{-1} - Y_{-2}| \in \Gamma(1, \lambda)$, $|Y_1 - Y_{-1}| \in \Gamma(2, \lambda)$ and $|Y_1 - Y_{-2}| \in \Gamma(3, \lambda)$ it follows that $E \max\{|Y_0 - Y_{-1}|, |Y_1 - Y_0|\}^2$, $E \max\{|Y_{-1} - Y_{-2}|, |Y_1 - Y_{-1}|\}^2$ and $E|Y_1 - Y_{-2}|^2$ are finite. Thus, by Lemma 4

$$ET_{0,1} \leq E \left[\tau_1(|Y_0 - Y_{-1}|, |Y_1 - Y_0|) + \tau_2(|Y_{-1} - Y_{-2}|, |Y_1 - Y_{-1}|) + \hat{\tau}_3(|Y_1 - Y_{-2}|) \right] < \infty.$$

From property (i) and the definition of $T_{m,n}$ it follows that

$$ET_{0,n} \leq nET_{0,1} < \infty, \quad n \geq 1.$$

Since $T_{0,n} \geq 0$, for all n , we can set $c = 0$.

(iii) & (iv) Follow from the definition of $T_{m,n}$.

(v) Consider the shift operator $S^k : \Omega \mapsto \Omega$ defined by

$$S^k \{(y_i, z_i)\}_i = \{(y_{i-k}, z_{i-k})\}_i.$$

Clearly

$$\{T_{nk, (n+1)k}(\omega)\}_n = \{T_{0,k}((S^k)^n \omega)\}_n.$$

The left and right shift operators are ergodic for any product measure. Thus, $(\Omega, \mathcal{F}, P, S^k)$ is an ergodic measure preserving dynamic system; correspondingly $\{T_{nk, (n+1)k}\}_n$ is an ergodic process and by the ergodic theorem we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} T_{ik, (i+1)k}(\omega) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} T_{0,k}((S^k)^i \omega) = ET_{0,k} \quad \text{a.s.}$$

Since assumptions (i) to (v) in the subadditive ergodic theorem are satisfied we conclude that (a) to (d) are true.

The definition of η and property (ii) above implies that, $\eta \leq ET_{0,1} < \infty$. Next, consider $\{T_{m,n}\}$ and $\{S_n\}$ defined on the same probability space. By Theorem 1 and the inequality

$$ES_n \leq ET_{0,n}, \quad n \geq 1,$$

it follows that

$$1/(\lambda\sigma)^2 = E \frac{S_n}{n} \leq \inf_{n \geq 1} E \frac{T_{0,n}}{n} = \eta.$$

Moreover,

$$\min(T_{0,n}, T_{0,-n}) \leq S_{2n} \leq \max(T_{0,n}, T_{0,-n}).$$

This yields,

$$\frac{1}{2} \min \left(\frac{T_{0,n}}{n}, \frac{T_{0,-n}}{n} \right) \leq \frac{S_{2n}}{2n} \leq \frac{1}{2} \max \left(\frac{T_{0,n}}{n}, \frac{T_{0,-n}}{n} \right), \quad (22)$$

where the expressions both on the left and the right side converge almost surely to $\eta/2$, and hence S_n/n converges almost surely to $\eta/2$. By properties (b) and (d) above,

$$\frac{T_{0,n}}{n} \xrightarrow{L_1} \eta \quad \text{and} \quad \frac{T_{0,-n}}{n} \xrightarrow{L_1} \eta,$$

hence

$$E \left| \max \left(\frac{T_{0,n}}{n}, \frac{T_{0,-n}}{n} \right) - \eta \right| \leq E \left| \frac{T_{0,n}}{n} - \eta \right| + E \left| \frac{T_{0,-n}}{n} - \eta \right| \rightarrow 0 \quad (23)$$

as $n \rightarrow \infty$, hence $\max(T_{0,n}/n, T_{0,-n}/n) \xrightarrow{L_1} \eta$. By substituting max with min above we see that also $\min(T_{0,n}/n, T_{0,-n}/n) \xrightarrow{L_1} \eta$. Subtracting $\eta/2$ from each term in (22) and considering absolute values and expectations, gives

$$E \left| \frac{S_{2n}}{2n} - \frac{\eta}{2} \right| \leq \frac{1}{2} E \left| \min \left(\frac{T_{0,n}}{n}, \frac{T_{0,-n}}{n} \right) - \eta \right| + \frac{1}{2} E \left| \max \left(\frac{T_{0,n}}{n}, \frac{T_{0,-n}}{n} \right) - \eta \right| \rightarrow 0$$

as $n \rightarrow \infty$, so $S_n/n \xrightarrow{L_1} \eta/2$. From the relation,

$$E \frac{S_n}{n} = \frac{1}{\lambda^2 \sigma^2}, \quad n \geq 1$$

and the well known fact that $S_n/n \xrightarrow{L_1} \eta/2$ implies $ES_n/n \rightarrow \eta/2$, we conclude that, $\eta/2$ is equal to $1/(\lambda\sigma)^2$. \square

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