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Abstract

We give a constructive characterization of morphisms between open sublocales of localic completions of locally compact metric (LCM) spaces, in terms of continuous functions. The category of open subspaces of LCM spaces is thereby shown to embed fully faithfully into the category of locales (or formal topologies).

Keywords: Locales, formal topologies, locally compact metric spaces

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1 Introduction

As is well-known the standard adjunction between locales and topological spaces [2] gives a fully faithful embedding of the category of sober spaces (which includes Hausdorff spaces) into locales

$$\Omega : \mathbf{Sob} \longrightarrow \mathbf{Locales}.$$

From a strict constructive point of view this embedding of categories is of little use since it cannot be proved, without employing axioms such as the Fan Theorem, that $\Omega(\mathbb{R})$ is isomorphic to the localic reals $\mathcal{R}$. For this reason one considers a different and more restricted embedding which sends $\mathbb{R}$ to a locale isomorphic to $\mathcal{R}$. In two previous papers [4, 5] we studied the embedding of locally compact metric spaces (in the sense of [1]) into locales or formal topologies. Vickers' construction of the localic completion of a metric space [7, 8] gives rise to a full and faithful functor $M : \mathbf{LCM} \longrightarrow \mathbf{FTop}$ from the category of locally compact metric (LCM) spaces in to the category of

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(inductively generated) formal topologies; see [4]. This means, in particular, that there is a bijection between the continuous maps $X \longrightarrow Y$ (i.e. locally uniformly continuous functions) and the continuous morphisms of formal topologies $M(X) \longrightarrow M(Y)$ (approximable mappings).

To study point-free versions of topological manifolds it is of interest to characterize the maps between open sublocales of formal Euclidean spaces

$$(\mathcal{R}^m)_{|U} \longrightarrow (\mathcal{R}^n)_{|V}.$$

We consider a more general version of this problem where the Euclidean spaces have been replaced by localic versions of LCM spaces. In this paper we study this correspondence when localic completions are restricted to open sublocales

$$M(X)_{|U} \longrightarrow M(Y)_{|V}. \tag{1}$$

This correspondence is not trivial from a constructive point of view. As shown in [5] the maps $M(X) \longrightarrow M(\mathbb{R})_{|(0,\infty)}$ correspond to continuous functions $X \longrightarrow \mathbb{R}$ that on each open ball has a positive uniform lower bound, rather than positive functions. Constructively, there is a distinction: Specker [6] gives a recursive example of a continuous positive function $[0,1] \longrightarrow \mathbb{R}$ that has no uniform positive lower bound. These considerations makes it clear that the set of maps $U_* \longrightarrow V_*$ between open subspaces of LCM spaces has to meet some extra conditions to be in 1-1 correspondence to maps in (1). In Section 2 we introduce and study the appropriate categories of metric spaces, called **OLCM** and **FLCM**. In Section 3 the open sublocales of $M(X)$ are studied. Section 4 establishes full and faithful functors **OLCM** $\longrightarrow$ **FLCM** $\longrightarrow$ **FTop**.

2 Open subspaces of LCM spaces

Bishop and Bridges [1] define a metric space X to be *locally compact* if it is inhabited and every bounded subspace is contained in a compact subspace. It follows that such a space X is complete (and separable). Below we define a category **OLCM** of open subspaces of locally compact metric (LCM) spaces. It is partly suggested by Definition 2.2.4 of [1], but its enunciation appears to be new.

The category of open subspaces of LCM spaces is given as follows. The objects are pairs (X, U) where $X = (X, d)$ is a LCM space and U is an open subset of X. A *continuous map* $f : (X, U) \longrightarrow (Y, V)$ between two objects is a function $f : U \longrightarrow V$ such that for any compact subset $K \Subset U$

(a) f is uniformly continuous on K,

(b) $f[K] \Subset V$.

Here $K \Subset U$ means that $K_r \subseteq U$ for some $r > 0$, where

$$K_r = \{x \in X : d(x, K) \leq r\}.$$

The distance $d(x, S)$ is well-defined whenever the set S is located and in particular when it is totally bounded. For $S \subseteq X$, let $\overline{S}$ be the closure of S in X, that is, the set of points in X that are limits of points in S. We have for located $S \subseteq X$:

$$S \Subset U \Longleftrightarrow \overline{S} \Subset U.$$

Moreover if S is totally bounded, then $\overline{S}$ is compact. Note that $f[S]$ is totally bounded whenever S is. Thus $\overline{f[K]}$ is compact, if K is compact. It follows that continuous maps are closed under composition, and form a category which we shall call **OLCM**.

Remarks 2.1 The category of locally compact metric spaces **LCM** may be regarded as a full subcategory of this category, given by the objects of the form (X, X) since the relation $S \Subset X$ is trivially true for any compact $S \subseteq X$.

The reciprocal map $(\cdot)^{-1} : (\mathbb{R}, \mathbb{R}_{\neq 0}) \longrightarrow (\mathbb{R}, \mathbb{R})$ is continuous: suppose $K \Subset \mathbb{R}_{\neq 0}$ is compact. Thus there is $r > 0$ so that

$$d(x, K) \leq r \Longrightarrow x \neq 0.$$

We have $K \subseteq (-\infty, -r] \cup [r, \infty)$. (Since if $y \in K$ and $y \in (-r, r)$, then we may take $x = 0$ and get $d(x, K) \leq r$. A contradiction.) The reciprocal map is uniformly continuous on $(-\infty, -r] \cup [r, \infty)$ and, a fortiori, on K. Thus (a) is valid and (b) is trivially true.

A continuous map $f : [0, 1] \longrightarrow (\mathbb{R}, \mathbb{R}_{\neq 0})$ is in particular required to satisfy $f([0, 1]) \Subset \mathbb{R}_{\neq 0}$. Thus $f([0, 1]) \subseteq (-\infty, -r] \cup [r, \infty)$ for some $r > 0$. In the recursive setting, this excludes the familiar counterexamples of Specker [6] and Julian and Richman [3] of a positive (uniformly) continuous function on [0,1] which has no positive uniform lower bound.

Next we prepare for the definition of a still wider category, **FLCM**, and for the definition of the localic completion. In a metric space $X = (X, d)$ a *formal open ball* is a symbol $\mathsf{b}(x, \delta)$, where $x \in X$ and δ is a positive rational number. These symbols are ordered by *formal inclusion* $\leq$

$$\mathsf{b}(x, \delta) \leq \mathsf{b}(y, \varepsilon) \Longleftrightarrow d(x, y) + \delta \leq \varepsilon.$$

Replacing $\leq$ by $<$ everywhere gives the corresponding definition of *strict formal inclusion* ($<$). The strict inclusion relation is extended to sets of symbols by saying that $U < V$ holds if, and only if, for each $a \in U$ there is $b \in V$ with $a < b$. The *radius* of a formal ball is $\rho(\mathsf{b}(x,\delta)) = \delta$. Each formal open ball represents a real open ball

$$\mathsf{b}(x,\delta)_* = B(x,\delta) = \{x \in X : d(x,y) < \delta\}.$$

This representation is of course not unique in general — consider $X = [0,1]$ and $\delta > 1$. For a set N of formal open balls, let $N_* = \bigcup\{b_* : b \in N\}$.

Lemma 2.2 *Let X be a locally compact metric space. For any formal balls $a < b$ of X, there is a compact subset $K \subseteq X$ with*

$$a_* \subseteq K \subseteq b_*.$$

Proof. Suppose $a < b$ are formal open balls where $a = \mathsf{b}(x,\delta)$. Then there is $\varepsilon > \delta$ with $\mathsf{b}(x,\varepsilon) < b$. It now suffices to find a compact $K \subseteq X$ with $B(x,\delta) \subseteq K \subseteq B(x,\varepsilon)$. Since X is locally compact there is a compact $L \supseteq B(x,\varepsilon)$. For $n \geq 1$, we let

$$N_n = \{x_{n,1}, \ldots, x_{n,m_n}\}$$

be a 2^{-n}-net for L. Pick α, β with $\delta < \alpha < \beta < \varepsilon$. Let $\lambda_n : \{1, \ldots, m_n\} \longrightarrow \{0,1\}$ be a function so that

(i) $\lambda_n(i) = 1$ implies $d(x, x_{n,i}) < \beta$

(ii) $\lambda_n(i) = 0$ implies $d(x, x_{n,i}) > \alpha$.

Let $N = \{x_{n,i} : n \geq 1, \lambda_n(i) = 1\}$ and let K be the set of limit points of this set. By definition $N \subseteq B(x,\beta)$ and hence $K \subseteq B(x,\varepsilon)$. Suppose now $z \in B(x,\delta)$ and $\rho > 0$. Let $n \geq 1$ be large enough that $2^{-n} \leq \min(\alpha - \delta, \rho)$. Then since $z \in L$ there is some index $i \in \{1, \ldots, m_n\}$ with $d(z, x_{n,i}) < 2^{-n}$. Then

$$d(x, x_{n,i}) \leq d(x,z) + d(z, x_{n,i}) < \delta + \alpha - \delta = \alpha.$$

Hence we must have $\lambda_n(i) = 1$, so $x_{n,i} \in N$. Since $\rho > 0$, this shows that $B(x,\delta) \subseteq K$. □

We may characterize the relation $\Subset$ in terms of formal inclusion.

Lemma 2.3 *Let X be a metric space. For a totally bounded $S \subseteq X$ and open set $U \subseteq X$: $S \Subset U$ if, and only if, there are formal open balls in X*

$$b_i < c_i \quad (i = 1, \ldots n),$$

with $S \subseteq \{b_1, \ldots, b_n\}_*$ and $\{c_1, \ldots, c_n\}_* \subseteq U$.

Proof. ($\Rightarrow$): Suppose that $S_r \subseteq U$ for some $r > 0$. Let $x_1, \ldots, x_n$ be a $r/2$-net for S. We may then take $b_i = \mathsf{b}(x_i, r/2)$ and $c_i = \mathsf{b}(x_i, r)$. We have $(c_i)_* = B(x_i, r) \subseteq U$, since if $d(y, x_i) < r$, then $d(y, S) \leq r$ and so $y \in U$.

($\Leftarrow$): Suppose that $b_i = \mathsf{b}(x_i, \delta_i)$ and $c_i = \mathsf{b}(y_i, \varepsilon_i)$, where $b_i < c_i$ and $S \subseteq \{b_1, \ldots, b_n\}_*$ and $\{c_1, \ldots, c_n\}_* \subseteq U$. Let $t > 0$ be so small that

$$d(x_i, y_i) + \delta_i + 2t < \varepsilon_i$$

for all $i = 1, \ldots, n$. Then we have $S_t \subseteq U$: Suppose $z \in S_t$. Thus in particular $d(z, u) \leq 2t$ for some $u \in S$. Hence $u \in (b_i)_*$ for some i, and

$$\begin{aligned} d(z, y_i) &\leq d(z, u) + d(u, x_i) + d(x_i, y_i) \\ &\leq 2t + \delta_i + d(x_i, y_i) < \varepsilon_i \end{aligned}$$

Thus $z \in (c_i)_* \subseteq U$. □

Motivated by this lemma we define yet another inclusion relation $<_*$. Let X be a metric space. For a subset $S \subseteq X$ and a set N of formal open balls of X define $S <_* N$ to hold if and only if there is a finitely enumerable (f.e.) set F of formal balls with $S \subseteq F_*$ and $F < N$. We pronounce $S <_* N$ as S *is formally well-included in* M. For any open $U \subseteq X$ define $H(U) = \{\mathsf{b}(x, \delta) : B(x, \delta) \subseteq U\}$. We have by Lemma 2.3 the following:

Corollary 2.4 *Let X be a metric space. For $S \subseteq X$ totally bounded and $U \subseteq X$ open that*

$$S \Subset U \Longleftrightarrow S <_* H(U). \ \square$$

Moreover for any set of formal open balls N of X and for any totally bounded $S \subseteq X$ we have

$$S <_* N \Longrightarrow S \Subset N_*. \tag{2}$$

Remark 2.5 The converse of (2) implies that $[0, 1]$ is point-wise covering compact, which is a non-constructive statement. Consider $X = S = [0, 1]$ and let N be a set of formal balls so that $N_* = X$. Clearly $S \Subset N_*$, and hence by the converse of (2) $S \subseteq F_*$ for some f.e. set $F < N$. Hence there is a f.e. set $G \subseteq N$ with $[0, 1] \subseteq G_*$. Thus $[0, 1]$ is point-wise covering compact, and we have a Brouwerian counter example to the reversal of the implication in (2).

2.1 Metric complements of located subsets

The function H is part of a Galois connection: for any set A of formal balls of X and any open $U \subseteq X$,

$$A_* \subseteq U \iff A \subseteq H(U).$$

Lemma 2.6 $H(U)_* = U$.

Proof. It is immediate that $H(U)_* \subseteq U$. Moreover if $x \in U$, then there is a positive rational δ so that $B(x, \delta) \subseteq U$ thus $\mathsf{b}(x, \delta) \in H(U)$ and so $x \in H(U)_*$. Hence also $H(U)_* = U$. □

It is direct from the Galois connection that $A \subseteq H(A_*)$. The reverse inclusion does not always hold. A set of formal balls A of X is *point-wise saturated* if $H(A_*) = A$. Any such set is saturated as well. By Lemma 2.6 it is clear that A is point-wise saturated if, and only if, $A = H(U)$ for some open $U \subseteq X$.

Lemma 2.7 *Let X be a metric space, let A be a set of formal balls of X and let $S \subseteq X$ be totally bounded. If A is point-wise saturated, then*

$$S <_* A \iff S \Subset A_*. \tag{3}$$

Proof. Suppose $A = H(A_*)$. Then by Lemma 2.4

$$S \Subset A_* \iff S <_* H(A_*) \iff S <_* A. \ \square$$

An important class of point-wise saturated sets arise in the following way. Let X be a metric space. Let $A \subseteq X$ be a located subset, and denote its *metric complement* by

$$C_A = \{x \in X \mid d(x, A) > 0\}.$$

The formal counterpart to C_A is

$$S_A =_{\text{def}} \{\mathsf{b}(x, \delta) \ : \ x \in X, \, d(x, A) \geq \delta\}.$$

It is follows from the theorem below that S_A is point-wise saturated.

Example 2.8 $C_{(-\infty,0]} = \{x \in \mathbb{R} \ : \ x > 0\}$ and $S_{(-\infty,0]} = \{\mathsf{b}(x, \delta) \ : \ x \in \mathbb{R}, \, x \geq \delta\}$.

Theorem 2.9 *For any located subset A of a metric space X, C_A is an open set and*

$$H(C_A) = S_A \qquad\qquad (S_A)_* = C_A.$$

Proof. C_A is open: if $x \in C_A$, then $d(x, A) > \delta$ for some $\delta > 0$. For $y \in B(x, \delta/2)$, we have either $d(y, A) > 0$ or $d(y, A) < \delta/2$. The latter is impossible, since it would imply that there is $z \in A$ with $d(y, z) < \delta/2$, and thus $d(x, z) < \delta$, which contradicts $d(x, A) > \delta$. Hence $B(x, \delta) \subseteq C_A$.

$H(C_A) = S_A$: First suppose $\mathsf{b}(x, \delta) \in H(C_A)$. Suppose that $d(x, A) < \delta$. Thus $d(x, y) < \delta$ for some $y \in A$. Thus $y \in H(C_A)$ by the assumption. Thereby $d(y, A) > 0$, which is impossible since $y \in A$. Hence $d(x, A) \geq \delta$, which means that $\mathsf{b}(x, \delta) \in S_A$. Conversely, assume $\mathsf{b}(x, \delta) \in S_A$. We show $B(x, \delta) \subseteq C_A$. Take y with $d(y, x) < \delta$ and ε such that $d(y, x) + \varepsilon < \delta$. Then either $d(y, A) > 0$ or $d(y, A) < \varepsilon$. The latter case is actually impossible, since then there would be $z \in A$ with $d(y, z) < \varepsilon$ and so

$$d(x, z) \leq d(y, x) + d(y, z) < d(y, x) + \varepsilon < \delta.$$

This contradicts $d(x, A) \geq \delta$. Hence $d(y, A) > 0$, i.e. $y \in C_A$.

$(S_A)_* = C_A$: By the first equation we have $H(C_A)_* = (S_A)_*$. Now Lemma 2.6 gives $C_A = H(C_A)_*$. □

2.2 The Category FLCM

We use the relation $<_*$ to define a new category **FLCM**. An object (X, P) is a locally compact metric space X together with a set P of formal balls of X. A morphism $f : (X, P) \longrightarrow (Y, Q)$ between two such objects is function $f : P_* \longrightarrow Q_*$ so that for any $a < P$ we have

(a) $f : a_* \longrightarrow Q_*$ is uniformly continuous,

(b) $f[a_*] <_* Q$.

Lemma 2.10 **FLCM** *is a category.*

Proof. The identity function id_{P_*} is uniformly continuous on any $a*$ with $a < P$. If $a < P$, then clearly $a_* <_* P$. Hence id_{P_*} is a morphism on (X, P). It now suffices to show that the morphisms are closed under composition. Suppose $f : (X, P) \longrightarrow (Y, Q)$ and $g : (Y, Q) \longrightarrow (Z, R)$ are morphisms. Let $a < P$ be some open ball. We have by (b) for f that there is a f.e. set of formal open balls $F = \{b_1, \dots, b_n\}$ with $f[a_*] \subseteq F_*$ and $F < Q$. Now

$$(g \circ f)[a_*] = g[f[a_*]] \subseteq g[(b_1)_*] \cup \cdots \cup g[(b_n)_*].$$

By (b) for g we get, for each $i = 1, \dots, n$, a f.e. set G_i with $g[(b_i)_*] \subseteq (G_i)_*$ and $G_i < R$. Thus $(g \circ f)[a_*] <_* R$ witnessed by the f.e. set $G = G_1 \cup \cdots \cup G_n$. Hence $g \circ f$ satisfies (b).

Notice also that g is uniformly continuous on each $(b_i)_*$. Hence it is uniformly continuous on the f.e. union F_* and therefore also on $f[a_*]$. Thus $g \circ f$ satisfies (a) as well. $\Box$

Theorem 2.11 *Let X and Y be LCM spaces. Let $U \subseteq X$ and $V \subseteq Y$ be open subsets. Then $f : (X, U) \longrightarrow (Y, V)$ is continuous in* **OLCM** *if, and only if, $f : (X, H(U)) \longrightarrow (Y, H(V))$ is a morphism in* **FLCM**.

Proof. ($\Rightarrow$) Suppose that $f : (X, U) \longrightarrow (Y, V)$ is continuous. Let $a < H(U)$. Thus there is a formal ball $c > a$ with $c_* \subseteq U$. Take a formal ball b with $a < b < c$. By Lemma 2.2 there is a compact $K \subseteq X$ with $a_* \subseteq K \subseteq b_*$. Lemma 2.3 then yields $K \Subset U$. By assumption f is uniformly continuous on K and hence also on a_*. This verifies condition (a). We have moreover $f[K] \Subset V$, and since $f[K]$ is totally bounded, there are by Lemma 2.3 formal balls $b_i < c_i$ $(i = 1, \ldots, n)$ with $f[K] \subseteq \{b_1, \ldots, b_n\}_*$ and $\{c_1, \ldots, c_n\}_* \subseteq V$. But this says that $f[K] <_* V$. Now $f[a_*] \subseteq f[K]$ so (b) is verified.

($\Leftarrow$) Suppose now that $f : (X, H(U)) \longrightarrow (Y, H(V))$ is a morphism in **FLCM**. Let K be a compact subset of X with $K \Subset U$. By Lemma 2.3, we find formal balls $b_i < c_i$ $(i = 1, \ldots, n)$ with $K \subseteq \{b_1, \ldots, b_n\}_*$ and $\{c_1, \ldots, c_n\}_* \subseteq U$. Thus $b_i < H(U)$ for each i. By assumption, f is uniformly continuous on each $(b_i)_*$. Hence f is uniformly continuous on $\{b_1, \ldots, b_n\}_*$, since n is finite. Thus f is uniformly continuous on K. By the assumption we have moreover that $f[(b_i)_*] <_* H(V)$ for each $i = 1, \ldots, n$. Thus there are f.e. sets of formal balls $F^i < G^i$ with $f[(b_i)_*] \subseteq (F^i)_*$ and $(G^i)_* \subseteq V$. Thus $f[K]_* \subseteq (\cup_{i=1}^n F_i)_*$ and $\cup_{i=1}^n F^i < \cup_{i=1}^n G^i$ both f.e. and $(\cup_{i=1}^n G^i)_* \subseteq V$. This shows that $f[K] \Subset V$. Hence $f : (X, U) \longrightarrow (Y, V)$ is continuous. $\Box$

Corollary 2.12 *The category* **OLCM** *is a full subcategory of* **FLCM** *via the embedding $(X, U) \mapsto (X, H(U))$.*

The structure of formal neighbourhoods P in the space (X, P) is essential. One might think that only the extent of the points matter, i.e. that for sets P and Q of formal balls of X, where X is a compact metric space,

$$P_* = Q_* \Longrightarrow (X, P) \cong (X, Q).$$

This is not so constructively. In fact, a special case of this implies that every compact metric space is covering compact, which is constructively false. See Proposition 4.6 below.

However, if P and Q are point-wise saturated and $P_* = Q_*$, then obviously $P = Q$.

3 Localic completion

Vickers [7, 8] gives a construction $M(X)$ of a formal topology from a metric space X — *the localic completion of* X — so that the canonical map $j_X : X \longrightarrow \mathrm{Pt}(M(X))$ is a metric completion of X. In particular j_X is a metric isomorphism, in case X is already complete.

Here the standard notion of formal topology based on preorders is employed. Basically it is a Grothendieck topology on a preorder instead of a category. A *formal topology* $\mathcal{X}$ is a triple $(X, \lhd, \leq)$ where X is the set of formal neighbourhoods, $\leq$ is a preorder relation on those and $\lhd$ is the formal cover relation, which is assumed to be set-presented. The relations satisfies the standard cover axioms. A continuous morphism between formal topologies $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is a relation $R \subseteq X \times Y$, which is an abstraction and a constructive version of the relation $U \subseteq f^{-1}[V]$ for a continuous function f. We shall here use the framework of [4]. There it was shown that the localic completion $M : \mathbf{LCM} \longrightarrow \mathbf{FTop}$ can be made into a full and faithful functor from the category of locally compact metric spaces to the category of formal topologies. Another result of [4] is that the inductively generated cover relation $\lhd$ of $M(X) = (M, \lhd, \leq)$ may be characterized by an explicitly defined relation $\lessdot$:

$$a \lhd U \Longleftrightarrow a \lessdot U. \tag{4}$$

This relation is defined as follows

$$a \lessdot U \Longleftrightarrow (\forall b, c \in M)[b < c < a \Rightarrow (\exists U_0 \in A(b,c))U_0 < U]. \tag{5}$$

We explain the notation. The formal neighbourhoods of M are the open formal balls $\mathsf{b}(x, \delta)$ of X and are ordered as described in the previous section. The set $A(b, c)$ consists of finitely enumerable sets $C \subseteq M$ such that

$$b \sqsubseteq C < c. \tag{6}$$

The relation $b \sqsubseteq C$ says that there is a number $\delta > 0$ (a "Lebesgue number") such that any formal ball of smaller radius than δ, which is included in b, is also included in some $q \in C$.

We also recall that the cover relation $M(X)$ is the smallest cover relation $\lhd$ satisfying, for all $p \in M$

(M1) $p \lhd \{q \in M : q < p\}$,

(M2) $p \lhd \{\mathsf{b}(x, \varepsilon) : x \in X\}$, for each rational $\varepsilon > 0$.

Notice that from (M2) follows by localization that

$$V \lhd V^{(\varepsilon)} = V \wedge \{\mathsf{b}(x,\varepsilon) : x \in X\}. \tag{7}$$

All balls in $V^{(\varepsilon)}$ have radius at most ε.

Lemma 3.1 *Let X be a LCM space. Then for $S \subseteq X$ and $U, V \subseteq M(X)$*

$$S <_* U \lhd V \Longrightarrow S <_* V.$$

Proof. Suppose $S <_* U \lhd V$. Then there is $F = \{b_1, \ldots, b_n\} \subseteq M(X)$ with $S_* \subseteq F_*$ and $F < U$. For each $i = 1, \ldots, n$, there is $a_i \in U$ with $b_i < a_i$. Pick c_i with $b_i < c_i < a_i$. Now $a_i \lhd V$, so $a_i \lessdot V$. Hence there is a f.e. $G_i \in A(b_i, c_i)$ with $G_i < V$. Thus $G = G_1 \cup \cdots \cup G_n$ is f.e. and $S_* \subseteq G_*$ and $G < V$. That is $S <_* V$. □

Two sets of neighbourhoods U and U' are said to be *equivalent* $(U \sim U')$ if $U \lhd U'$ and $U' \lhd U$.

Lemma 3.2 *Let X and Y be LCM spaces. Let $U, U' \subseteq M(X)$ and $V, V' \subseteq M(Y)$ and suppose that $U \sim U'$ and $V \sim V'$. Then $f : (X, U) \longrightarrow (Y, V)$ continuous implies that $f : (X, U') \longrightarrow (Y, V')$ is continuous.*

Proof. Suppose $f : (X, U) \longrightarrow (Y, V)$ continuous. Consider an arbitrary $p < U'$. Then there is $a \in U'$ and q with $p < q < a$. As $U \sim U'$, we have $a \lhd U$. Hence also $a \lessdot U$. Thus there is a f.e. $F = \{p_1, \ldots, p_n\} \in A(p, q)$ with $F < U$. By assumption f is uniformly continuous on each $(p_i)_*$. Hence f is uniformly continuous also on $p_* \subseteq F_*$. Moreover by assumption, $f[(p_i)_*] <_* V$. It follows that $f[F_*] <_* V$. By Lemma 3.1 and $V \lhd V'$, we get $f[F_*] <_* V'$. Since $f[p_*] \subseteq f[F_*]$ we get $f[p_*] <_* V'$. Thereby we have proved that $f : (X, U') \longrightarrow (Y, V')$ is continuous. □

Corollary 3.3 *Let X be a LCM space and let $U \subseteq M(X)$ be a set of formal balls. Then the identity function i on U_* gives an isomorphism*

$$i : (X, U) \longrightarrow (X, \tilde{U})$$

in **FLCM**. *Here $\tilde{U} = \{a \in M(X) : a \lhd U\}$, the saturation of U.*

By this corollary it is enough to consider objects (X, U) in **FLCM** where $U \subseteq M(X)$ is a saturated subset with respect to the cover relation of $M(X)$. Let **SLCM** denote the full subcategory of **FLCM** determined by such objects. The functor J : **FLCM** $\longrightarrow$ **SLCM** given by $J(X, U) = (X, \tilde{U})$ and $J(f) = f$ and the inclusion functor in the reverse direction, form by the corollary an equivalence of categories. From Lemma 3.2 follows

Corollary 3.4 J : **FLCM** $\longrightarrow$ **SLCM** *is full and faithful.*

3.1 Open sublocales

Let $X = (X, \lhd, \leq)$ be a formal topology, and let $G \subseteq X$ be a set of neighbourhoods. Then the *open subspace* $X_{|G}$ *of* X *determined by* G is defined as follows. Let $X_{|G} = (X, \lhd', \leq)$, which is as X except that we change the cover relation $\lhd$ of X to be the one defined by

$$a \lhd' U \Longleftrightarrow_{\mathrm{def}} a \wedge G \lhd U.$$

Note that $U_1 \lhd' U_2$ iff $U_1 \wedge G \lhd U_2 \wedge G$. Hence only the parts covered by G counts when comparing two open sets. Also if $G \sim G'$ then

$$a \wedge G \lhd U \Longleftrightarrow a \wedge G' \lhd U.$$

It is thus sufficient to consider Gs that are saturated subsets of X. Such subsets are necessarily down-closed, i.e. $G_{\leq} = G$.

We have the following useful results

Lemma 3.5 *Let X be a formal topology and let $G \subseteq X$ be an arbitrary set of neighbourhoods. If X is regular, then so is $X_{|G}$.*

Proof. Suppose that $X = (X, \leq, \lhd)$ is regular. We prove that $X_{|G} = (X, \leq, \lhd)$ is regular. We recall the definitions $a^{\perp} = \{u \in X : u \wedge a \lhd \emptyset\}$ and

$$a \lll b \Longleftrightarrow_{\mathrm{def}} X \lhd a^{\perp} \cup \{b\}$$

and $\mathrm{wc}(b) = \{a \in X : a \lll b\}$. Recall that X regular means that for any b

$$b \lhd \mathrm{wc}(b).$$

Now let $a^{\perp'}$, $\lll'$ and $\mathrm{wc}'(b)$ be the corresponding terms for $X_{|G}$. We wish to prove $b \lhd' \mathrm{wc}'(b)$, i.e. that for an arbitrary $d \leq b$ with $d \leq G$ we have $d \lhd \mathrm{wc}'(b)$. By regularity of X: $d \lhd \mathrm{wc}(d)$. Now $\mathrm{wc}(d) \subseteq \mathrm{wc}'(b)$, as for any a:

$$a^{\perp} \cup \{d\} \lhd a^{\perp'} \cup \{b\}.$$

Thus $d \lhd \mathrm{wc}(d) \subseteq \mathrm{wc}'(b)$. □

Lemma 3.6 *Suppose that $F, G : X \longrightarrow Y$ are continuous morphisms between formal topologies, where Y is regular. If $F \subseteq G$, then $F = G$.*

Proof. Suppose that $F \subseteq G$. Since Y is regular we have for any b that $G^{-1}b \lhd G^{-1}[\mathrm{wc}(b)]$. To prove $G \subseteq F$ it will thus be enough to prove

$$G^{-1}[\mathrm{wc}(b)] \lhd F^{-1}b.$$

Suppose that $a\,G\,c$ with $c \lll b$. Thus $Y \lhd c^{\perp} \cup \{b\}$. Hence by (A3) for F and localization we obtain

$$a \lhd F^{-1}[c^{\perp} \cup \{b\}] \wedge a.$$

To prove $a \lhd F^{-1}b$ it is enough to show

$$F^{-1}[c^{\perp} \cup \{b\}] \wedge a \lhd F^{-1}b.$$

Suppose that $u \leq a$, $u\,F\,d$ where $d \in c^{\perp} \cup \{b\}$. In case $d = b$, $u \in F^{-1}b$ and so $u \lhd F^{-1}b$ is clear. In case $d \in c^{\perp}$, it holds that $d \wedge c \lhd \emptyset$. Now $u\,F\,d$ implies $u\,G\,d$ by assumption and as $u \leq a$ we gave also $u\,G\,c$. Hence

$$u \lhd\; G^{-1}[d \wedge c] \lhd G^{-1}[\emptyset] \lhd \emptyset.$$

Thus, trivially, also $u \lhd F^{-1}b$. □

Lemma 3.7 *Let X and Y be formal topologies and let $U \subseteq X$ and $V \subseteq Y$ be saturated subsets. Suppose that $F, G : X_{|U} \longrightarrow Y_{|V}$ are continuous mappings with $F \cap (U \times V) \subseteq G$. Then $F \subseteq G$.*

Proof. Suppose that $a\,F\,b$, with $a \in X$ and $b \in Y$ arbitrary. Then we have $b \lhd_{Y_{|V}} b \wedge V$. Hence by (A1) for F, and the definition of the covering relation in $M_{|U}$

$$a \wedge U \lhd_X U \wedge F^{-1}[b \wedge V].$$

But the assumption $F \cap (U \times V) \subseteq G$ implies that

$$U \wedge F^{-1}[b \wedge V] \subseteq U \wedge G^{-1}[b \wedge V].$$

Hence $a \wedge U \lhd_X U \wedge G^{-1}[b \wedge V] \lhd_X G^{-1}b$ and consequently $a\,G\,b$. □

3.2 Open sublocales of localic completions

Let X be a complete metric space, and let $G \subseteq M(X)$ be a subset. The canonical metric isomorphism $j_X : X \longrightarrow \mathrm{Pt}(M(X))$ restricts to a metric isomorphism

$$j_{X,G} = G_* \longrightarrow G^* = \mathrm{Pt}(M(X)_{|G}) = \mathrm{Pt}(M(X)_{|\tilde{G}})$$

i.e. $j_{X,G}(x) = \{\mathsf{b}(y, \delta) \in M(X) : d(x, y) < \delta\}$. Note that

$$b \in j(x) \Longleftrightarrow x \in b_*.$$

Let X be a LCM space. For a saturated subset $G \subseteq M(X)$, the open sublocale $M(X)_{|G}$ has the following characterization of its cover relation

$$a \lhd' U \Longleftrightarrow a \wedge G \lessdot U.$$

Note that since G is down-closed with respect to $<$ we get

$$a \wedge G \lessdot U \iff (\forall b, c \in G)[b < c < a \Rightarrow (\exists U_0 \in A(b,c))U_0 < U]. \quad (8)$$

Further, note that if $c \in G$ then each set in $A(b,c)$ is a subset of G.

Suppose that Y is an arbitrary metric space and that $H \subseteq M(Y)$ is a saturated subset. For $F : M(X)_{|G} \longrightarrow M(Y)_{|H}$ the relation $a \lhd' F^{-1}V$, where $a \in G$ and $V \subseteq H$, is characterized by

$$(\forall b, c \in G)[b < c < a \Rightarrow (\exists \{u_1, \ldots, u_n\} \in A(b,c))(\exists \{v_1, \ldots, v_n\} < V)(\forall i)\, u_i\, F\, v_i]. \quad (9)$$

This follows since $F^{-1}V \lhd' F^{-1}\{v \in M(Y) : v < V\}$.

Theorem 3.8 *Let X be a LCM space and let G be a subset of $M(X)$. Let Y be a complete metric space and H be a subset of $M(Y)$. There are thus metric isomorphisms*

$$j_{X,G} : G_* \longrightarrow \mathrm{Pt}(M(X)_{|\tilde{G}}) \qquad j_{Y,H} : H_* \longrightarrow \mathrm{Pt}(M(Y)_{|\tilde{H}})$$

Suppose $F : M(X)_{|\tilde{G}} \longrightarrow M(Y)_{|\tilde{H}}$ is a continuous morphism. Then $f : (X,G) \longrightarrow (Y,H)$ defined as the composition $j_{Y,H}^{-1} \circ \mathrm{Pt}(F) \circ j_{X,G}$ is a continuous map in **FLCM**. *Moreover for $a \in \tilde{G}$, $b \in \tilde{H}$*

$$a\, F\, b \text{ implies } f[a_*] \subseteq b_*.$$

Proof. We make the abbreviations $A = M(X)_{|\tilde{G}}$ and $B = M(Y)_{|\tilde{H}}$. Let $p < G$. We prove that f is uniformly continuous on p_*: Let $\varepsilon > 0$. Then by (7) we have $B \lhd_B H \lhd_B H^{(\varepsilon/2)}$. Thus for any $a \in A$

$$a \lhd_A F^{-1}[B] \lhd_A F^{-1}[H^{(\varepsilon/2)}].$$

Now $p < G$, so there some $a \in G$ with $p < a$. Pick further b, c with $p < b < c < a$. By the characterization (8) we get $U = \{u_1, \ldots, u_n\} \in A(b,c)$ and $V = \{v_1, \ldots, v_n\} < H^{(\varepsilon/2)}$ with $u_i\, F\, v_i$ for all $i = 1, \ldots, n$. Since $b \sqsubseteq U < c$, there is a rational $\theta > 0$ such that for all $q \in A$

$$\rho(q) \leq \theta \text{ and } q \leq b \Longrightarrow q \leq U. \quad (10)$$

Write $p = \mathsf{b}(z, \alpha)$ and $b = \mathsf{b}(z', \alpha)$. Since $p < b$, we can find $\theta' > 0$ with

$$d(z, z') + \alpha + \theta' < \alpha'. \quad (11)$$

Then pick $\delta > 0$ so that $\delta < \theta/2, \theta'/2$.

To prove uniform continuity of f on p_* we show that for $x, y \in p_*$:

$$d(x, y) < \delta \Longrightarrow d(f(x), f(y)) \leq \varepsilon.$$

Suppose $x, y \in p_*$ with $d(x, y) < \delta$. Put $q = \mathsf{b}(x, 2\delta)$. Then $\rho(q) < \theta$ and $q \leq b$ by (11) and $x \in p_*$. Thus by (10) there is some i with $q \leq u_i$. As $u_i \, F \, v_i$, we have a fortiori $q \, F \, v_i$. Now $x, y \in q_*$ so $f(x), f(y) \in (v_i)_*$. Hence $d(f(x), f(y)) \leq \varepsilon$ as the radius of v_i is no greater than $\varepsilon/2$.

We verify the second condition for a morphism of **FLCM**. Suppose $p < G$. Let $\varepsilon = 1$ (in fact, any positive rational number would work) and let $p < b < c < a$, U and V be as obtained above. From $b \sqsubseteq U$ follows that $p_* \subseteq b_* \subseteq U_*$. We have $f[(u_i)_*] \subseteq (v_i)_*$ and hence $f[p_*] \subseteq V_*$. Since $V < H^{(\varepsilon/2)} \leq H$, we have indeed $f[p_*] <_* H$. □

4 The embedding functor

For a continuous map $f : (X, U) \longrightarrow (Y, V)$ in **FLCM** we define two relations $D_f, B_f \subseteq U \times V$

$$a \, D_f \, b \Longleftrightarrow (\exists b' \in V) f[a_*] \subseteq b'_* \,\&\, b' < b$$

and

$$a \, B_f \, b \Longleftrightarrow (\exists a' \in U) a < a' \,\&\, a' \, D_f \, b.$$

Lemma 4.1 *Let $f : (X, U) \longrightarrow (Y, V)$ be continuous map in* **FLCM** *where U and V are saturated sets of formal balls. Suppose that $a < b < U$ and $C, E \subseteq V$ satisfies*

$$f[b_*] <_* C \text{ and } C \sqsubseteq E.$$

Then there is $S \in A(a, b)$ with

$$(\forall d \in S)(\exists e \in E) \, d \, B_f \, e$$

Proof. From $f[b_*] <_* C$ follows that there are formal balls $c_1, \ldots, c_n \in C$ and $c'_1, \ldots, c'_n \in V$ with

$$f[b_*] \subseteq \{c'_1, \ldots, c'_n\}_* \quad c'_i < c_i \quad \{c_1, \ldots, c_n\} \sqsubseteq E.$$

Pick a rational $\gamma > 0$ so small that $c_i \sqsubseteq_\gamma E$ for all $i = 1, \ldots, n$. Write

$$c'_i = \mathsf{b}(x_i, \alpha_i) \qquad c_i = \mathsf{b}(y_i, \beta_i)$$

As $c'_i < c_i$ we can find a rational $\theta_i > 0$ with

$$d(x_i, y_i) + \alpha_i + \theta_i < \beta_i. \tag{12}$$

Then take $\varepsilon = \min(\theta_1, \ldots, \theta_n, \gamma)$. Now f is uniformly continuous on b_*, so there is $\delta > 0$ with

$$(\forall v, w \in b_*)[d(v, w) < \delta \Longrightarrow d(f(v), f(w)) < \varepsilon/2]. \tag{13}$$

Since X is locally compact and $a < b$ there is a $S \in A(a, b)$ where each ball in S has radius smaller than δ. (See Lemma 4.7 in [4].)

Now pick any $d = \mathsf{b}(z, \sigma) \in S$. Thus $d < b$ and $z \in b_*$. Since $f[b_*] \subseteq \{c'_1, \ldots, c'_n\}_*$, we get $f(z) \in (c'_i)_*$ for some $i = 1, \ldots, n$. Thus $d(f(z), x_i) < \alpha_i$. Hence by (12)

$$d(f(z), y_i) + \varepsilon \leq d(f(z), x_i) + d(x_i, y_i) + \varepsilon < \alpha_i + d(x_i, y_i) + \varepsilon < \beta_i.$$

This implies that $\mathsf{b}(f(z), \varepsilon) < c_i$. But $\varepsilon \leq \gamma$ and since $c_i \sqsubseteq_\gamma E$, there is some $e \in E$ with

$$\mathsf{b}(f(z), \varepsilon) \leq e. \tag{14}$$

Next let $d' = \mathsf{b}(z, \sigma')$ where σ' is sufficiently small that $\sigma < \sigma' < \delta$ and $d < d' < b$. We claim that

$$f[d'_*] \subseteq \mathsf{b}(f(z), \varepsilon/2) \tag{15}$$

This together with $d < d'$ and $\mathsf{b}(f(z), \varepsilon/2) < \mathsf{b}(f(z), \varepsilon)$ and (14) yields

$$d\, B_f\, e$$

as required. We prove (15). Let $v \in f[d'_*]$. Thus $v = f(u)$ for some u with $d(u, z) < \sigma' < \delta$. By (13) we have $d(f(u), f(z)) < \varepsilon/2$. Thus $v \in \mathsf{b}(f(z), \varepsilon/2)_*$. $\square$

Let $f : (X, U) \longrightarrow (Y, V)$ be a continuous map in **FLCM** where U and V are saturated sets of formal balls. We shall define a morphism of formal topologies

$$A_f : M(X)_{|U} \longrightarrow M(Y)_{|V}$$

by

$$a\, A_f\, b \Longleftrightarrow_{\text{def}} a \lhd_{M(X)_{|U}} D_f^{-1}[b \wedge V].$$

The right hand side is equivalent to $a \wedge U \lhd_{M(X)} D_f^{-1}[b \wedge V]$

Lemma 4.2 *If $f : (X, U) \longrightarrow (Y, V)$ a continuous map in* **FLCM** *, where U and V are saturated, then $A_f : M(X)_{|U} \longrightarrow M(Y)_{|V}$ is a continuous morphism between formal topologies.*

Proof. Let $M_1 = M(X)_{|U}$ and $M_2 = M(Y)_{|V}$. We verify the continuity conditions for $A = A_f$. Explicitly these are

(A1) $a\,A\,b$ and $b \lhd_{M_2} W \Rightarrow a \lhd_{M_1} A^{-1}W$,

(A2) $a \lhd_{M_1} W$ and $(\forall c \in W) c\,A\,b \Rightarrow a\,A\,b$,

(A3) $M_1 \lhd_{M_1} A^{-1}[M_2]$,

(A4) $a\,A\,b_1$ and $a\,A\,b_2 \Rightarrow a \lhd_{M_1} A^{-1}[b_1 \wedge b_2]$.

We notice that (A2) is immediate by the transitivity of $\lhd_{M_1}$. To verify the other conditions it useful to note that

$$a \lhd_{M_1} A^{-1}W \Longleftrightarrow a \lhd_{M_1} D^{-1}[W \wedge V] \tag{16}$$

where $D = D_f$.

Proof of (A1): Suppose $a\,A\,b$ and $b \lhd_{M_2} W$. By (16) it is sufficient to establish $a \lhd_{M_1} D^{-1}[W \wedge V]$. Since $a\,A\,b$ means $a \lhd_{M_1} D^{-1}[b \wedge V]$, it is enough to prove

$$D^{-1}[b \wedge V] \lhd_{M_1} D^{-1}[W \wedge V].$$

Take $q \in D^{-1}[b \wedge V]$. By the axiom (M1) we need only to prove $q' \lhd_{M_1} D^{-1}[W \wedge V]$ for an arbitrary $q' < q$. Let $q' < q$. Pick q'' with $q' < q'' < q$. There is $d \in b \wedge V$ with $q\,D\,d$. Hence there is $c < d$ with $f[q_*] \subseteq c_*$. Since U and V are saturated, they are also down-closed. Thus $q, q' \in U$ and $c, d \in V$. Now $b \lhd_{M_2} W$ and $d \in b \wedge V$ implies $d \lhd_{M(Y)} W \wedge V$. By (4) we thus have $d \lessdot W \wedge V$. Pick c', c'' so that $c < c' < c'' < d$. Hence there is $E \in A(c', c'')$ with $E < W \wedge V$. We have $f[q''_*] \subseteq f[q_*] <_* \{c'\}$ and $\{c'\} \sqsubseteq E$. Then we can apply Lemma 4.1 and get $S \in A(q', q'')$ with

$$(\forall u \in S)(\exists e \in E) u\,B_f\,e.$$

Thus

$$S \subseteq D^{-1}[W \wedge V].$$

Since $q' \sqsubseteq S$, we have $q' \lhd_{M(X)} S$ and thus $q' \lhd_{M(X)} D^{-1}[W \wedge V]$.

Proof of (A3): By (16) it is sufficient to show $M_1 \lhd_{M_1} D^{-1}[M_2 \wedge V]$. But since U and V are both down-closed, this is equivalent to verifying $U \lhd_{M(X)} D^{-1}V$. By (4) this is in turn equivalent to $U \lessdot D^{-1}V$. Take an arbitrary $a \in U$ and take arbitrary b and c with $b < c < a$. Now $c < U$, so by the continuity of f there is $P = \{p_1, \ldots, p_n\} < V$ with $f[c_*] \subseteq P_*$. Thus we find $E = \{q_1, \ldots, q_n\} \subseteq V$ with $p_i < q_i$ and further $C = \{p'_1, \ldots, p'_n\}$ with $p_i < p'_i < q_i$. This implies that $f[c_*] <_* C$ and $C \sqsubseteq E$. By Lemma 4.1 we get $S \in A(b, c)$ with

$$(\forall d \in S)(\exists e \in E)\, d\,B_f\,e.$$

But $d\, B_f\, e$ and $e \in E \subseteq V$ implies $d < D^{-1}V$. Hence $b \sqsubseteq S < D^{-1}V$. Thus also $b \lhd_{M_1} D^{-1}V$. Now since $a \lhd_{M_1} \{b \in U : b < a\}$ by axiom (M1) we get by transitivity $a \lhd_{M_1} D^{-1}V$. Hence $U \lhd_{M_1} D^{-1}V$.

Proof of (A4): Suppose $a\, A\, b_1$ and $a\, A\, b_2$. By (16) and localization,

$$a \lhd_{M_1} D^{-1}[b_1 \wedge V] \wedge D^{-1}[b_2 \wedge V] = D^{-1}[b_1 \wedge V] \cap D^{-1}[b_2 \wedge V].$$

By (16) it is enough to show $a \lhd_{M_1} D^{-1}[b_1 \wedge b_2 \wedge V]$. This is done as soon as we have shown

$$D^{-1}[b_1 \wedge V] \cap D^{-1}[b_2 \wedge V] \lhd_{M(X)} D^{-1}[b_1 \wedge b_2 \wedge V].$$

Let d be an arbitrary element in the set on the left hand side. Thus there are $c_i' < c_i \in b_i \wedge V$ with $f[d_*] \subseteq (c_i')_*$. Moreover V is down-closed so $c_i \in V$ and $c_i \leq b_i$. Write $c_i = \mathsf{b}(z_i, \gamma_i)$ and $c_i' = \mathsf{b}(z_i', \gamma_i')$. Pick $\varepsilon > 0$ small enough that

$$d(z_i', z_i) + \gamma_i' + 2\varepsilon < \gamma_i, \tag{17}$$

for $i = 1, 2$. We wish to prove $d \lhd_{M(X)} D^{-1}[b_1 \wedge b_2 \wedge V]$. We use (4). Suppose that $d' < d'' < d$. It is enough to find $C \in A(d', d'')$ with $C < D^{-1}[b_1 \wedge b_2 \wedge V]$. Pick p so that $d'' < p < d$. We have $d \in U$, so $p < U$. The function f is thus uniformly continuous on p_* and we find $\delta > 0$ with

$$(\forall x, y \in p_*)[d(x, y) < \delta \Longrightarrow d(f(x), f(y)) < \varepsilon]. \tag{18}$$

Now as X is locally compact there is $C = \{\mathsf{b}(u_1, \alpha_1), \ldots, \mathsf{b}(u_n, \alpha_n)\} \in A(d', d'')$ where $\alpha_j < \delta$ for all $j = 1, \ldots, n$. (Lemma 4.7 of [4].) We may find a small $\beta > 0$ so that $\mathsf{b}(u_j, \alpha_j + \beta) < d''$ and $\alpha_j + \beta < \delta$ for all $j = 1, \ldots, n$. To prove $C < D^{-1}[b_1 \wedge b_2 \wedge V]$ it is now sufficient to demonstrate that $\mathsf{b}(u_j, \alpha_j + \beta) \in D^{-1}[b_1 \wedge b_2 \wedge V]$ for each j. Now $u_j \in d''_* \subseteq p_*$ and (18) gives

$$f[\mathsf{b}(u_j, \alpha_j + \beta)_*] \subseteq \mathsf{b}(f(u_j), \varepsilon)_*.$$

Obviously $\mathsf{b}(f(u_j), \varepsilon) < \mathsf{b}(f(u_j), 2\varepsilon)$, so if we can show $\mathsf{b}(f(u_j), 2\varepsilon) \in b_1 \wedge b_2 \wedge V$ we are done. To do this we show that $\mathsf{b}(f(u_j), 2\varepsilon) < c_i = \mathsf{b}(z_i, \gamma_i)$ for $i = 1, 2$. This is the same as proving

$$d(f(u_j), z_i) + 2\varepsilon < \gamma_i. \tag{19}$$

Since $f(u_j) \in f[p_*] \subseteq f[d_*] \subseteq c_i'$ and $d(f(u_j), z_i') \leq \gamma_i'$ we have

$$d(f(u_j), z_i) + 2\varepsilon \leq d(f(u_j), z_i') + d(z_i', z_i) + 2\varepsilon \leq \gamma_i' + d(z_i', z_i) + 2\varepsilon.$$

Thus by (17) we get the desired (19). □

Theorem 4.3 *Let $f : (X, U) \longrightarrow (Y, V)$ be a continuous map in* **FLCM**, *where U and V are saturated. Then the diagram*

$$\begin{array}{ccc} (X,U) & \xrightarrow{j_{X,U}} & \mathrm{Pt}(M(X)_{|U}) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Pt}(A_f)} \\ (Y,V) & \xrightarrow[j_{Y,V}]{} & \mathrm{Pt}(M(Y)_{|V}) \end{array}$$

commutes.

Proof. Let $x \in U_*$. Then using that points split covers we get

$$\begin{aligned} \mathrm{Pt}(A_f)(j_{X,U}(x)) &= \{b \in M(Y)_{|V} : (\exists a \in j_{X,U}(x))\, a\, A_f\, b\} \\ &= \{b \in M(Y)_{|V} : (\exists a \in j_{X,U}(x))\, a \in U \ \&\ a\, D_f\, b\} \\ &= \{b \in M(Y)_{|V} : (\exists a \in U)(\exists c \in V)\, x \in a_* \ \&\ f[a_*] \subseteq c_* \ \&\ c < b\} \end{aligned}$$

For any $b \in \mathrm{Pt}(A_f)(j_{X,U}(x))$, we have $f(x) \in b_*$ and so $b \in j_{Y,V}(f(x))$. Conversely, suppose that $b \in j_{Y,V}(f(x))$. Thus $f(x) \in b_*$. Since $x \in U_*$, we have also $f(x) \in V_*$. Take $d \in V$ with $f(x) \in d_*$. Chose $\varepsilon > 0$ so small that $q = \mathsf{b}(f(x), \varepsilon) < b$ and $q < d$. Let $\delta > 0$ be so small that both $p = \mathsf{b}(x, \delta) < U$ and (by continuity of f) $f[p_*] \subseteq q_*$. Therefore $b \in \mathrm{Pt}(A_f)(j_{X,U}(x))$. $\square$

Theorem 4.4 *Let X and Y be LCM spaces. Suppose that $U \subseteq M(X)$ and $V \subseteq M(Y)$ are saturated sets. Let $F : M(X)_{|U} \longrightarrow M(Y)_{|V}$ be a continuous map. Then $F = A_f$ where $f : (X, U) \longrightarrow (Y, V)$ is given by*

$$f = j_{Y,V}^{-1} \circ \mathrm{Pt}(F) \circ j_{X,U}.$$

Proof. Since $M(Y)$ is a regular formal topology the open subspace $M_2 = M(Y)_{|V}$ is regular too (Lemma 3.5). Thus to prove $F = A_f$ it is, by Lemma 3.6, sufficient to check that $F \subseteq A_f$. By Lemma 3.7 it is enough to verify that $F \cap (U \times V) \subseteq A_f$. Assume that $a\, F\, b$ where $a \in U$ and $b \in V$. By axiom (M1) $b \lhd_{M_2} \{p : p < b\}$ where $M_2 = M(Y)_{|V}$. Thus $a \lhd_{M_1} F^{-1}\{p : p < b\}$ and hence by localization $a \lhd_{M_1} U \wedge F^{-1}\{p : p < b\}$. We prove

$$U \wedge F^{-1}\{p : p < b\} \subseteq A_f^{-1} b.$$

If c is in the left hand side, we have $c \in U$ and $c\, F\, p$ for some $p < b$. Hence by Theorem 3.8, $f[c_*] \subseteq p_*$. Thus $c\, D_f\, b$, and hence $c \in A_f^{-1} b$. Thus $a \lhd_{M_1} A_f^{-1} b$. Therefore $a\, A_f\, b$. $\square$

Corollary 4.5 *There is a full and faithful functor* $M : \mathbf{SLCM} \longrightarrow \mathbf{FTop}$ *given by*

$$M(X,U) = M(X)_{|U} \text{ and } M(f) = A_f.$$

By composition

$$MJ : \mathbf{FLCM} \longrightarrow \mathbf{FTop}$$

is a full and faithful functor as well. Further composition gives a fully faithful functor

$$MJK : \mathbf{OLCM} \longrightarrow \mathbf{FTop},$$

where $K(X,U) = (X, H(U))$ *and* $K(f) = f$.

Proof. Functoriality of M: To prove this we employ the functoriality of Pt. Let (X,U) be an object in **SLCM**. By Theorem 4.4 we have $\mathrm{id}_{M(X,U)} = A_h$ where

$$h = j^{-1} \circ \mathrm{Pt}(\mathrm{id}_{M(X,U)}) \circ j = j^{-1} \circ \mathrm{id}_{\mathrm{Pt}(M(X,U))} \circ j = j^{-1} \circ j = \mathrm{id}_{(X,U)}.$$

Let $g : (X,U) \longrightarrow (Y,V)$ and $f : (Y,V) \longrightarrow (Z,W)$ be continuous functions in **SLCM**. Using functoriality of Pt and Theorem 4.3 one obtains

$$j^{-1} \circ \mathrm{Pt}(A_f \circ A_g) \circ j = j^{-1} \circ \mathrm{Pt}(A_f) \circ j^{-1} \circ j \circ \mathrm{Pt}(A_g) \circ j = f \circ g.$$

By Theorem 4.4 we have $A_f \circ A_g = A_h$ where h is the left hand side in the equation above. Thus $h = f \circ g$. This establishes functoriality of M.

Fullness: For any $F : M(X,U) \longrightarrow M(Y,V)$ we have by Theorem 4.4 some $f : (X,U) \longrightarrow (Y,V)$ so that $F = A_f = M(f)$.

Faithfulness: If $M(f) = M(g)$, then $A_f = A_g$ and so by Theorem 4.3

$$f = j^{-1} \circ \mathrm{Pt}(A_f) \circ j = j^{-1} \circ \mathrm{Pt}(A_g) \circ j = g.$$

The functors $J : \mathbf{FLCM} \longrightarrow \mathbf{SLCM}$ and $K : \mathbf{OLCM} \longrightarrow \mathbf{FLCM}$ both are full and faithful according Corollary 3.4 and Corollary 2.12 respectively. □

Finally, we can prove what was stated at the end of Section 2: that the point-wise structure of the objects in **FLCM** is not enough, unless we assume non-constructive axioms. We have:

Proposition 4.6 *Let* X *be a compact metric space. Then* X *is covering compact if for any* $P \subseteq M(X)$,

$$P_* = X \Longrightarrow (X,P) \cong (X, H(X)).$$

Proof. Assume that the condition holds. Suppose that $P \subseteq M(X)$ and suppose that $P_* = X$. Then $(X, P) \cong (X, H(X))$ by the condition. Thus since M is functor we have $M(X, P) \cong M(X, H(X)) \cong M(X)$. Now $M(X)$ is compact, since X compact, so $M(X, P)$ must be compact as well. We have

$$M(X) \lhd_{M(X,P)} P.$$

By compactness of $M(X, P)$ there is a f.e. $F \subseteq P$ with $M(X) \lhd_{M(X,P)} F$. Hence $P \lhd_{M(X)} F$, but then also $P_* \subseteq F_*$. Thus P has a f.e. point-wise subcover F. This shows that X is covering compact. □

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