5 Colouring

How many colours do we need to colour the countries of a map in such a way that adjacent countries are coloured differently? How many days have to be scheduled for committee meetings of a parliament if every committee intends to meet for one day and some members of parliament serve on several committees? How can we find a school timetable of minimum total length, based on the information of how often each teacher has to teach each class?

vertex colouring

A *vertex colouring* of a graph $G = (V, E)$ is a map $c: V \to S$ such that $c(v) \neq c(w)$ whenever v and w are adjacent. The elements of the set S are called the available *colours*. All that interests us about S is its size: typically, we shall be asking for the smallest integer k such that G has a k-*colouring*, a vertex colouring $c: V \to \{1, \ldots, k\}$. This k is the (*vertex-*) *chromatic number* of G; it is denoted by $\chi(G)$. A graph G with $\chi(G) = k$ is called k-*chromatic*; if $\chi(G) \leqslant k$, we call G k-*colourable*.

chromatic number $\chi(G)$

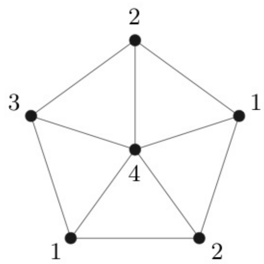

Fig. 5.0.1. A vertex colouring $V \to \{1, \ldots, 4\}$

colour classes

Note that a k-colouring is nothing but a vertex partition into k independent sets, now called *colour classes*; the non-trivial 2-colourable graphs, for example, are precisely the bipartite graphs. Historically, the colouring terminology comes from the map colouring problem stated above, which leads to the problem of determining the maximum chromatic number of planar graphs. The committee scheduling problem, too, can be phrased as a vertex colouring problem—how?

edge colouring

An *edge colouring* of $G = (V, E)$ is a map $c: E \to S$ with $c(e) \neq c(f)$ for any adjacent edges e, f. The smallest integer k for which a k-*edge-colouring* exists, i.e. an edge colouring $c: E \to \{1, \dots, k\}$, is the *edge-chromatic number*, or *chromatic index*, of G; it is denoted by $\chi'(G)$. The third of our introductory questions can be modelled as an edge colouring problem in a bipartite multigraph (how?).

chromatic index $\chi'(G)$

Clearly, every edge colouring of G is a vertex colouring of its line graph $L(G)$, and vice versa; in particular, $\chi'(G) = \chi(L(G))$. The problem of finding good edge colourings may thus be viewed as a restriction of the more general vertex colouring problem to this special class of graphs. As we shall see, this relationship between the two types of colouring problem is reflected by a marked difference in our knowledge about their solutions: while there are only very rough estimates for χ, its sister χ' always takes one of two values, either Δ or $\Delta + 1$.

5.1 Colouring maps and planar graphs

If any result in graph theory has a claim to be known to the world outside, it is the following *four colour theorem* (which implies that every map can be coloured with at most four colours):

Theorem 5.1.1. (Four Colour Theorem)
Every planar graph is 4-colourable.

Some remarks about the proof of the four colour theorem and its history can be found in the notes at the end of this chapter. Here, we prove the following weakening:

Proposition 5.1.2. (Five Colour Theorem)
Every planar graph is 5-colourable.

(4.1.1)
(4.2.10)

Proof. Let G be a plane graph with $n \geqslant 6$ vertices and m edges. We assume inductively that every plane graph with fewer than n vertices can be 5-coloured. By Corollary 4.2.10,

n, m

$$d(G) = 2m/n \leqslant 2\,(3n-6)/n < 6\,;$$

v, H, c

let $v \in G$ be a vertex of degree at most 5. By the induction hypothesis, the graph $H := G - v$ has a vertex colouring $c: V(H) \to \{1, \dots, 5\}$. If c uses at most 4 colours for the neighbours of v, we can extend it to a 5-colouring of G. Let us assume, therefore, that v has exactly 5 neighbours, and that these have distinct colours.

Let D be an open disc around v, so small that it meets only those five straight edge segments of G that contain v. Let us enumerate these segments according to their cyclic position in D as $s_1, \dots, s_5$, and let vv_i be the edge containing s_i ($i = 1, \dots, 5$; Fig. 5.1.1). Without loss of generality we may assume that $c(v_i) = i$ for each i.

D

$s_1, \dots, s_5$

$v_1, \dots, v_5$

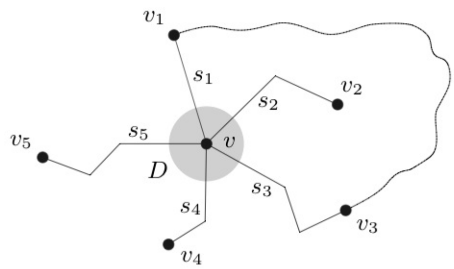

Fig. 5.1.1. The proof of the five colour theorem

Let us show first that every v_1–v_3 path $P \subseteq H$ separates v_2 from v_4 in H. Clearly, this is the case if and only if the cycle $C := vv_1Pv_3v$ separates v_2 from v_4 in G. We prove this by showing that v_2 and v_4 lie in different faces of C.

P

C

Let us pick an inner point x_2 of s_2 in D and an inner point x_4 of s_4 in D. Then in $D \setminus (s_1 \cup s_3) \subseteq \mathbb{R}^2 \setminus C$ every point can be linked by a polygonal arc to x_2 or to x_4. This implies that x_2 and x_4 (and hence also v_2 and v_4) lie in different faces of C: otherwise D would meet only one of the two faces of C, which would contradict the fact that v lies on the frontier of both these faces (Theorem 4.1.1).

Given $i, j \in \{1, \dots, 5\}$, let $H_{i,j}$ be the subgraph of H induced by the vertices coloured i or j. We may assume that the component C_1 of $H_{1,3}$ containing v_1 also contains v_3. Indeed, if we interchange the colours 1 and 3 at all the vertices of C_1, we obtain another 5-colouring of H; if $v_3 \notin C_1$, then v_1 and v_3 are both coloured 3 in this new colouring, and we may assign colour 1 to v. Thus, $H_{1,3}$ contains a v_1–v_3 path P. As shown above, P separates v_2 from v_4 in H. Since $P \cap H_{2,4} = \emptyset$, this means that v_2 and v_4 lie in different components of $H_{2,4}$. In the component containing v_2, we now interchange the colours 2 and 4, thus recolouring v_2 with colour 4. Now v no longer has a neighbour coloured 2, and we may give it this colour. □

$H_{i,j}$

As a backdrop to the two famous theorems above, let us cite another well-known result:

Theorem 5.1.3. (Grötzsch 1959)
Every planar graph not containing a triangle is 3-colourable.

5.2 Colouring vertices

How do we determine the chromatic number of a given graph? How can we *find* a vertex-colouring with as few colours as possible? How does the chromatic number relate to other graph invariants, such as average degree, connectivity or girth?

Straight from the definition of the chromatic number we may derive the following upper bound:

Proposition 5.2.1. *Every graph G with m edges satisfies*

$$\chi(G) \leqslant \tfrac{1}{2} + \sqrt{2m + \tfrac{1}{4}}\,.$$

Proof. Let c be a vertex colouring of G with $k = \chi(G)$ colours. Then G has at least one edge between any two colour classes: if not, we could have used the same colour for both classes. Thus, $m \geqslant \frac{1}{2}k(k-1)$. Solving this inequality for k, we obtain the assertion claimed. □

greedy algorithm

One obvious way to colour a graph G with not too many colours is the following *greedy algorithm*: starting from a fixed vertex enumeration $v_1, \dots, v_n$ of G, we consider the vertices in turn and colour each v_i with the first available colour—e.g., with the smallest positive integer not already used to colour any neighbour of v_i among $v_1, \dots, v_{i-1}$. In this way, we never use more than $\Delta(G) + 1$ colours, even for unfavourable choices of the enumeration $v_1, \dots, v_n$. If G is complete or an odd cycle, then this is even best possible.

In general, though, this upper bound of $\Delta + 1$ is rather generous, even for greedy colourings. Indeed, when we come to colour the vertex v_i in the above algorithm, we only need a supply of $d_{G[v_1,\dots,v_i]}(v_i) + 1$ rather than $d_G(v_i) + 1$ colours to proceed; recall that, at this stage, the algorithm ignores any neighbours v_j of v_i with $j > i$. Hence in most graphs, there will be scope for an improvement of the $\Delta + 1$ bound by choosing a particularly suitable vertex ordering to start with: one that picks vertices of large degree early (when most neighbours are ignored) and vertices of small degree last. Locally, the number $d_{G[v_1,\dots,v_i]}(v_i) + 1$ of colours required will be smallest if v_i has minimum degree in $G[v_1, \dots, v_i]$. But this is easily achieved: we just choose v_n first, with $d(v_n) = \delta(G)$, then choose as v_{n-1} a vertex of minimum degree in $G - v_n$, and so on.

colouring number $\mathrm{col}(G)$

The least number k such that G has a vertex enumeration in which each vertex is preceded by fewer than k of its neighbours is called the *colouring number* $\mathrm{col}(G)$ of G. The enumeration we just discussed shows that $\mathrm{col}(G) \leqslant \max_{H \subseteq G} \delta(H) + 1$. But for $H \subseteq G$ clearly also $\mathrm{col}(G) \geqslant \mathrm{col}(H)$ and $\mathrm{col}(H) \geqslant \delta(H) + 1$, since the 'back-degree' of the last vertex in any enumeration of H is just its ordinary degree in H, which is at least $\delta(H)$. So we have proved the following:

Proposition 5.2.2. *Every graph G satisfies*

$$\chi(G) \leqslant \operatorname{col}(G) = \max\{\,\delta(H) \mid H \subseteq G\,\} + 1\,.$$

□

[7.3]
[9.2.1]
[9.2.3]
[11.2.3]

Corollary 5.2.3. *Every graph G has a subgraph of minimum degree at least $\chi(G) - 1$.* □

The colouring number of a graph is closely related to its arboricity; see the remark following Theorem 2.4.4.

As we have seen, every graph G satisfies $\chi(G) \leqslant \Delta(G) + 1$, with equality for complete graphs and odd cycles. In all other cases, this general bound can be improved a little:

Theorem 5.2.4. (Brooks 1941)
Let G be a connected graph. If G is neither complete nor an odd cycle, then

$$\chi(G) \leqslant \Delta(G)\,.$$

Proof. We apply induction on $|G|$. If $\Delta(G) \leqslant 2$, then G is a path or a cycle, and the assertion is trivial. We therefore assume that $\Delta := \Delta(G) \geqslant 3$, and that the assertion holds for graphs of smaller order. Suppose that $\chi(G) > \Delta$. Δ

Let $v \in G$ be a vertex and $H := G - v$. Then $\chi(H) \leqslant \Delta$: by induction, every component H' of H satisfies $\chi(H') \leqslant \Delta(H') \leqslant \Delta$ unless H' is complete or an odd cycle, in which case $\chi(H') = \Delta(H') + 1 \leqslant \Delta$ as every vertex of H' has maximum degree in H' and one such vertex is also adjacent to v in G. v, H

Since H can be Δ-coloured but G cannot, we have the following:

Every Δ-colouring of H uses all the colours $1, \dots, \Delta$ on the neighbours of v; in particular, $d(v) = \Delta$. (1)

Given any Δ-colouring of H, let us denote the neighbour of v coloured i by v_i, $i = 1, \dots, \Delta$. For all $i \neq j$, let $H_{i,j}$ denote the subgraph of H spanned by all the vertices coloured i or j. $v_1, \dots, v_\Delta$ $H_{i,j}$

For all $i \neq j$, the vertices v_i and v_j lie in a common component $C_{i,j}$ of $H_{i,j}$. (2) $C_{i,j}$

Otherwise we could interchange the colours i and j in one of those components; then v_i and v_j would be coloured the same, contrary to (1).

$C_{i,j}$ is always a v_i–v_j path. (3)

Indeed, let P be a v_i–v_j path in $C_{i,j}$. As $d_H(v_i) \leqslant \Delta - 1$, the neighbours of v_i have pairwise different colours: otherwise we could recolour v_i, contrary to (1). Hence the neighbour of v_i on P is its only neighbour in $C_{i,j}$, and similarly for v_j. Thus if $C_{i,j} \neq P$, then P has an inner vertex with three identically coloured neighbours in H; let u be the first such vertex on P (Fig. 5.2.1). Since at most $\Delta - 2$ colours are used on the neighbours of u, we may recolour u. But this makes $P\mathring{u}$ into a component of $H_{i,j}$, contradicting (2).

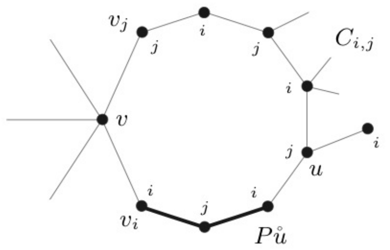

Fig. 5.2.1. The proof of (3) in Brooks's theorem

For distinct i, j, k, the paths $C_{i,j}$ and $C_{i,k}$ meet only in v_i. (4)

For if $v_i \neq u \in C_{i,j} \cap C_{i,k}$, then u has two neighbours coloured j and two coloured k, so we may recolour u. In the new colouring, v_i and v_j lie in different components of $H_{i,j}$, contrary to (2).

$v_1, \dots, v_\Delta$ c u c'

The proof of the theorem now follows easily. If the neighbours of v are pairwise adjacent, then each has Δ neighbours in $N(v) \cup \{v\}$ already, so $G = G[N(v) \cup \{v\}] = K^{\Delta+1}$. As G is complete, there is nothing to show. We may thus assume that $v_1 v_2 \notin G$, where $v_1, \dots, v_\Delta$ derive their names from some fixed Δ-colouring c of H. Let $u \neq v_2$ be the neighbour of v_1 on the path $C_{1,2}$; then $c(u) = 2$. Interchanging the colours 1 and 3 in $C_{1,3}$, we obtain a new colouring c' of H; let v'_i, $H'_{i,j}$, $C'_{i,j}$ etc. be defined with respect to c' in the obvious way. As a neighbour of $v_1 = v'_3$, our vertex u now lies in $C'_{2,3}$, since $c'(u) = c(u) = 2$. By (4) for c, however, the path $\mathring{v}_1 C_{1,2}$ retained its original colouring, so $u \in \mathring{v}_1 C_{1,2} \subseteq C'_{1,2}$. Hence $u \in C'_{2,3} \cap C'_{1,2}$, contradicting (4) for c'. □

As we have seen, a graph G of large chromatic number must have large maximum degree and colouring number, both at least $\chi(G) - 1$. What else does large $\chi(G)$ imply about G, in terms of other invariants or in structural terms? Are there, as in Kuratowski's theorem, some 'canonical' highly chromatic types of subgraph that must occur in every graph of large enough chromatic number?

One obvious possible cause for larger chromatic number, $\chi(G) \geqslant k$ say, is the presence of a K^k subgraph. This is a local property of G, compatible with arbitrary values of global invariants such as ε and κ. Hence, an assumption of $\chi(G) \geqslant k$ does not tell us anything about those invariants for G itself. It does, however, imply the existence of a subgraph where those invariants are large: by Corollary 5.2.3, G has a subgraph H with $\delta(H) \geqslant k-1$, and hence by Theorem 1.4.3, also a subgraph H' with $\kappa(H') \geqslant \lceil \frac{1}{4}k \rceil$.

But the presence of just any such subgraph is not equivalent to $\chi(G)$ being large, not even in a weak qualitative sense: as complete bipartite graphs show, no assumption of high[1] values of δ or κ alone can force χ to exceed 2, let alone to get arbitrarily large.

In particular, the collection of graphs of minimum degree at least $k-1$ or connectivity at least $\lceil \frac{1}{4}k \rceil$ cannot, as a whole, play the role of an easily identifiable Kuratowski-type set of minimal k-chromatic graphs. It may have a subclass that can. But no such set can be finite. Indeed, the following fundamental theorem of Erdős implies that for no k does there exist a finite set $\mathcal{H}$ of graphs of chromatic number at least 3 such that every graph of chromatic number at least k has a subgraph in $\mathcal{H}$:

Theorem 5.2.5. (Erdős 1959) [9.2.3]
For every integer k there exists a graph G with girth $g(G) > k$ and chromatic number $\chi(G) > k$.

Theorem 5.2.5 was first proved non-constructively using random graphs, and we shall give this proof in Chapter 11.2. Constructing graphs of large chromatic number and girth directly is not easy; cf. Exercise 24 for the simplest case.

The message of Erdős's theorem is that, contrary to our initial guess, large chromatic number can occur as a purely global phenomenon: note that locally, around each vertex, a graph of large girth looks just like a tree, and in particular is 2-colourable there. But what exactly can cause high chromaticity as a global phenomenon remains a mystery.

Nevertheless, there exists a simple—though not always short—procedure to construct all the graphs of chromatic number at least k. For each $k \in \mathbb{N}$, let us define the class of *k-constructible* graphs recursively as follows: *k-constructible*

(i) K^k is k-constructible.

(ii) If G is k-constructible and two vertices x, y of G are non-adjacent, then also $(G+xy)/xy$ is k-constructible.

(iii) If G_1, G_2 are k-constructible and there are vertices x, y_1, y_2 such that $G_1 \cap G_2 = \{x\}$ and $xy_1 \in E(G_1)$ and $xy_2 \in E(G_2)$, then also $(G_1 \cup G_2) - xy_1 - xy_2 + y_1y_2$ is k-constructible (Fig. 5.2.2).

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[1] High in absolute terms. In Chapter 7 we shall study the effect of edge densities that let ε get large also relative to the order of the graph. That is a much stronger assumption.

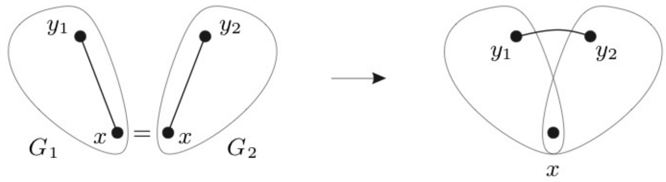

Fig. 5.2.2. The *Hajós construction* (iii)

One easily checks inductively that all k-constructible graphs—and hence their supergraphs—are at least k-chromatic. For example, any colouring of the graph $(G+xy)/xy$ in (ii) induces a colouring of G, and hence by inductive assumption uses at least k colours. Similarly, in any colouring of the graph constructed in (iii) the vertices y_1 and y_2 do not both have the same colour as x, so this colouring induces a colouring of either G_1 or G_2 and hence uses at least k colours.

It is remarkable, though, that the converse holds too:

Theorem 5.2.6. (Hajós 1961)
Let G be a graph and $k \in \mathbb{N}$. Then $\chi(G) \geqslant k$ if and only if G has a k-constructible subgraph.

Proof. Let G be a graph with $\chi(G) \geqslant k$; we show that G has a k-constructible subgraph. Suppose not; then $k \geqslant 3$. Adding some edges if necessary, let us make G edge-maximal with the property that none of its subgraphs is k-constructible. Now G is not a complete r-partite graph for any r: for then $\chi(G) \geqslant k$ would imply $r \geqslant k$, and G would contain the k-constructible graph K^k.

Since G is not a complete multipartite graph, non-adjacency is not an equivalence relation on $V(G)$. So there are vertices y_1, x, y_2 such that
x, y_1, y_2
$y_1x, xy_2 \notin E(G)$ but $y_1y_2 \in E(G)$. Since G is edge-maximal without a k-constructible subgraph, each edge xy_i lies in some k-constructible
H_1, H_2
subgraph H_i of $G+xy_i$ $(i=1,2)$.

H_2'
v' *etc.*

Let H_2' be an isomorphic copy of H_2 that contains x and $H_2 - H_1$ but is otherwise disjoint from G, together with an isomorphism $v \mapsto v'$ from H_2 to H_2' that fixes $H_2 \cap H_2'$ pointwise. Then $H_1 \cap H_2' = \{x\}$, so

$$H := (H_1 \cup H_2') - xy_1 - xy_2' + y_1y_2'$$

is k-constructible by (iii). One vertex at a time, let us identify in H each vertex $v' \in H_2' - G$ with its partner v; since vv' is never an edge of H, each of these identifications amounts to a construction step of type (ii). Eventually, we obtain the graph

$$(H_1 \cup H_2) - xy_1 - xy_2 + y_1y_2 \subseteq G\,;$$

this is the desired k-constructible subgraph of G. □

Does Hajós's theorem solve our Kuratowski-type problem for highly chromatic graphs, which was to find a class of graphs of chromatic number at least k with the property that every such graph has a subgraph in this class? Formally, it does—albeit with an infinite characterizing set (the set of k-constructible graphs). Unlike Kuratowski's characterization of planar graphs, however, this does not—at least not obviously—make Hajós's theorem a 'good characterization' of the graphs of chromatic number $< k$, in the sense of complexity theory. See the notes for details.

5.3 Colouring edges

Clearly, every graph G satisfies $\chi'(G) \geqslant \Delta(G)$. For bipartite graphs, we have equality here:

Proposition 5.3.1. (König 1916) [5.4.5]
Every bipartite graph G satisfies $\chi'(G) = \Delta(G)$.

Proof. We apply induction on $\|G\|$. For $\|G\| = 0$ the assertion holds. (1.6.1)
Now assume that $\|G\| \geqslant 1$, and that the assertion holds for graphs with fewer edges. Let $\Delta := \Delta(G)$, pick an edge $xy \in G$, and choose a Δ-edge-colouring of $G - xy$ by the induction hypothesis. Let us refer to the edges coloured α as *α-edges*, etc. Δ, xy α-edge

In $G - xy$, each of x and y is incident with at most $\Delta - 1$ edges. Hence there are $\alpha, \beta \in \{1, \dots, \Delta\}$ such that x is not incident with an α-edge and y is not incident with a β-edge. If $\alpha = \beta$, we can colour the edge xy with this colour and are done; so we may assume that $\alpha \neq \beta$, and that x is incident with a β-edge. α, β

Let us extend this edge to a maximal walk W from x whose edges are coloured β and α alternately. Since no such walk contains a vertex twice (why not?), W exists and is a path. Moreover, W does not contain y: if it did, it would end in y on an α-edge (by the choice of β) and thus have even length, so $W + xy$ would be an odd cycle in G (cf. Proposition 1.6.1). We now recolour all the edges on W, swapping α with β. By the choice of α and the maximality of W, adjacent edges of $G - xy$ are still coloured differently. We have thus found a Δ-edge-colouring of $G - xy$ in which neither x nor y is incident with a β-edge. Colouring xy with β, we extend this colouring to a Δ-edge-colouring of G. □

Theorem 5.3.2. (Vizing 1964)
Every graph G satisfies

$$\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G) + 1 .$$

Proof. We prove the second inequality by induction on $\|G\|$. For $\|G\| = 0$ it is trivial. For the induction step let $G = (V, E)$ with $\Delta := \Delta(G) > 0$ be V, E Δ

given, and assume that the assertion holds for graphs with fewer edges. *colouring* Instead of '$(\Delta+1)$-edge-colouring' let us just say 'colouring'.

For every edge $e \in G$ there exists a colouring of $G - e$, by the induction hypothesis. In such a colouring, the edges at a given vertex v use at most $d(v) \leqslant \Delta$ colours, so some colour $\beta \in \{1, \dots, \Delta+1\}$ is *missing* *missing at v*. For any other colour α, there is a unique maximal walk (possibly trivial) starting at v, whose edges are coloured alternately α *α/β-path* and β. This walk is a path; we call it the *α/β-path from v*.

Suppose that G has no colouring. Then the following holds:

Given $xy \in E$, and any colouring of $G - xy$ in which the colour α is missing at x and the colour β is missing at y, the α/β-path from y ends in x. (1)

Otherwise we could interchange the colours α and β along this path and colour xy with α, obtaining a colouring of G (contradiction).

Let $xy_0 \in G$ be an edge. By induction, $G_0 := G - xy_0$ has a α colouring c_0. Let α be a colour missing at x in this colouring. Further, let $y_0, \dots, y_k$ be a maximal sequence of distinct neighbours of x in G such that $c_0(xy_{i+1})$ is missing in c_0 at y_i for every $i < k$. For each of G_i the graphs $G_i := G - xy_i$ we define a colouring c_i, setting

c_i
$$c_i(e) := \begin{cases} c_0(xy_{j+1}) & \text{for } e = xy_j \text{ with } j \in \{0, \dots, i-1\} \\ c_0(e) & \text{otherwise;} \end{cases}$$

note that in each of these colourings the same colours are missing at x as in c_0.

β Now let β be a colour missing at y_k in c_0. By (1), the α/β-path P from y_k in G_k (with respect to c_k) ends in x, with an edge yx coloured β since α is missing at x. Since y cannot serve as y_{k+1}, by the maximality of the sequence $y_0, \dots, y_k$, we thus have $y = y_i$ for some $0 \leqslant i < k$ (Fig. 5.3.1). By definition of c_k, therefore, $\beta = c_k(xy_i) = c_0(xy_{i+1})$. By

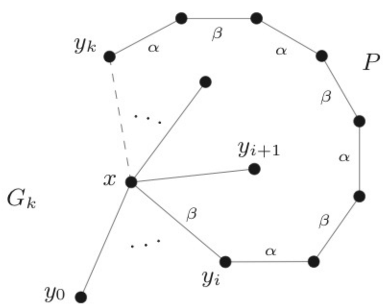

Fig. 5.3.1. The α/β-path P in $G_k = G - xy_k$

the choice of y_{i+1} this means that β was missing at y_i in c_0, and hence also in c_i. Now the α/β-path P' from y_i in G_i starts with $y_i P y_k$, since the edges of $P\mathring{x}$ are coloured the same in c_i as in c_k. But in c_0, and hence in c_i, there is no edge at y_k coloured β. Therefore P' ends in y_k, contradicting (1). □

Vizing's theorem divides the finite graphs into two classes according to their chromatic index; graphs satisfying $\chi' = \Delta$ are called (imaginatively) *class 1*, those with $\chi' = \Delta + 1$ are *class 2*.

5.4 List colouring

In this section, we take a look at a relatively recent generalization of the concepts of colouring studied so far. This generalization may seem a little far-fetched at first glance, but it turns out to supply a fundamental link between the classical (vertex and edge) chromatic numbers of a graph and its other invariants.

Suppose we are given a graph $G = (V, E)$, and for each vertex of G a list of colours permitted at that particular vertex: when can we colour G (in the usual sense) so that each vertex receives a colour from its list? More formally, let $(S_v)_{v \in V}$ be a family of sets. We call a vertex colouring c of G with $c(v) \in S_v$ for all $v \in V$ a colouring *from the lists* S_v. The graph G is called *k-list-colourable*, or *k-choosable*, if, for every family $(S_v)_{v \in V}$ with $|S_v| = k$ for all v, there is a vertex colouring of G from the lists S_v. The least integer k for which G is k-choosable is the *list-chromatic number*, or *choice number* $\mathrm{ch}(G)$ of G.

k-choosable

choice number $\mathrm{ch}(G)$

List-colourings of edges are defined analogously. The least integer k such that G has an edge colouring from any family of lists of size k is the *list-chromatic index* $\mathrm{ch}'(G)$ of G; formally, we just set $\mathrm{ch}'(G) := \mathrm{ch}(L(G))$, where $L(G)$ is the line graph of G.

$\mathrm{ch}'(G)$

In principle, showing that a given graph is k-choosable is more difficult than proving it to be k-colourable: the latter is just the special case of the former where all lists are equal to $\{1, \dots, k\}$. Thus,

$$\mathrm{ch}(G) \geqslant \chi(G) \quad \text{and} \quad \mathrm{ch}'(G) \geqslant \chi'(G)$$

for all graphs G.

In spite of these inequalities, many of the known upper bounds for the chromatic number have turned out to be valid for the choice number, too. Examples for this phenomenon include Brooks's theorem and Proposition 5.2.2; in particular, graphs of large choice number still have subgraphs of large minimum degree. On the other hand, it is easy to construct graphs for which the two invariants are wide apart (Exercise 26).