

Apartness and formal topology

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Abstract

The theory of formal spaces and the more recent theory of apartness spaces have *a priori* not much more in common than that each of them was initiated as a constructive approach to general topology. We nonetheless try to do the first steps in relating these competing theories to each other.

Formal topology was put forward in the mid 1980s by Sambin [8] in order to make available to Martin–Löf’s type theory [7] the concepts of classical topology that are worth keeping to such a constructive and predicative framework. In the meantime formal topology has proved a fairly universal setting for doing topology in a point–free way. We refer to [9] for a recent and exhaustive survey of formal topology.

The theory of apartness spaces was started by Bridges and Vîţă [4] nearly twenty years later to reformulate set–theoretic topology as an extension of Bishop’s constructive analysis [2, 3]. The subsequent development of the theory of apartness spaces has also shed some light on its classical counterpart, the theory of proximity or nearness spaces. A comprehensive overview will be available soon [5].

In formal topology ‘basic neighbourhood’ is a primitive concept, whereas ‘point’ is a derived notion; as sets of basic neighbourhoods, points have to be handled with particular care to meet the needs of a predicative framework like Martin–Löf type theory. In the theory of apartness spaces, it is the other way round: as in classical topology, points are given as such, and (basic) neighbourhoods are sets of points. Since, however, it is hard to detect any truly impredicative move in the practice of Bishop’s constructive mathematics in general, we dare to undertake the following attempt to link formal topology and the theory of apartness spaces to each other.

1 Basic definitions

We recall the standard definitions associated with formal topologies and morphisms between them (approximable mappings).

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Definition 1.1 Let A be a set, and let \triangleleft be a relation between elements of A and subsets of A , i.e. $\triangleleft \subseteq A \times \mathcal{P}(A)$. Extend \triangleleft to a relation between subsets of A by setting $U \triangleleft V$ if and only if $a \triangleleft V$ for all $a \in U$. For a preorder (A, \leq) and a subset $U \subseteq A$, its *downwards closure* U_{\leq} consists of those $a \in A$ such that $a \leq b$ for some $b \in U$.

Definition 1.2 A *formal topology* is a pre-ordered set $A = (A, \leq)$ of so-called *basic neighbourhoods* together with a relation $\triangleleft \subseteq A \times \mathcal{P}(A)$, the *covering relation*, satisfying the four conditions

$$\begin{aligned} \text{(R)} \quad & a \in U \text{ implies } a \triangleleft U, & \text{(L)} \quad & a \triangleleft U, a \triangleleft V \text{ implies } a \triangleleft U_{\leq} \cap V_{\leq}, \\ \text{(T)} \quad & a \triangleleft U, U \triangleleft V \text{ implies } a \triangleleft V, & \text{(E)} \quad & a \leq b \text{ implies } a \triangleleft \{b\}. \end{aligned}$$

Write $U \wedge V = U_{\leq} \cap V_{\leq}$, and then abbreviate $\{a\} \wedge \{b\}$ by $a \wedge b$.

Definition 1.3 A subset $\alpha \subseteq A$ is *filtering* if for all $a, b \in \alpha$ there is $c \in \alpha$ with $c \leq a$ and $c \leq b$. We use *non-void* and *inhabited* set as synonyms. A (*formal*) *point* is a non-void subset $\alpha \subseteq A$ which is filtering, and such that $U \cap \alpha$ is non-void whenever $a \triangleleft U$ for some $a \in \alpha$. The collection of points is denoted by $\text{Pt}(A)$. The collection U^* of points associated with a set U of basic neighbourhoods is given by

$$U^* = \{\alpha \in \text{Pt}(A) : (\exists a \in U) a \in \alpha\}.$$

The latter construct has the following properties:

$$\text{(O1)} \quad \left(\bigcup_{i \in I} U_i\right)^* = \bigcup_{i \in I} U_i^*$$

$$\text{(O2)} \quad (U \wedge V)^* = U^* \cap V^*$$

$$\text{(O3)} \quad U \triangleleft V \implies U^* \subseteq V^*$$

Moreover, $\emptyset^* = \emptyset$ and $A^* = \text{Pt}(A)$; whence the collections of the shape U^* form an ordinary topology on $\text{Pt}(A)$. A neighbourhood basis for this topology, in the sense of [3], is given by the $\{a\}^*$ with $a \in A$. The collection of points $\text{Pt}(A)$ need not be a set in predicative systems like Martin-Löf type theory.

Approximable mappings

Definition 1.4 For a relation $F \subseteq A \times B$ the *inverse image* of $V \subseteq B$ under the relation F is, as usual,

$$F^{-1}[V] = \{a \in A : (\exists b \in V) a F b\}.$$

The relation F is naturally extended to subsets as follows. For $U \subseteq A$, let $U F b$ mean $(\forall u \in U) u F b$, and for $V \subseteq B$, we let $a F V$ mean $a \triangleleft F^{-1}[V]$.

Definition 1.5 Let (A, \leq, \triangleleft) and $(B, \leq', \triangleleft')$ be formal topologies. A relation $F \subseteq A \times B$ is an *approximable mapping* or *continuous morphism* from (A, \leq, \triangleleft) to $(B, \leq', \triangleleft')$ if

- (a1) $a F b, b \triangleleft V$ implies $a F V$,
- (a2) $a \triangleleft U, U F b$ implies $a F b$,
- (a3) $a F B$ for all $a \in A$,
- (a4) $a F V, a F W$ implies $a F (V_{\leq'} \cap W_{\leq'})$.

Using (a4) one proves

Proposition 1.6 *For an approximable mapping F we have*

$$a F b, a' F b' \implies a \wedge a' \triangleleft F^{-1}[b \wedge b']. \quad \square$$

Definition 1.7 The *point function* $\text{Pt}(F) : \text{Pt}(A) \rightarrow \text{Pt}(B)$ associated with an approximable mapping F from A to B assigns

$$\text{Pt}(F)(\alpha) = \{b \in B : (\exists a \in \alpha) a F b\}$$

to every point α of A .

This is a well-defined function which is pointwise continuous with respect to the topologies on $\text{Pt}(A)$ and $\text{Pt}(B)$ recalled above.

2 Apartness relations

Definition 2.1 Two basic neighbourhoods a and b are *apart* ($a \perp b$) if $a \wedge b \triangleleft \emptyset$. More generally, two sets U and V of basic neighbourhoods are *apart* ($U \perp V$) if $a \perp b$ for all $a \in U$ and $b \in V$.

Using axioms (L) and (T), it is easy to see that $a \perp b$ is equivalent to

$$\{c : c \triangleleft \{a\} \text{ and } c \triangleleft \{b\}\} \triangleleft \emptyset.$$

Moreover, $U \perp V$ is equivalent to $U \wedge V \triangleleft \emptyset$, and

$$U \perp V \implies F^{-1}[U] \perp F^{-1}[V]$$

for every approximable mapping F as a corollary to Proposition 1.6.

Definition 2.2 Two points α and β are *apart* ($\alpha \neq \beta$) if $a \perp b$ for some $a \in \alpha$ and some $b \in \beta$. A point α and a set S of points are *apart* ($\alpha \bowtie S$) if there is $a \in \alpha$ such that for each $\beta \in S$ there is $b \in \beta$ with $a \perp b$. For a set of points S , let $\sim S$ to be the set of points α with $\alpha \neq \beta$ for all $\beta \in S$, and let $-S$ be the set of points α such that $\alpha \bowtie S$.

Proposition 2.3 *For every formal topology, the apartness relation between points is irreflexive and symmetric.*

Proof. To see that the apartness relation is irreflexive, suppose that $\alpha \neq \alpha$, for some point α . Then $a \perp b$ for some $a \in \alpha$ and $b \in \beta$. Since the point is filtering, there is $c \in \alpha$ with $c \leq a$ and $c \leq b$. Thus $c \in a \wedge b \triangleleft \emptyset$. But this contradicts that c is in α , since no basic neighbourhood of a point can be covered by the empty set. \square

3 Regular topologies

Definition 3.1 For a formal topology $A = (A, \leq, \triangleleft)$ define $a^\perp = \{b \in A : a \perp b\}$. Let a and b be basic neighbourhoods. Then a is *well inside* b ($a \ll b$) if for every basic neighbourhood c

$$c \triangleleft a^\perp \cup \{b\}.$$

The formal topology A is *regular* if for each b

$$b \triangleleft \{a \in A : a \ll b\}.$$

It is known [1] that the points of a regular formal topology A form a set as soon as the topology is set-presented. The latter is a mild condition requiring that there is a family of subsets

$$C(a, i) \subseteq A \quad (i \in I(a), a \in A)$$

so that $a \triangleleft U$ if and only if $C(a, i) \subseteq U$ for some $i \in I(a)$.

Lemma 3.2 *In a regular formal topology we have*

$$b \in \alpha \implies b \in \beta \text{ or } \alpha \neq \beta$$

for all points α, β and every basic neighbourhood b .

Proof. Suppose that $b \in \alpha$. By regularity, there is some $a \in \alpha$ with $a \ll b$. Take any $c \in \beta$. For this basic neighbourhood we have $c \triangleleft a^\perp \cup \{b\}$, since $a \ll b$. Thus either $b \in \beta$, or there is some $d \in a^\perp$ with $d \in \beta$. But the latter implies $\alpha \neq \beta$. \square

Proposition 3.3 *In a regular formal topology the inequality is tight: that is, $\alpha = \beta$ whenever $\neg(\alpha \neq \beta)$.*

Proof. Suppose $\neg(\alpha \neq \beta)$. If $a \in \alpha$, then $a \in \beta$ by Lemma 3.2. Hence $\alpha \subseteq \beta$. Reversing the roles of the points, we get $\beta \subseteq \alpha$. \square

Proposition 3.4 *In a formal topology whose inequality is tight, all points are maximal: that is, $\alpha = \beta$ whenever $\alpha \subseteq \beta$.*

Proof. This follows since the combination $\alpha \subseteq \beta$ and $\alpha \neq \beta$ is impossible. \square

3.1 Point-to-set apartness

Theorem 3.5 *If A is a regular formal topology, then $X = \text{Pt}(A)$ satisfies the following axioms of a (point-to-set) apartness space, with $x, y \in X$ and $S, T \subseteq X$:*

$$(A1) \quad x \neq y \implies x \bowtie \{y\}$$

$$(A2) \quad x \bowtie S \implies x \notin S$$

(A3) $x \bowtie (S \cup T) \iff x \bowtie S$ and $x \bowtie T$

(A4) $x \in -S$ and $-S \subseteq \sim T \implies x \bowtie T$

(A5) $x \bowtie S \implies (\forall z \in X)(x \neq z \text{ or } z \bowtie S)$

In fact, (A1–A3) hold for an arbitrary formal topology.

Proof. (A1) is true by definition. (A2) follows since $\neg x \neq x$. (A3) uses the fact that points are filtered with respect to \leq .

(A4) Suppose that $\alpha \bowtie S$ and

$$-S \subseteq \sim T. \quad (1)$$

Then for some $a \in \alpha$

$$(\forall \beta \in S)(\exists b \in \beta) a \perp b \quad (2)$$

By regularity take $a' \ll a$ with $a' \in \alpha$. Consider an arbitrary point $\gamma \in T$, and let c be a basic neighbourhood in that point. Since a' is well inside a ,

$$c \triangleleft \{a\} \cup (a')^\perp.$$

There are two cases: if $a \in \gamma$, then $\gamma \bowtie S$ by (2), and thus $\gamma \in \sim T$ by (1), which is a contradiction. The other case must hold, i.e. for some $b \in (a')^\perp$ we have $b \in \gamma$. Since γ was arbitrary in T and a' fixed, we have $\alpha \bowtie T$.

(A5) Now suppose $\alpha \bowtie S$. Then take $a, a' \in \alpha$ with $a' \ll a$ and such that a satisfies (2). For each $\gamma \in X$ there is some basic $c \in \gamma$. Hence

$$c \triangleleft \{a\} \cup (a')^\perp.$$

Two possibilities arise: either $a \in \gamma$, and then $\gamma \bowtie S$ by (2), or there is some $d \in (a')^\perp$ with $d \in \gamma$; in this case $\alpha \neq \gamma$. \square

With the same proof as for (A4), the stronger property (see [6])

(PA4) $x \in -S$ and $-S \subseteq \neg T \implies x \bowtie T$,

where $\neg T = \{x \in X : \neg x \in T\}$, can actually be verified for regular formal topologies.

Theorem 3.6 For an approximable mapping $F : A \rightarrow B$ between regular formal topologies, the corresponding point mapping

$$f = \text{Pt}(F) : \text{Pt}(A) \rightarrow \text{Pt}(B)$$

is continuous in the sense of (point-to-set)apartness spaces, i.e.

$$f(\alpha) \bowtie f(S) \implies \alpha \bowtie S.$$

Proof. Suppose that $f(\alpha) \bowtie f(S)$ where $S \subseteq \text{Pt}(B)$. Thus there is some $b \in f(\alpha)$ with

$$(\forall \beta \in S)(\exists b' \in \beta) b \perp b'.$$

Thus there is some $a \in \alpha$ where $a F b$. Then, by the above, we find for every $\beta \in S$ some $a' \in \beta$ such that $a' F b'$ and $b \perp b'$. Thus using Proposition 1.6

$$a \wedge a' \triangleleft F^{-1}[b \wedge b'] \triangleleft F^{-1}[\emptyset] \triangleleft \emptyset.$$

We have thereby $a \perp a'$. Since $\beta \in S$ was arbitrary, this yields $\alpha \bowtie S$. \square

3.2 Set-to-set apartness

The natural apartness between two subsets S, T of a metric space $X = (X, \rho)$ is

$$S \bowtie T \iff (\exists \varepsilon > 0)(\forall x \in S)(\forall y \in T)\rho(x, y) \geq \varepsilon$$

and satisfies the following axioms [10]:

$$(B1) \quad S \bowtie \emptyset$$

$$(B2) \quad S \bowtie T \implies S \cap T = \emptyset$$

$$(B3) \quad S \bowtie T \implies T \bowtie S$$

$$(B4) \quad S \bowtie (T \cup R) \iff S \bowtie T \text{ and } S \bowtie R$$

$$(B5) \quad x \in -S \implies (\exists T)(x \in -T \text{ and } X = -S \cup T)$$

with $-S = \{z : \{z\} \bowtie S\}$.

Definition 3.7 For a formal topology A define the *set-to-set apartness* by setting

$$S \bowtie' T \iff (\exists U, V \subseteq A)[S \subseteq U^*, T \subseteq V^* \text{ and } U \perp V]$$

for $S, T \subseteq \text{Pt}(A)$.

For $S = \{\alpha\}$, this then agrees with the point-to-set apartness considered above.

Lemma 3.8 $\{\alpha\} \bowtie' T \iff (\exists a \in \alpha)(\forall \beta \in T)(\exists b \in \beta) a \perp b$.

Proof. (\implies) is straightforward. To prove (\impliedby) let $U = \{a\}$ and

$$V = \{b : (\exists \beta \in T)(b \in \beta \text{ and } a \perp b)\}. \quad \square$$

Proposition 3.9 For a regular formal topology A the relation \bowtie' satisfies the axioms (B1–B5) of a set-to-set apartness. In fact, (B1–B4) hold for an arbitrary formal topology.

Proof. The verification of (B1–B3) is straightforward using (O1–O3). (B4, \implies) is direct. To prove (B4, \impliedby) suppose that $S \subseteq U_1^*$, $T \subseteq V_1^*$, $S \subseteq U_2^*$ and $R \subseteq V_2^*$ where $U_1 \perp V_1$ and $U_2 \perp V_2$. Since $U_1^* \cap U_2^* = (U_1 \wedge U_2)^*$ and $(V_1 \cup V_2)^* = V_1^* \cup V_2^*$, we need to show $(U_1 \wedge U_2) \perp (V_1 \cup V_2)$. Indeed, we have

$$\begin{aligned} (U_1 \wedge U_2) \wedge (V_1 \cup V_2) &= (U_1 \wedge U_2)_{\leq} \cap (V_1 \cup V_2)_{\leq} \\ &= ((U_1)_{\leq} \cap (U_2)_{\leq}) \cap ((V_1)_{\leq} \cup (V_2)_{\leq}) \\ &= ((U_1)_{\leq} \cap (U_2)_{\leq}) \cap (V_1)_{\leq} \cup ((U_1)_{\leq} \cap (U_2)_{\leq}) \cap (V_2)_{\leq} \\ &\subseteq (U_1)_{\leq} \cap (V_1)_{\leq} \cup (U_2)_{\leq} \cap (V_2)_{\leq} \\ &= U_1 \wedge V_1 \cup U_2 \wedge V_2 \end{aligned}$$

and the last member has an empty cover since $U_1 \perp V_1$ and $U_2 \perp V_2$.

As for (B5) assume that $\alpha \in -S$. Let $a \in \alpha$ be such that for every $\beta \in S$ there is some $b \in \beta$ with $a \perp b$. Take $a' \ll a$ where $a' \in \alpha$. Put

$$T = ((a')^\perp)^*.$$

Then for each $\beta \in T$ there is, by definition, some $b \in \beta$ such that $b \perp a'$. Hence $\alpha \in -T$. Now consider an arbitrary γ , and some $c \in \gamma$. Then $c \triangleleft \{a\} \cup (a')^\perp$. Thus $a \in \gamma$, in which case $\gamma \in -S$, or there is some $b \in \gamma$ with $b \perp a'$, in which case $\gamma \in T$ — by the definition of T . This verifies (B5). \square

The approximable mappings turn out to be strongly continuous in the sense of [10].

Lemma 3.10 *If $F : A \rightarrow B$ is an approximable mapping between formal topologies, then*

- (a) $U, V \subseteq B$ and $U \perp V \implies F^{-1}(U) \perp F^{-1}(V)$,
- (b) $W \subseteq B$ and $R \subseteq W^* \implies f^{-1}(R) \subseteq F^{-1}(W)^*$,

where $f = \text{Pt}(F) : \text{Pt}(A) \rightarrow \text{Pt}(B)$.

Proof. Part (a) follows from Proposition 1.6. Part (b) follows by the definition of f . \square

Theorem 3.11 *For an approximable mapping $F : A \rightarrow B$ between formal topologies the point function $f = \text{Pt}(F)$ satisfies*

$$f(S) \bowtie' f(T) \implies S \bowtie' T \quad (S, T \subseteq \text{Pt}(A)).$$

Proof. $f(S) \bowtie' f(T)$ says that $U \perp V$ for certain $U, V \subseteq B$ with $f(S) \subseteq U^*$ and $f(T) \subseteq V^*$. By part (a) of Lemma 3.10 we have $F^{-1}(U) \perp F^{-1}(V)$ and thus, by part (b),

$$f^{-1}(f(S)) \bowtie' f^{-1}(f(T)),$$

because $f^{-1}(f(S)) \subseteq F^{-1}(U)^*$ and $f^{-1}(f(T)) \subseteq F^{-1}(V)^*$. By the union axiom (B4) we get $S \bowtie' T$, because $S = S \cup f^{-1}(f(S))$ and likewise for T . \square

Remark 3.12 The above results hold classically also for ordinary topological spaces, given that $S \bowtie' T$ is defined as the statement that S and T are separated by disjoint open sets. Indeed, then (B1–B4) hold in an arbitrary topological space, and (B5) is true if the space is regular. Strong continuity is moreover valid for every continuous map.

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