Lecture Notes on

GRAPH THEORY

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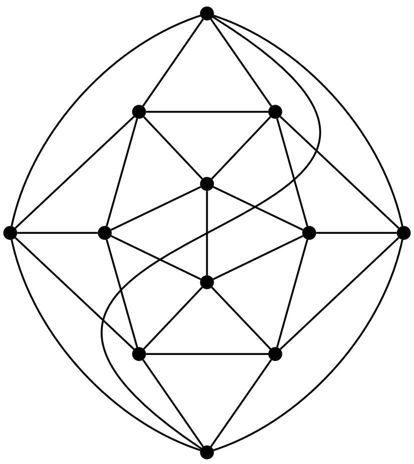

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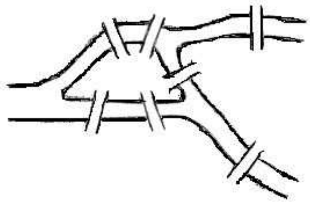
Introduction

Graph theory can be said to have its beginning in 1736 when EULER considered the (general case of the) **Königsberg bridge problem**: Is there a walking route that crosses each of the seven bridges of Königsberg exactly once? (Solutio Problematis ad geometriam situs pertinentis, *Commentarii Academiae Scientiarum Imperialis Petropolitanae* 8 (1736), pp. 128-140.)



It took 200 years before the first book on graph theory was written. This was done by KÖNIG in 1936. ("Theorie der endlichen und unendlichen Graphen", Teubner, Leipzig, 1936. Translation in English, 1990.) Since then graph theory has developed into an extensive and popular branch of mathematics, which has been applied to many problems in mathematics, computer science, and other scientific and not-so-scientific areas. For the history of early graph theory, see

N.L. BIGGS, R.J. LLOYD AND R.J. WILSON, "Graph Theory 1736 – 1936", Clarendon Press, 1986.

There seem to be no standard notations or even definitions for graph theoretical objects. This is natural, because the names one uses for these objects reflect the applications. So, for instance, if we consider a communications network (say, for email) as a graph, then the computers, which take part in this network, are called nodes rather than vertices or points. On the other hand, other names are used for molecular structures in chemistry, flow charts in programming, human relations in social sciences, and so on.

These lectures study *finite graphs* and majority of the topics is included in

J.A. BONDY AND U.S.R. MURTY, "Graph Theory with Applications", Macmillan, 1978.

R. DIESTEL, "Graph Theory", Springer-Verlag, 1997.

F. HARARY, "Graph Theory", Addison-Wesley, 1969.

D.B. WEST, "Introduction to Graph Theory", Prentice Hall, 1996.

R.J. WILSON, "Introduction to Graph Theory", Longman, (3rd ed.) 1985.

In these lectures we study *combinatorial aspects* of graphs. For more *algebraic* topics and methods, see

N. BIGGS, "Algebraic Graph Theory", Cambridge University Press, (2nd ed.) 1993.
and for *computational aspects*, see

S. EVEN, "Graph Algorithms", Computer Science Press, 1979.

In these lecture notes we mention several open problems that have gained respect among the researchers. Indeed, graph theory has the advantage that it contains easily formulated open problems that can be stated early in the theory. Finding a solution to any one of these problems is on another layer of difficulty.

Sections with a star ($*$) in their heading are optional.

Notations and notions

• For a finite set X, $|X|$ denotes its size (cardinality, the number of its elements).
• Let
$$[1, n] = \{1, 2, \dots, n\},$$
and in general,
$$[i, n] = \{i, i+1, \dots, n\}$$
for integers $i \le n$.
• For a real number x, the **floor** and the **ceiling** of x are the integers
$$\lfloor x \rfloor = \max\{k \in \mathbb{Z} \mid k \le x\} \text{ and } \lceil x \rceil = \min\{k \in \mathbb{Z} \mid x \le k\}.$$
• A family $\{X_1, X_2, \dots, X_k\}$ of subsets $X_i \subseteq X$ of a set X is a **partition** of X, if
$$X = \bigcup_{i \in [1,k]} X_i \quad \text{and} \quad X_i \cap X_j = \emptyset \text{ for all different } i \text{ and } j \; .$$
• For two sets X and Y,
$$X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$$
is their **Cartesian product**.
• For two sets X and Y,
$$X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$$
is their **symmetric difference**. Here $X \setminus Y = \{x \mid x \in X, x \notin Y\}$.
• Two numbers $n, k \in \mathbb{N}$ (often $n = |X|$ and $k = |Y|$ for sets X and Y) have the **same parity**, if both are even, or both are odd, that is, if $n \equiv k \pmod 2$. Otherwise, they have opposite parity.

Graph theory has abundant examples of **NP-complete problems**. Intuitively, a problem is in P [1] if there is an efficient (practical) algorithm to find a solution to it. On the other hand, a problem is in NP [2], if it is first efficient to guess a solution and then efficient to check that this solution is correct. It is conjectured (and not known) that P $\neq$ NP. This is one of the great problems in modern mathematics and theoretical computer science. If the guessing in NP-problems can be replaced by an efficient systematic search for a solution, then P=NP. For any one NP-complete problem, if it is in P, then necessarily P=NP.

[1] Solvable – by an algorithm – in polynomially many steps on the size of the problem instances.
[2] Solvable *nondeterministically* in polynomially many steps on the size of the problem instances.

1.1 Graphs and their plane figures

Let V be a *finite* set, and denote by

$$E(V) = \{\{u, v\} \mid u, v \in V,\ u \neq v\}\ .$$

the subsets of V of two distinct elements.

DEFINITION. A pair $G = (V, E)$ with $E \subseteq E(V)$ is called a **graph** (**on** V). The elements of V are the **vertices**, and those of E the **edges** of the graph. The vertex set of a graph G is denoted by V_G and its edge set by E_G. Therefore $G = (V_G, E_G)$.

In literature, graphs are also called *simple graphs*; vertices are called *nodes* or *points*; edges are called *lines* or *links*. The list of alternatives is long (but still finite).

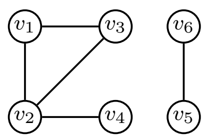
A pair $\{u, v\}$ is usually written simply as uv. Notice that then $uv = vu$. In order to simplify notations, we also write $v \in G$ instead of $v \in V_G$.

DEFINITION. For a graph G, we denote

$$\nu_G = |V_G| \ \text{ and } \ \varepsilon_G = |E_G|\ .$$

The number ν_G of the vertices is called the **order** of G, and ε_G is the **size** of G. For an edge $e = uv \in E_G$, the vertices u and v are its **ends**. Vertices u and v are **adjacent** or **neighbours**, if $e = uv \in E_G$. Two edges $e_1 = uv$ and $e_2 = uw$ having a common end, are **adjacent** with each other.

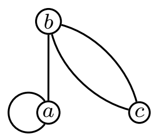
A graph G can be represented as a plane figure by drawing a line (or a curve) between the points u and v (representing vertices) if $e = uv$ is an edge of G. The figure on the right is a drawing of the graph G with $V_G = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E_G = \{v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_5v_6\}$.



Often we shall omit the identities (names v) of the vertices in our figures, in which case the vertices are drawn as anonymous circles.

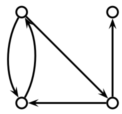
Graphs can be generalized by allowing **loops** vv and **parallel** (or **multiple**) **edges** between vertices to obtain a **multigraph** $G = (V, E, \psi)$, where $E = \{e_1, e_2, \ldots, e_m\}$ is a set (of symbols), and $\psi\colon E \to E(V) \cup \{vv \mid v \in V\}$ is a function that attaches an unordered pair of vertices to each $e \in E$: $\psi(e) = uv$.

Note that we can have $\psi(e_1) = \psi(e_2)$. This is drawn in the figure of G by placing two (parallel) edges that connect the common ends. On the right there is (a drawing of) a multigraph G with vertices $V = \{a, b, c\}$ and edges $\psi(e_1) = aa$, $\psi(e_2) = ab$, $\psi(e_3) = bc$, and $\psi(e_4) = bc$.



Later we concentrate on (simple) graphs.

DEFINITION. We also study **directed graphs** or **digraphs** $D = (V, E)$, where the edges have a direction, that is, the edges are ordered: $E \subseteq V \times V$. In this case, $uv \neq vu$.



The directed graphs have representations, where the edges are drawn as arrows. A digraph can contain edges uv and vu of *opposite directions*.

Graphs and digraphs can also be coloured, labelled, and weighted:

DEFINITION. A function $\alpha\colon V_G \to K$ is a **vertex colouring** of G by a set K of colours. A function $\alpha\colon E_G \to K$ is an **edge colouring** of G. Usually, $K = [1, k]$ for some $k \geq 1$.

If $K \subseteq \mathbb{R}$ (often $K \subseteq \mathbb{N}$), then α is a **weight function** or a **distance function**.

Isomorphism of graphs

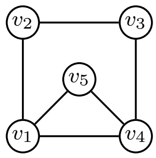
DEFINITION. Two graphs G and H are **isomorphic**, denoted by $G \cong H$, if there exists a bijection $\alpha\colon V_G \to V_H$ such that

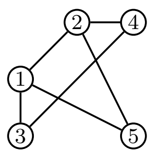
$$uv \in E_G \iff \alpha(u)\alpha(v) \in E_H$$

for all $u, v \in G$.

Hence G and H are isomorphic if the vertices of H are renamings of those of G. Two isomorphic graphs enjoy the same graph theoretical properties, and *they are often identified*. In particular, all isomorphic graphs have the same plane figures (excepting the identities of the vertices). This shows in the figures, where we tend to replace the vertices by small circles, and talk of 'the graph' although there are, in fact, infinitely many of such graphs.

Example 1.1. The following graphs are isomorphic. Indeed, the required isomorphism is given by $v_1 \mapsto 1$, $v_2 \mapsto 3$, $v_3 \mapsto 4$, $v_4 \mapsto 2$, $v_5 \mapsto 5$.





Isomorphism Problem. *Does there exist an efficient algorithm to check whether any two given graphs are isomorphic or not?*

The following table lists the number $2^{\binom{n}{2}}$ of graphs on a given set of n vertices, and the number of nonisomorphic graphs on n vertices. It tells that at least for computational purposes an efficient algorithm for checking whether two graphs are isomorphic or not would be greatly appreciated.

n	1	2	3	4	5	6	7	8	9
graphs	1	2	8	64	1024	32 768	2 097 152	268 435 456	$2^{36} > 6 \cdot 10^{10}$
nonisomorphic	1	2	4	11	34	156	1044	12 346	274 668

Other representations

Plane figures catch graphs for our eyes, but if a problem on graphs is to be *programmed*, then these figures are (to say the least) unsuitable. Matrices of integers are ideal for computers, since every respectable programming language has array structures for these, and computers are good in crunching numbers.

Let $V_G = \{v_1, \dots, v_n\}$ be ordered. The **adjacency matrix** of G is the $n \times n$-matrix M with entries $M_{ij} = 1$ or $M_{ij} = 0$ according to whether $v_i v_j \in E_G$ or not. For instance, the graphs of Example 1.1 has an adjacency matrix on the right. Notice that the adjacency matrix is always symmetric (with respect to its diagonal consisting of zeros).

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

A graph has usually many different adjacency matrices, one for each ordering of its set V_G of vertices. The following result is obvious from the definitions.

Theorem 1.1. *Two graphs G and H are isomorphic if and only if they have a common adjacency matrix. Moreover, two isomorphic graphs have exactly the same set of adjacency matrices.*

Graphs can also be represented by sets. For this, let $\mathcal{X} = \{X_1, X_2, \dots, X_n\}$ be a family of subsets of a set X, and define the **intersection graph** $G_{\mathcal{X}}$ as the graph with vertices $X_1, \dots, X_n$, and edges $X_i X_j$ for all i and j ($i \neq j$) with $X_i \cap X_j \neq \emptyset$.

Theorem 1.2. *Every graph is an intersection graph of some family of subsets.*

Proof. Let G be a graph, and define, for all $v \in G$, a set

$$X_v = \{\{v, u\} \mid vu \in E_G\}.$$

Then $X_u \cap X_v \neq \emptyset$ if and only if $uv \in E_G$. □

Let $s(G)$ be the smallest size of a base set X such that G can be represented as an intersection graph of a family of subsets of X, that is,

$$s(G) = \min\{|X| \mid G \cong G_{\mathcal{X}} \text{ for some } \mathcal{X} \subseteq 2^X\} .$$

How small can $s(G)$ be compared to the order ν_G (or the size ε_G) of the graph? It was shown by KOU, STOCKMEYER AND WONG (1976) that it is algorithmically difficult to determine the number $s(G)$ – the problem is NP-complete.

Example 1.2. As yet another example, let $A \subseteq \mathbb{N}$ be a finite set of natural numbers, and let $G_A = (A, E)$ be the graph defined on $V_{G_A} = A$ such that $rs \in E$ $(= E_{G_A})$ if and only if r and s (for $r \neq s$) have a common divisor > 1. As an exercise, we state: *All graphs can be represented in the form G_A for some set A of natural numbers.*

1.2 Subgraphs

Ideally, in a problem the local properties of a graph determine a solution. In such a situation we deal with (small) parts of the graph (subgraphs), and a solution can be found to the problem by combining the information determined by the parts. For instance, as we shall see later on, the existence of an Euler tour is very local, it depends only on the number of the neighbours of the vertices.

Degrees of vertices

DEFINITION. Let $v \in G$ be a vertex a graph G. The **neighbourhood** of v is the set

$$N_G(v) = \{u \in G \mid vu \in E_G\} .$$

The **degree** of v is the number of its neighbours:

$$d_G(v) = |N_G(v)| .$$

If $d_G(v) = 0$, then v is said to be **isolated** in G, and if $d_G(v) = 1$, then v is a **leaf** of the graph. The **minimum degree** and the **maximum degree** of G are defined as

$$\delta(G) = \min\{d_G(v) \mid v \in G\} \quad \text{and} \quad \Delta(G) = \max\{d_G(v) \mid v \in G\} .$$

The following lemma, due to EULER (1736), tells that if several people shake hands, then the number of hands shaken is even.

Lemma 1.1 (Handshaking lemma). *For each graph G,*

$$\sum_{v \in G} d_G(v) = 2 \cdot \varepsilon_G .$$

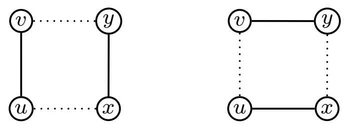
Moreover, the number of vertices of odd degree is even.

Proof. Every edge $e \in E_G$ has two ends. The second claim follows immediately from the first one. □

Lemma 1.1 holds equally well for multigraphs, when $d_G(v)$ is defined as the number of edges that have v as an end, and when *a loop vv is counted twice.*

Note that the degrees of a graph G do not determine G. Indeed, there are graphs $G = (V, E_G)$ and $H = (V, E_H)$ on the same set of vertices that are *not* isomorphic, but for which $d_G(v) = d_H(v)$ for all $v \in V$.

DEFINITION. For a graph G, a 2-**switch** with respect to the edges $uv, xy \in E_G$ with $ux, vy \notin E_G$ replaces the edges uv and xy by ux and vy. Denote $G \xrightarrow{2s} H$ if there exists a finite sequence of 2-switches that carries G to H.

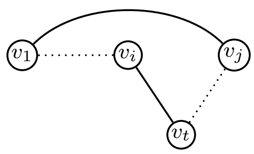


Note that if $G \xrightarrow{2s} H$ then also $H \xrightarrow{2s} G$ if we can apply the sequence in reverse order.

Before proving Berge's switching theorem we need the following tool.

Lemma 1.2. *Let G be a graph of order n with a degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$, where $d_G(v_i) = d_i$. Then there is a graph G' such that $G \xrightarrow{2s} G'$ with $N_{G'}(v_1) = \{v_2, \ldots, v_{d_1+1}\}$.*

Proof. Denote $d = \Delta(G)$ $(= d_1)$. Suppose that there exists a vertex v_i with $2 \leq i \leq d+1$ such that $v_1v_i \notin E_G$. Since $d_G(v_1) = d$, there exists a v_j with $j \geq d+2$ such that $v_1v_j \in E_G$. Here $d_i \geq d_j$, since $j > i$. Since $v_1v_j \in E_G$, there exists a v_t $(2 \leq t \leq n)$ such that $v_iv_t \in E_G$, but $v_jv_t \notin E_G$. We can now perform a 2-switch with respect to the vertices v_1, v_j, v_i, v_t. This gives a new graph H, where $v_1v_i \in E_H$ and $v_1v_j \notin E_H$, and the other neighbours of v_1 remain to be its neighbours.



When we repeat this process for all indices i with $v_1v_i \notin E_G$ for $2 \leq i \leq d+1$, we obtain a graph G' as required. □

Theorem 1.3 (BERGE (1973)). *Two graphs G and H on a common vertex set V satisfy $d_G(v) = d_H(v)$ for all $v \in V$ if and only if H can be obtained from G by a sequence of 2-switches.*

Proof. If $G \xrightarrow{2s} H$, then clearly H has the same degrees as G.

In converse, we use induction on the order ν_G. Let G and H have the same degrees. By Lemma 1.2, we have a vertex v and graph G' and H' such that $G \xrightarrow{2s} G'$ and $H \xrightarrow{2s} H'$ with $N_{G'}(v) = N_{H'}(v)$. Now the graphs $G'-v$ and $H'-v$ have the same degrees. By the induction hypothesis, $G'-v \xrightarrow{2s} H'-v$, and thus also $G' \xrightarrow{2s} H'$. Finally, we observe that also $H' \xrightarrow{2s} H$ by the 'reverse 2-switches', and this proves the claim. □

DEFINITION. Let $d_1, d_2, \ldots, d_n$ be a descending sequence of nonnegative integers, that is, $d_1 \geq d_2 \geq \cdots \geq d_n$. Such a sequence is said to be **graphical**, if there exists a graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ such that $d_i = d_G(v_i)$ for all i.

Using the next result recursively one can decide whether a sequence of integers is graphical or not.

Theorem 1.4 (HAVEL **(1955),** HAKIMI **(1962)).** *A sequence $d_1, d_2, \ldots, d_n$ (with $d_1 \geq 1$ and $n \geq 2$) is graphical if and only if*

$$d_2 - 1,\ d_3 - 1, \ldots, d_{d_1+1} - 1,\ d_{d_1+2},\ d_{d_1+3}, \ldots, d_n. \tag{1.1}$$

is graphical (when put into nonincreasing order).

Proof. ($\Leftarrow$) Consider G of order $n-1$ with vertices (and degrees)

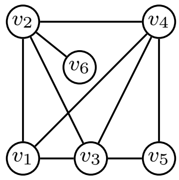
$$d_G(v_2) = d_2 - 1, \ldots, d_G(v_{d_1+1}) = d_{d_1+1} - 1,$$
$$d_G(v_{d_1+2}) = d_{d_1+2}, \ldots, d_G(v_n) = d_n$$

as in (1.1). Add a new vertex v_1 and the edges v_1v_i for all $i \in [2, d_{d_1+1}]$. Then in this new graph H, $d_H(v_1) = d_1$, and $d_H(v_i) = d_i$ for all i.

($\Rightarrow$) Assume $d_G(v_i) = d_i$. By Lemma 1.2 and Theorem 1.3, we can suppose that $N_G(v_1) = \{v_2, \ldots, v_{d_1+1}\}$. But now the degree sequence of $G{-}v_1$ is in (1.1). $\square$

Example 1.3. Consider the sequence $s = 4, 4, 4, 3, 2, 1$. By Theorem 1.4,

$$\begin{aligned} s \text{ is graphical} \iff\ & 3, 3, 2, 1, 1 \text{ is graphical} \\ & 2, 1, 1, 0 \text{ is graphical} \\ & 0, 0, 0 \text{ is graphical}. \end{aligned}$$

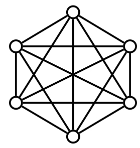


The last sequence corresponds to a discrete graph $\overline{K}_3$, and hence also our original sequence s is graphical. Indeed, the graph G on the right has this degree sequence.

Special graphs

DEFINITION. A graph $G = (V, E)$ is **trivial**, if it has only one vertex, *i.e.*, $\nu_G = 1$; otherwise G is **nontrivial**.

The graph $G = K_V$ is the **complete graph** on V, if every two vertices are adjacent: $E = E(V)$. All complete graphs of order n are isomorphic with each other, and they will be denoted by K_n.



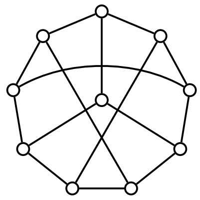
K_6

The **complement** of G is the graph $\overline{G}$ on V_G, where $E_{\overline{G}} = \{e \in E(V) \mid e \notin E_G\}$. The complements $G = \overline{K}_V$ of the complete graphs are called **discrete graphs**. In a discrete graph $E_G = \emptyset$. Clearly, all discrete graphs of order n are isomorphic with each other.

A graph G is said to be **regular**, if every vertex of G has the same degree. If this degree is equal to r, then G is r-**regular** or **regular of degree** r.

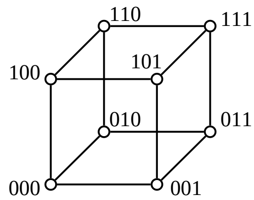
Note that a discrete graph is 0-regular, and a complete graph K_n is $(n-1)$-regular. In particular, $\varepsilon_{K_n} = n(n-1)/2$, and therefore $\varepsilon_G \le n(n-1)/2$ for all graphs G that have order n.

Example 1.4. The graph on the right is the **Petersen graph** that we will meet several times (drawn differently). It is a 3-regular graph of order 10.



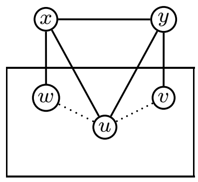
Example 1.5. Let $k \ge 1$ be an integer, and consider the set $\mathbb{B}^k$ of all binary strings of length k. For instance, $\mathbb{B}^3 = \{000, 001, 010, 100, 011, 101, 110, 111\}$. Let Q_k be the graph, called the k-**cube**, with $V_{Q_k} = \mathbb{B}^k$, where $uv \in E_{Q_k}$ if and only if the strings u and v differ in exactly one place.

The order of Q_k is $\nu_{Q_k} = 2^k$, the number of binary strings of length k. Also, Q_k is k-regular, and so, by the handshaking lemma, $\varepsilon_{Q_k} = k \cdot 2^{k-1}$. On the right we have the 3-cube, or simply the cube.



Example 1.6. Let $n \ge 4$ be any even number. We show by induction that there exists a 3-regular graph G with $\nu_G = n$. Notice that all 3-regular graphs have even order by the handshaking lemma.

If $n = 4$, then K_4 is 3-regular. Let G be a 3-regular graph of order $2m-2$, and suppose that $uv, uw \in E_G$. Let $V_H = V_G \cup \{x, y\}$, and $E_H = (E_G \setminus \{uv, uw\}) \cup \{ux, xv, uy, yw, xy\}$. Then H is 3-regular of order $2m$.



Subgraphs

DEFINITION. A graph H is a **subgraph** of a graph G, denoted by $H \subseteq G$, if $V_H \subseteq V_G$ and $E_H \subseteq E_G$. A subgraph $H \subseteq G$ **spans** G (and H is a **spanning subgraph** of G), if every vertex of G is in H, i.e., $V_H = V_G$.

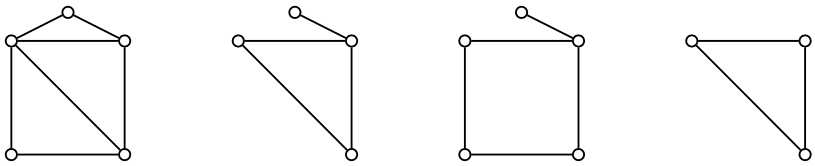
Also, a subgraph $H \subseteq G$ is an **induced subgraph**, if $E_H = E_G \cap E(V_H)$. In this case, H is **induced** by its set V_H of vertices.

In an induced subgraph $H \subseteq G$, the set E_H of edges consists of all $e \in E_G$ such that $e \in E(V_H)$. To each nonempty subset $A \subseteq V_G$, there corresponds a unique induced subgraph

$$G[A] = (A, E_G \cap E(A)) .$$

To each subset $F \subseteq E_G$ of edges there corresponds a unique spanning subgraph of G,

$$G[F] = (V_G, F) \ .$$



G subgraph spanning induced

For a set $F \subseteq E_G$ of edges, let

$$G{-}F = G[E_G \setminus F]$$

be the subgraph of G obtained by removing (only) the edges $e \in F$ from G. In particular, $G{-}e$ is obtained from G by removing $e \in E_G$.

Similarly, we write $G + F$, if each $e \in F$ (for $F \subseteq E(V_G)$) is added to G.

For a subset $A \subseteq V_G$ of vertices, we let $G{-}A \subseteq G$ be the subgraph induced by $V_G \setminus A$, that is,

$$G{-}A = G[V_G \setminus A] \ ,$$

and, *e.g.*, $G{-}v$ is obtained from G by removing the vertex v together with the edges that have v as their end.

Many problems concerning (induced) subgraphs are algorithmically difficult. For instance, to find a maximal complete subgraph (a subgraph K_m of maximum order) of a graph is unlikely to be even in NP.

Reconstruction Problem. The famous open problem, **Kelly-Ulam problem** or the **Reconstruction Conjecture**, states that *a graph of order at least* 3 *is determined up to isomorphism by its vertex deleted subgraphs* $G{-}v$ *(*$v \in G$*)*: if there exists a bijection $\alpha\colon V_G \to V_H$ such that $G{-}v \cong H{-}\alpha(v)$ for all v, then $G \cong H$.

1.3 Paths and cycles

The most fundamental notions in graph theory are practically oriented. Indeed, many graph theoretical questions ask for optimal solutions to problems such as: find a shortest path (in a complex network) from a given point to another. This kind of problems can be difficult, or at least nontrivial, because there are usually choices what branch to choose when leaving an intermediate point.

Walks

DEFINITION. Let $e_i = u_i u_{i+1} \in E_G$ be edges of G for $i \in [1, k]$. Here e_i and e_{i+1} are compatible in the sense that e_i is adjacent to e_{i+1} for all $i \in [1, k-1]$. The sequence

$$W = e_1 e_2 \dots e_k$$

is a **walk of length** k **from** u_1 **to** u_{k+1}.

We write, more informally,

$$W\colon u_1 \longrightarrow u_2 \longrightarrow \dots \longrightarrow u_k \longrightarrow u_{k+1} \quad \text{or} \quad W\colon u_1 \xrightarrow{k} u_{k+1}\ .$$

Write $u \xrightarrow{\star} v$ to say that there is a walk of some length from u to v. Here we understand that $W\colon u \xrightarrow{\star} v$ *is always a specific walk*, $W = e_1 e_2 \dots e_k$, although we sometimes do not care to mention the edges e_i it uses. The length of a walk W is denoted by $|W|$.

DEFINITION. Let $W = e_1 e_2 \dots e_k$ ($e_i = u_i u_{i+1}$) be a walk.

W is **closed**, if $u_1 = u_{k+1}$.

W is a **path**, if $u_i \neq u_j$ for all $i \neq j$.

W is a **cycle**, if it is closed, and $u_i \neq u_j$ for $i \neq j$ except that $u_1 = u_{k+1}$.

W is a **trivial path**, if its length is 0. A trivial path has no edges.

For a walk $W\colon u = u_1 \longrightarrow \dots \longrightarrow u_{k+1} = v$, also

$$W^{-1}\colon v = u_{k+1} \longrightarrow \dots \longrightarrow u_1 = u$$

is a walk in G, called the **inverse walk** of W.

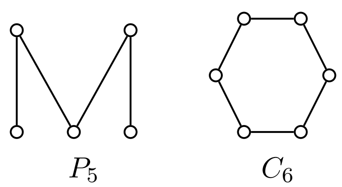
A vertex u is an **end** of a path P, if P starts or ends in u.

The **join** of two walks $W_1\colon u \xrightarrow{\star} v$ and $W_2\colon v \xrightarrow{\star} w$ is the walk $W_1 W_2\colon u \xrightarrow{\star} w$. (Here the end v must be common to the walks.)

Paths P and Q are **disjoint**, if they have no vertices in common, and they are **independent**, if they can share only their ends.

Clearly, the inverse walk P^{-1} of a path P is a path (the **inverse path** of P). The join of two paths need not be a path.

A (sub)graph, which is a path (cycle) of length $k-1$ (k, resp.) having k vertices is denoted by P_k (C_k, resp.). If k is even (odd), we say that the path or cycle is **even** (**odd**). Clearly, all paths of length k are isomorphic. The same holds for cycles of fixed length.



Lemma 1.3. *Each walk $W\colon u \xrightarrow{\star} v$ with $u \neq v$ contains a path $P\colon u \xrightarrow{\star} v$, that is, there is a path $P\colon u \xrightarrow{\star} v$ that is obtained from W by removing edges and vertices.*

Proof. Let $W\colon u = u_1 \longrightarrow \ldots \longrightarrow u_{k+1} = v$. Let $i < j$ be indices such that $u_i = u_j$. If no such i and j exist, then W, itself, is a path. Otherwise, in $W = W_1W_2W_3\colon u \xrightarrow{\star} u_i \xrightarrow{\star} u_j \xrightarrow{\star} v$ the portion $U_1 = W_1W_3\colon u \xrightarrow{\star} u_i = u_j \xrightarrow{\star} v$ is a shorter walk. By repeating this argument, we obtain a sequence $U_1, U_2, \ldots, U_m$ of walks $u \xrightarrow{\star} v$ with $|W| > |U_1| > \cdots > |U_m|$. When the procedure stops, we have a path as required. (Notice that in the above it may very well be that W_1 or W_3 is a trivial walk.) □

DEFINITION. If there exists a walk (and hence a path) from u to v in G, let

$$d_G(u, v) = \min\{k \mid u \xrightarrow{k} v\}$$

be the **distance** between u and v. If there are no walks $u \xrightarrow{\star} v$, let $d_G(u, v) = \infty$ by convention. A graph G is **connected**, if $d_G(u, v) < \infty$ for all $u, v \in G$; otherwise, it is **disconnected**. The maximal connected subgraphs of G are its **connected components**. Denote

$$c(G) = \text{ the number of connected components of } G\ .$$

If $c(G) = 1$, then G is, of course, connected.

The maximality condition means that a subgraph $H \subseteq G$ is a connected component if and only if H is connected and there are no edges leaving H, i.e., for every vertex $v \notin H$, the subgraph $G[V_H \cup \{v\}]$ is disconnected. Apparently, every connected component is an induced subgraph, and

$$N_G^*(v) = \{u \mid d_G(v, u) < \infty\}$$

is *the* connected component of G that contains $v \in G$. In particular, the connected components form a partition of G.

Shortest paths

DEFINITION. Let G^α be an edge weighted graph, that is, G^α is a graph G together with a weight function $\alpha\colon E_G \to \mathbb{R}$ on its edges. For $H \subseteq G$, let

$$\alpha(H) = \sum_{e \in E_H} \alpha(e)$$

be the (total) **weight** of H. In particular, if $P = e_1e_2 \ldots e_k$ is a path, then its weight is $\alpha(P) = \sum_{i=1}^{k} \alpha(e_i)$. The **minimum weighted distance** between two vertices is

$$d_G^\alpha(u, v) = \min\{\alpha(P) \mid P\colon u \xrightarrow{\star} v\}\ .$$

In extremal problems we seek for optimal subgraphs $H \subseteq G$ satisfying specific conditions. In practice we encounter situations where G might represent

- a distribution or transportation network (say, for mail), where the weights on edges are *distances*, travel *expenses*, or *rates of flow* in the network;
- a system of channels in (tele)communication or computer architecture, where the weights present the rate of *unreliability* or *frequency of action* of the connections;
- a model of chemical bonds, where the weights measure molecular *attraction*.

In these examples we look for a subgraph with the smallest weight, and which connects two given vertices, or all vertices (if we want to travel around). On the other hand, if the graph represents a network of pipelines, the weights are volumes or capacities, and then one wants to find a subgraph with the maximum weight.

We consider the minimum problem. For this, let G be a graph with an integer weight function $\alpha\colon E_G \to \mathbb{N}$. In this case, call $\alpha(uv)$ the **length** of uv.

The **shortest path problem**: *Given a connected graph G with a weight function $\alpha\colon E_G \to \mathbb{N}$, find $d_G^{\alpha}(u,v)$ for given $u, v \in G$.*

Assume that G is a connected graph. Dijkstra's algorithm solves the problem for every pair u, v, where u is a fixed starting point and $v \in G$. Let us make the convention that $\alpha(uv) = \infty$, if $uv \notin E_G$.

Dijkstra's algorithm:

(i) Set $u_0 = u$, $t(u_0) = 0$ and $t(v) = \infty$ for all $v \neq u_0$.

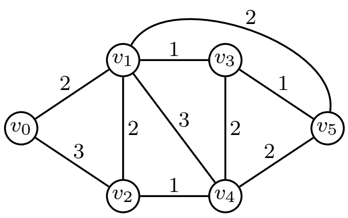
(ii) For $i \in [0, \nu_G - 1]$: for each $v \notin \{u_1, \dots, u_i\}$,

$$\text{replace } \; t(v) \; \text{ by } \; \min\{t(v), t(u_i) + \alpha(u_i v)\} \; .$$

Let $u_{i+1} \notin \{u_1, \dots, u_i\}$ be *any* vertex with the least value $t(u_{i+1})$.

(iii) Conclusion: $d_G^{\alpha}(u,v) = t(v)$.

Example 1.7. Consider the following weighted graph G. Apply Dijkstra's algorithm to the vertex v_0.



- $u_0 = v_0$, $t(u_0) = 0$, others are ∞.
- $t(v_1) = \min\{\infty, 2\} = 2$, $t(v_2) = \min\{\infty, 3\} = 3$, others are ∞. Thus $u_1 = v_1$.
- $t(v_2) = \min\{3, t(u_1) + \alpha(u_1 v_2)\} = \min\{3, 4\} = 3$, $t(v_3) = 2+1 = 3$, $t(v_4) = 2+3 = 5$, $t(v_5) = 2+2 = 4$. Thus choose $u_2 = v_3$.
- $t(v_2) = \min\{3, \infty\} = 3$, $t(v_4) = \min\{5, 3+2\} = 5$, $t(v_5) = \min\{4, 3+1\} = 4$. Thus set $u_3 = v_2$.
- $t(v_4) = \min\{5, 3+1\} = 4$, $t(v_5) = \min\{4, \infty\} = 4$. Thus choose $u_4 = v_4$.
- $t(v_5) = \min\{4, 4+1\} = 4$. The algorithm stops.

We have obtained:

$$t(v_1) = 2, \; t(v_2) = 3, \; t(v_3) = 3, \; t(v_4) = 4, \; t(v_5) = 4 \; .$$

These are the minimal weights from v_0 to each v_i.

The steps of the algorithm can also be rewritten as a table:

v_1	**2**	-	-	-	-
v_2	3	3	**3**	-	-
v_3	∞	**3**	-	-	-
v_4	∞	5	5	**4**	-
v_5	∞	4	4	4	**4**

The correctness of Dijkstra's algorithm can verified be as follows.

Let $v \in V$ be any vertex, and let $P\colon u_0 \xrightarrow{\star} u \xrightarrow{\star} v$ be a shortest path from u_0 to v, where u is any vertex $u \neq v$ on such a path, possibly $u = u_0$. Then, clearly, the first part of the path, $u_0 \xrightarrow{\star} u$, is a shortest path from u_0 to u, and the latter part $u \xrightarrow{\star} v$ is a shortest path from u to v. Therefore, the length of the path P equals the sum of the weights of $u_0 \xrightarrow{\star} u$ and $u \xrightarrow{\star} v$. Dijkstra's algorithm makes use of this observation iteratively.

2

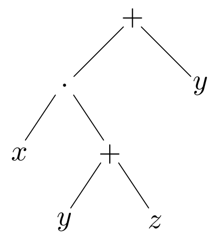
Connectivity of Graphs

2.1 Bipartite graphs and trees

In problems such as the shortest path problem we look for minimum solutions that satisfy the given requirements. The solutions in these cases are usually subgraphs without cycles. Such connected graphs will be called trees, and they are used, *e.g.*, in search algorithms for databases. For concrete applications in this respect, see

T.H. CORMEN, C.E. LEISERSON AND R.L. RIVEST, "Introduction to Algorithms", MIT Press, 1993.

Certain structures with operations are representable as trees. These trees are sometimes called *construction trees*, *decomposition trees*, *factorization trees* or *grammatical trees*. Grammatical trees occur especially in linguistics, where syntactic structures of sentences are analyzed. On the right there is a tree of operations for the arithmetic formula $x \cdot (y + z) + y$.



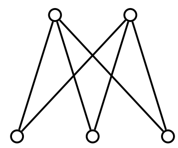
Bipartite graphs

DEFINITION. A graph G is called **bipartite**, if V_G has a partition to two subsets X and Y such that each edge $uv \in E_G$ connects a vertex of X and a vertex of Y. In this case, (X, Y) is a **bipartition** of G, and G is (X, Y)**-bipartite**.

A bipartite graph G (as in the above) is a **complete** (m, k)-**bipartite graph**, if $|X| = m$, $|Y| = k$, and $uv \in E_G$ for all $u \in X$ and $v \in Y$.

All complete (m, k)-bipartite graphs are isomorphic. Let $K_{m,k}$ denote such a graph.

A subset $X \subseteq V_G$ is **stable**, if $G[X]$ is a discrete graph.



$K_{2,3}$

The following result is clear from the definitions.

Theorem 2.1. *A graph G is bipartite if and only if V_G has a partition to two stable subsets.*

Example 2.1. The k-cube Q_k of Example 1.5 is bipartite for all k. Indeed, consider $A = \{u \mid u \text{ has an even number of } 1'\text{s}\}$ and $B = \{u \mid u \text{ has an odd number of } 1'\text{s}\}$. Clearly, these sets partition $\mathbb{B}^k$, and they are stable in Q_k.

Theorem 2.2. *A graph G is bipartite if and only if it has no odd cycles.*

Proof. ($\Rightarrow$) Let G be (X, Y)-bipartite. For a cycle $C\colon v_1 \to \ldots \to v_{k+1} = v_1$ of length k, $v_1 \in X$ implies $v_2 \in Y$, $v_3 \in X, \ldots, v_{2i} \in Y$, $v_{2i+1} \in X$. Consequently, $k + 1 = 2m + 1$ is odd, and $k = |C|$ is even.

($\Leftarrow$) Suppose that all cycles in G are even. First, we observe that it suffices to show the claim for connected graphs. Indeed, if G is disconnected, then each cycle of G is contained in one of the connected components, $G_1, \ldots, G_p$, of G. If G_i is (X_i, Y_i)-bipartite, then $(X_1 \cup X_2 \cup \cdots \cup X_p, Y_1 \cup Y_2 \cup \cdots \cup Y_p)$ is a bipartition of G.

Assume thus that G is connected. Let $v \in G$ be a chosen vertex, and define

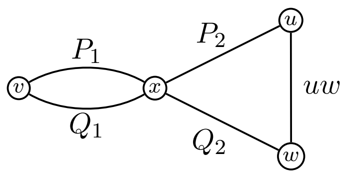
$$X = \{x \mid d_G(v, x) \text{ is even}\}, \qquad Y = \{y \mid d_G(v, y) \text{ is odd}\} .$$

Since G is connected, $V_G = X \cup Y$. Also, by the definition of distance, $X \cap Y = \emptyset$.

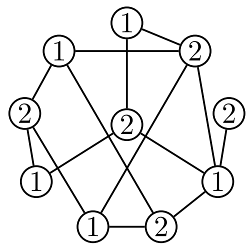
Let $u, w \in G$ be both in X or both in Y, and let $P\colon v \xrightarrow{\star} u$ and $Q\colon v \xrightarrow{\star} w$ be (among the) shortest paths from v to u and w.

Assume that x is the last common vertex of P and Q: $P = P_1P_2$, $Q = Q_1Q_2$, where $P_2\colon x \xrightarrow{\star} u$ and $Q_2\colon x \xrightarrow{\star} w$ are independent. Since P and Q are shortest paths, P_1 and Q_1 are shortest paths $v \xrightarrow{\star} x$. Consequently, $|P_1| = |Q_1|$.

So $|P_2|$ and $|Q_2|$ have the same parity, i.e., $|P_2|+|Q_2|$ is even, and so $uw \notin E_G$. Hence $G[X]$ and $G[Y]$ are discrete induced subgraphs, and G is bipartite as claimed. □



Checking whether a graph is bipartite is easy. Indeed, this can be done by using two 'opposite' colours, say 1 and 2. Start from any vertex v_1, and colour it by 1. Then colour the neighbours of v_1 by 2, and proceed by colouring all neighbours of an already coloured vertex by an opposite colour.



If the whole graph can be coloured, then G is (X, Y)-bipartite, where X consists of those vertices with colour 1, and Y of those vertices with colour 2; otherwise, at some point one of the vertices gets both colours, and in this case, G is not bipartite.

Example 2.2 (ERDÖS (1965)). We show that each graph G has a bipartite subgraph $H \subseteq G$ such that $\varepsilon_H \geq \frac{1}{2}\varepsilon_G$.
Indeed, let $V_G = X \cup Y$ be a partition such that the number of edges between X and Y is as large as possible. Denote

$$F = E_G \cap \{uv \mid u \in X, v \in Y\} ,$$

and let $H = G[F]$. Obviously H is a spanning subgraph, and it is bipartite.

By the maximum condition,

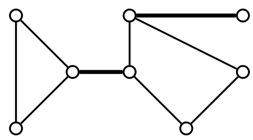
$$d_H(v) \geq \frac{1}{2} d_G(v) \ ,$$

since, otherwise, v is on the wrong side. (That is, if $v \in X$, then the pair $X' = X \setminus \{v\}$, $Y' = Y \cup \{v\}$ does better that the pair X, Y.) Now

$$\varepsilon_H = \frac{1}{2} \sum_{v \in H} d_H(v) \geq \frac{1}{2} \sum_{v \in G} \frac{1}{2} d_G(v) = \frac{1}{2} \varepsilon_G \, .$$

Bridges

DEFINITION. An edge $e \in E_G$ is a **bridge** of the graph G, if $G{-}e$ has more connected components than G, that is, if $c(G{-}e) > c(G)$.



In particular, and most importantly, an edge e in a connected G is a bridge if and only if $G{-}e$ is disconnected. On the right the two horizontal lines are bridges. The rest are not.

Theorem 2.3. *An edge $e \in E_G$ is a bridge if and only if e is not in any cycle of G.*

Proof. First of all, note that $e = uv$ is a bridge if and only if u and v belong to different connected components of $G{-}e$.

($\Rightarrow$) If there is a cycle in G containing e, then there is a cycle $C = eP \colon u \longrightarrow v \xrightarrow{\star} u$, where $P \colon v \xrightarrow{\star} u$ is a path in $G{-}e$, and so e is not a bridge.

($\Leftarrow$) Assume that $e = uv$ is not a bridge. Hence u and v are in the same connected component of $G{-}e$. If $P \colon v \xrightarrow{\star} u$ is a path in $G{-}e$, then $eP \colon u \longrightarrow v \xrightarrow{\star} u$ is a cycle in G that contains e. □

Lemma 2.1. *Let e be a bridge in a connected graph G.*

(i) Then $c(G{-}e) = 2$.

(ii) Let H be a connected component of $G{-}e$. If $f \in E_H$ is a bridge of H, then f is a bridge of G.

Proof. For (i), let $e = uv$. Since e is a bridge, the ends u and v are not connected in $G{-}e$. Let $w \in G$. Since G is connected, there exists a path $P \colon w \xrightarrow{\star} v$ in G. This is a path of $G{-}e$, unless $P \colon w \xrightarrow{\star} u \rightarrow v$ contains $e = uv$, in which case the part $w \xrightarrow{\star} u$ is a path in $G{-}e$.

For (ii), if $f \in E_H$ belongs to a cycle C of G, then C does not contain e (since e is in no cycle), and therefore C is inside H, and f is not a bridge of H. □

Trees

DEFINITION. A graph is called **acyclic**, if it has no cycles. An acyclic graph is also called a **forest**. A **tree** is a connected acyclic graph.

By Theorem 2.3 and the definition of a tree, we have

Corollary 2.1. *A connected graph is a tree if and only if all its edges are bridges.*

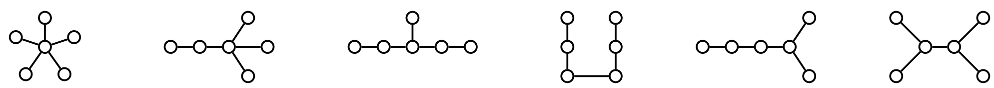
Example 2.3. The following enumeration result for trees has many different proofs, the first of which was given by CAYLEY in 1889: *There are n^{n-2} trees on a vertex set V of n elements.* We omit the proof.

On the other hand, there are only a few trees *up to isomorphism*:

n	1	2	3	4	5	6	7	8
trees	1	1	1	2	3	6	11	23

n	9	10	11	12	13	14	15	16
trees	47	106	235	551	1301	3159	7741	19 320

The nonisomorphic trees of order 6 are:



We say that a path $P\colon u \xrightarrow{\star} v$ is **maximal** in a graph G, if there are no $e \in E_G$ for which Pe or eP is a path. Such paths exist, because ν_G is finite.

Lemma 2.2. *Let $P\colon u \xrightarrow{\star} v$ be a maximal path in a graph G. Then $N_G(v) \subseteq P$. Moreover, if G is acyclic, then $d_G(v) = 1$.*

Proof. If $e = vw \in E_G$ with $w \notin P$, then also Pe is a path, which contradicts the maximality assumption for P. Hence $N_G(v) \subseteq P$. For acyclic graphs, if $wv \in E_G$, then w belongs to P, and wv is necessarily the last edge of P in order to avoid cycles. □

Corollary 2.2. *Each tree T with $\nu_T \geq 2$ has at least two leaves.*

Proof. Since T is acyclic, both ends of a maximal path have degree one. □

Theorem 2.4. *The following are equivalent for a graph T.*

(i) *T is a tree.*
(ii) *Any two vertices are connected in T by a unique path.*
(iii) *T is acyclic and $\varepsilon_T = \nu_T - 1$.*

Proof. Let $\nu_T = n$. If $n = 1$, then the claim is trivial. Suppose thus that $n \geq 2$.

(i)⇒(ii) Let T be a tree. Assume the claim does not hold, and let $P, Q \colon u \xrightarrow{\star} v$ be two different paths between the same vertices u and v. Suppose that $|P| \geq |Q|$. Since $P \neq Q$, there exists an edge e which belongs to P but not to Q. Each edge of T is a bridge, and therefore u and v belong to different connected components of $T{-}e$. Hence e must also belong to Q; a contradiction.

(ii)⇒(iii) We prove the claim by induction on n. Clearly, the claim holds for $n = 2$, and suppose it holds for graphs of order less than n. Let T be any graph of order n satisfying (ii). In particular, T is connected, and it is clearly acyclic.

Let $P \colon u \xrightarrow{\star} v$ be a maximal path in T. By Lemma 2.2, we have $d_T(v) = 1$. In this case, $P \colon u \xrightarrow{\star} w \longrightarrow v$, where vw is the unique edge having an end v. The subgraph $T{-}v$ is connected, and it satisfies the condition (ii). By induction hypothesis, $\varepsilon_{T-v} = n - 2$, and so $\varepsilon_T = \varepsilon_{T-v} + 1 = n - 1$, and the claim follows.

(iii)⇒(i) Assume (iii) holds for T. We need to show that T is connected. Indeed, let the connected components of T be $T_i = (V_i, E_i)$, for $i \in [1, k]$. Since T is acyclic, so are the connected graphs T_i, and hence they are trees, for which we have proved that $|E_i| = |V_i| - 1$. Now, $\nu_T = \sum_{i=1}^{k} |V_i|$, and $\varepsilon_T = \sum_{i=1}^{k} |E_i|$. Therefore,

$$n - 1 = \varepsilon_T = \sum_{i=1}^{k}(|V_i| - 1) = \sum_{i=1}^{k} |V_i| - k = n - k \ ,$$

which gives that $k = 1$, that is, T is connected. □

Example 2.4. Consider a cup tournament of n teams. If during a round there are k teams left in the tournament, then these are divided into $\lfloor k \rfloor$ pairs, and from each pair only the winner continues. If k is odd, then one of the teams goes to the next round without having to play. How many plays are needed to determine the winner?

So if there are 14 teams, after the first round 7 teams continue, and after the second round 4 teams continue, then 2. So 13 plays are needed in this example.

The answer to our problem is $n - 1$, since the cup tournament is a tree, where a play corresponds to an edge of the tree.

Spanning trees

Theorem 2.5. *Each connected graph has a* **spanning tree***, that is, a spanning graph that is a tree.*

Proof. Let $H \subseteq G$ be a minimal connected spanning subgraph, that is, a connected spanning subgraph of G such that $H{-}e$ is disconnected for all $e \in E_H$. Such a subgraph is obtained from G by removing nonbridges:

- To start with, let $H_0 = G$.
- For $i \geq 0$, let $H_{i+1} = H_i - e_i$, where e_i is a not a bridge of H_i. Since e_i is not a bridge, H_{i+1} is a connected spanning subgraph of H_i and thus of G.
- $H = H_k$, when only bridges are left.

By Corollary 2.1, H is a tree. □

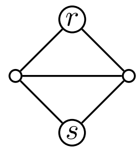
Corollary 2.3. *For each connected graph G, $\varepsilon_G \geq \nu_G - 1$. Moreover, a connected graph G is a tree if and only if $\varepsilon_G = \nu_G - 1$.*

Proof. Let T be a spanning tree of G. Then $\varepsilon_G \geq \varepsilon_T = \nu_T - 1 = \nu_G - 1$. The second claim is also clear. □

Example 2.5. In **Shannon's switching game** a positive player P and a negative player N play on a graph G with two special vertices: a **source** s and a **sink** r. P and N alternate turns so that P designates an edge by $+$, and N by $-$. Each edge can be designated at most once. It is P's purpose to designate a path $s \xrightarrow{\star} r$ (that is, to designate all edges in one such path), and N tries to block all paths $s \xrightarrow{\star} r$ (that is, to designate at least one edge in each such path). We say that a game (G, s, r) is

- **positive**, if P has a winning strategy no matter who begins the game,
- **negative**, if N has a winning strategy no matter who begins the game,
- **neutral**, if the winner depends on who begins the game.

The game on the right is neutral.



LEHMAN proved in 1964 that *Shannon's switching game (G, s, r) is positive if and only if there exists $H \subseteq G$ such that H contains s and r and H has two spanning trees with no edges in common.*

In the other direction the claim can be proved along the following lines. Assume that there exists a subgraph H containing s and r and that has two spanning trees with no edges in common. Then P plays as follows. If N marks by $-$ an edge from one of the two trees, then P marks by $+$ an edge in the other tree such that this edge reconnects the broken tree. In this way, P always has two spanning trees for the subgraph H with only edges marked by $+$ in common.

In converse the claim is considerably more difficult to prove.

There remains the problem to characterize those Shannon's switching games (G, s, r) that are neutral (negative, respectively).

The connector problem

To build a network connecting n nodes (towns, computers, chips in a computer) it is desirable to decrease the cost of construction of the links to the minimum. This is the **connector problem**. In graph theoretical terms we wish to find an **optimal spanning subgraph** of a weighted graph. Such an optimal subgraph is clearly a spanning tree, for, otherwise a deletion of any nonbridge will reduce the total weight of the subgraph.

Let then G^α be a graph G together with a weight function $\alpha\colon E_G \to \mathbb{R}^+$ (positive reals) on the edges. Kruskal's algorithm (also known as the **greedy algorithm**) provides a solution to the connector problem.

Kruskal's algorithm: For a connected and weighted graph G^α of order n:

(i) Let e_1 be an edge of smallest weight, and set $E_1 = \{e_1\}$.

(ii) For each $i = 2, 3, \dots, n-1$ in this order, choose an edge $e_i \notin E_{i-1}$ of smallest possible weight such that e_i does not produce a cycle when added to $G[E_{i-1}]$, and let $E_i = E_{i-1} \cup \{e_i\}$.

The final outcome is $T = (V_G, E_{n-1})$.

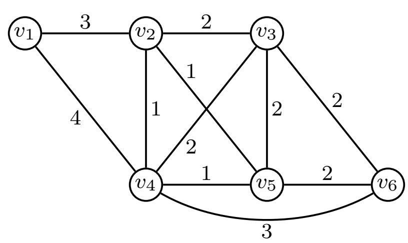
By the construction, $T = (V_G, E_{n-1})$ is a spanning tree of G, because it contains no cycles, it is connected and has $n-1$ edges. We now show that T has the minimum total weight among the spanning trees of G.

Suppose T_1 is any spanning tree of G. Let e_k be the first edge produced by the algorithm that is not in T_1. If we add e_k to T_1, then a cycle C containing e_k is created. Also, C must contain an edge e that is not in T. When we replace e by e_k in T_1, we still have a spanning tree, say T_2. However, by the construction, $\alpha(e_k) \le \alpha(e)$, and therefore $\alpha(T_2) \le \alpha(T_1)$. Note that T_2 has more edges in common with T than T_1.

Repeating the above procedure, we can transform T_1 to T by replacing edges, one by one, such that the total weight does not increase. We deduce that $\alpha(T) \le \alpha(T_1)$.

The outcome of Kruskal's algorithm need not be unique. Indeed, there may exist several optimal spanning trees (with the same weight, of course) for a graph.

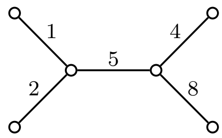
Example 2.6. When applied to the weighted graph on the right, the algorithm produces the sequence: $e_1 = v_2v_4$, $e_2 = v_4v_5$, $e_3 = v_3v_6$, $e_4 = v_2v_3$ and $e_5 = v_1v_2$. The total weight of the spanning tree is thus 9.
Also, the selection $e_1 = v_2v_5$, $e_2 = v_4v_5$, $e_3 = v_5v_6$, $e_4 = v_3v_6$, $e_5 = v_1v_2$ gives another optimal solution (of weight 9).



Problem. Consider trees T with weight functions $\alpha\colon E_T \to \mathbb{N}$. Each tree T of order n has exactly $\binom{n}{2}$ paths. (Why is this so?) *Does there exist a weighted tree T^α of order n such that the (total) weights of its paths are $1, 2, \dots, \binom{n}{2}$?*

In such a weighted tree T^α different paths have different weights, and each $i \in [1, \binom{n}{2}]$ is a weight of one path. Also, α must be injective.

No solutions are known for any $n \geq 7$.

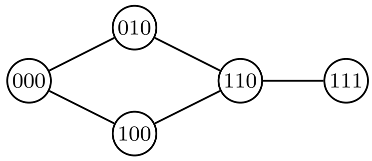


TAYLOR (1977) proved: *if T of order n exists, then necessarily $n = k^2$ or $n = k^2 + 2$ for some $k \geq 1$.*

Example 2.7. A computer network can be presented as a graph G, where the vertices are the node computers, and the edges indicate the direct links. Each computer v has an *address* $a(v)$, a bit string (of zeros and ones). The **length** of an address is the number of its bits. A message that is sent to v is preceded by the address $a(v)$. The **Hamming distance** $h(a(v), a(u))$ of two addresses of the same length is the number of places, where $a(v)$ and $a(u)$ differ. For example, $h(00010, 01100) = 3$ and $h(10000, 00000) = 1$.

It would be a good way to address the vertices so that the Hamming distance of two vertices is the same as their distance in G. In particular, if two vertices were adjacent, their addresses should differ by one symbol. This would make it easier for a node computer to forward a message.

A graph G is said to be **addressable**, if it has an addressing a such that $d_G(u, v) = h(a(u), a(v))$.



We prove that *every tree T is addressable. Moreover, the addresses of the vertices of T can be chosen to be of length $\nu_T - 1$.*

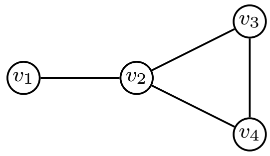
The proof goes by induction. If $\nu_T \leq 2$, then the claim is obvious. In the case $\nu_T = 2$, the addresses of the vertices are simply 0 and 1.

Let then $V_T = \{v_1, \dots, v_{k+1}\}$, and assume that $d_T(v_1) = 1$ (a leaf) and $v_1 v_2 \in E_T$. By the induction hypothesis, we can address the tree $T - v_1$ by addresses of length $k - 1$. We change this addressing: let a_i be the address of v_i in $T - v_1$, and change it to $0a_i$. Set the address of v_1 to $1a_2$. It is now easy to see that we have obtained an addressing for T as required.

The triangle K_3 is not addressable. In order to gain more generality, we modify the addressing for general graphs by introducing a special symbol $*$ in addition to 0 and 1. A **star address** will be a sequence of these three symbols. The Hamming distance remains as it was, that is, $h(u, v)$ is the number of places, where u and v have a different symbol 0 or 1. The special symbol $*$ does not affect $h(u, v)$. So, $h(10{*}{*}01, 0{*}{*}101) = 1$ and $h(1{*}{*}{*}{*}{*}, {*}00{*}{*}{*}) = 0$. We still want to have $h(u, v) = d_G(u, v)$.

We star address this graph as follows:

$$a(v_1) = 0000\,, \qquad a(v_2) = 10*0\,,$$
$$a(v_3) = 1*01\,, \qquad a(v_4) = **11\,.$$



These addresses have length 4. Can you design a star addressing with addresses of length 3?

WINKLER proved in 1983 a rather unexpected result: *The minimum star address length of a graph G is at most $\nu_G - 1$.*

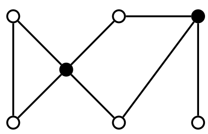
For the proof of this, see VAN LINT AND WILSON, "A Course in Combinatorics".

2.2 Connectivity

Spanning trees are often optimal solutions to problems, where cost is the criterion. We may also wish to construct graphs that are as simple as possible, but where two vertices are always connected by at least two independent paths. These problems occur especially in different aspects of fault tolerance and reliability of networks, where one has to make sure that a break-down of one connection does not affect the functionality of the network. Similarly, in a reliable network we require that a break-down of a node (computer) should not result in the inactivity of the whole network.

Separating sets

DEFINITION. A vertex $v \in G$ is a **cut vertex**, if $c(G-v) > c(G)$. A subset $S \subseteq V_G$ is a **separating set**, if $G-S$ is disconnected. We also say that S **separates** the vertices u and v and it is a (u, v)-**separating set**, if u and v belong to different connected components of $G-S$.



If G is connected, then v is a cut vertex if and only if $G-v$ is disconnected, that is, $\{v\}$ is a separating set. The following lemma is immediate.

Lemma 2.3. *If $S \subseteq V_G$ separates u and v, then every path $P\colon u \xrightarrow{\star} v$ visits a vertex of S.*

Lemma 2.4. *If a connected graph G has no separating sets, then it is a complete graph.*

Proof. If $\nu_G \leq 2$, then the claim is clear. For $\nu_G \geq 3$, assume that G is not complete, and let $uv \notin E_G$. Now $V_G \setminus \{u, v\}$ is a separating set. The claim follows from this. □

DEFINITION. The (**vertex**) **connectivity number** $\kappa(G)$ of G is defined as

$$\kappa(G) = \min\{k \mid k = |S|,\ G-S \text{ disconnected or trivial}, S \subseteq V_G\}\ .$$

A graph G is k-**connected**, if $\kappa(G) \geq k$.

In other words,

- $\kappa(G) = 0$, if G is disconnected,
- $\kappa(G) = \nu_G - 1$, if G is a complete graph, and
- otherwise $\kappa(G)$ equals the minimum size of a separating set of G.

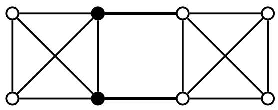
Clearly, if G is connected, then it is 1-connected.

DEFINITION. An **edge cut** F of G consists of edges so that $G{-}F$ is disconnected. Let

$$\kappa'(G) = \min\{k \mid k = |F|,\ G{-}F \text{ disconnected}, F \subseteq E_G\} .$$

For trivial graphs, let $\kappa'(G) = 0$. A graph G is k-**edge connected**, if $\kappa'(G) \geq k$. A minimal edge cut $F \subseteq E_G$ is a **bond** ($F \setminus \{e\}$ is not an edge cut for any $e \in F$).

Example 2.8. Again, if G is disconnected, then $\kappa'(G) = 0$. On the right, $\kappa(G) = 2$ and $\kappa'(G) = 2$. Notice that the minimum degree is $\delta(G) = 3$.



Lemma 2.5. *Let G be connected. If $e = uv$ is a bridge, then either $G = K_2$ or one of u or v is a cut vertex.*

Proof. Assume that $G \neq K_2$ and thus that $\nu_G \geq 3$, since G is connected. Let $G_u = N^*_{G-e}(u)$ and $G_v = N^*_{G-e}(v)$ be the connected components of $G{-}e$ containing u and v. Now, either $\nu_{G_u} \geq 2$ (and u is a cut vertex) or $\nu_{G_v} \geq 2$ (and v is a cut vertex). □

Lemma 2.6. *If F be a bond of a connected graph G, then $c(G{-}F) = 2$.*

Proof. Since $G{-}F$ is disconnected, and F is minimal, the subgraph $H = G{-}(F \setminus \{e\})$ is connected for given $e \in F$. Hence e is a bridge in H. By Lemma 2.1, $c(H{-}e) = 2$, and thus $c(G{-}F) = 2$, since $H{-}e = G{-}F$. □

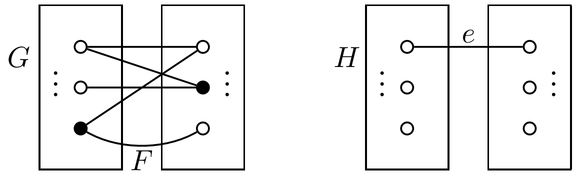
Theorem 2.6 (WHITNEY (1932)). *For any graph G,*

$$\kappa(G) \leq \kappa'(G) \leq \delta(G) .$$

Proof. Assume G is nontrivial. Clearly, $\kappa'(G) \leq \delta(G)$, since if we remove all edges with an end v, we disconnect G. If $\kappa'(G) = 0$, then G is disconnected, and in this case also $\kappa(G) = 0$. If $\kappa'(G) = 1$, then G is connected and contains a bridge. By Lemma 2.5, either $G = K_2$ or G has a cut vertex. In both of these cases, also $\kappa(G) = 1$.

Assume then that $\kappa'(G) \geq 2$. Let F be an edge cut of G with $|F| = \kappa'(G)$, and let $e = uv \in F$. Then F is a bond, and $G{-}F$ has two connected components.

Consider the connected subgraph $H = G{-}(F \setminus \{e\}) = (G{-}F) + e$, where e is a bridge.



Now for each $f \in F \setminus \{e\}$ choose an end different from u and v. (The choices for different edges need not be different.) Note that since $f \neq e$, either end of f is different from u or v. Let S be the collection of these choices. Thus $|S| \leq |F| - 1 = \kappa'(G) - 1$, and $G{-}S$ does not contain edges from $F \setminus \{e\}$.

If $G{-}S$ is disconnected, then S is a separating set and so $\kappa(G) \leq |S| \leq \kappa'(G) - 1$ and we are done. On the other hand, if $G{-}S$ is connected, then either $G{-}S = K_2$ $(= e)$, or either u or v (or both) is a cut vertex of $G{-}S$ (since $H{-}S = G{-}S$, and therefore $G{-}S \subseteq H$ is an induced subgraph of H). In both of these cases, there is a vertex of $G{-}S$, whose removal results in a trivial or a disconnected graph. In conclusion, $\kappa(G) \leq |S| + 1 \leq \kappa'(G)$, and the claim follows. □

Menger's theorem

Theorem 2.7 (M**ENGER (1927)).** *Let $u, v \in G$ be nonadjacent vertices of a connected graph G. Then the minimum number of vertices separating u and v is equal to the maximum number of independent paths from u to v.*

Proof. If a subset $S \subseteq V_G$ is (u, v)-separating, then every path $u \xrightarrow{\star} v$ of G visits S. Hence $|S|$ is at least the number of independent paths from u to v.

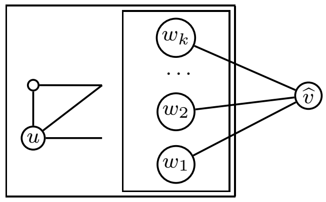
Conversely, we use induction on $m = \nu_G + \varepsilon_G$ to show that if $S = \{w_1, w_2, \dots, w_k\}$ is a (u, v)-separating set of the smallest size, then G has at least (and thus exactly) k independent paths $u \xrightarrow{\star} v$.

The case for $k = 1$ is clear, and this takes care of the small values of m, required for the induction.

(1) Assume first that u and v have a common neighbour $w \in N_G(u) \cap N_G(v)$. Then necessarily $w \in S$. In the smaller graph $G{-}w$ the set $S \setminus \{w\}$ is a minimum (u, v)-separating set, and the induction hypothesis yields that there are $k - 1$ independent paths $u \xrightarrow{\star} v$ in $G{-}w$. Together with the path $u \longrightarrow w \longrightarrow v$, there are k independent paths $u \xrightarrow{\star} v$ in G as required.

(2) Assume then that $N_G(u) \cap N_G(v) = \emptyset$, and denote by $H_u = N^*_{G-S}(u)$ and $H_v = N^*_{G-S}(v)$ the connected components of $G{-}S$ for u and v.

(2.1) Suppose next that $S \nsubseteq N_G(u)$ and $S \nsubseteq N_G(v)$.



Let $\widehat{v}$ be a new vertex, and define G_u to be the graph on $H_u \cup S \cup \{\widehat{v}\}$ having the edges of $G[H_u \cup S]$ together with $\widehat{v}w_i$ for all $i \in [1,k]$. The graph G_u is connected and it is smaller than G. Indeed, in order for S to be a minimum separating set, all $w_i \in S$ have to be adjacent to some vertex in H_v. This shows that $\varepsilon_{G_u} \le \varepsilon_G$, and, moreover, the assumption (2.1) rules out the case $H_v = \{v\}$. So $|H_v| \ge 2$ and $\nu_{G_u} < \nu_G$.

If S' is any $(u,\widehat{v})$-separating set of G_u, then S' will separate u from all $w_i \in S \setminus S'$ in G. This means that S' separates u and v in G. Since k is the size of a minimum (u,v)-separating set, we have $|S'| \ge k$. We noted that G_u is smaller than G, and thus by the induction hypothesis, there are k independent paths $u \xrightarrow{\star} \widehat{v}$ in G_u. This is possible only if there exist k paths $u \xrightarrow{\star} w_i$, one for each $i \in [1,k]$, that have only the end u in common.

By the present assumption, also u is nonadjacent to some vertex of S. A symmetric argument applies to the graph G_v (with a new vertex $\widehat{u}$), which is defined similarly to G_u. This yields that there are k paths $w_i \xrightarrow{\star} v$ that have only the end v in common. When we combine these with the above paths $u \xrightarrow{\star} w_i$, we obtain k independent paths $u \xrightarrow{\star} w_i \xrightarrow{\star} v$ in G.

(2.2) There remains the case, where for *all* (u,v)-separating sets S of k elements, either $S \subseteq N_G(u)$ or $S \subseteq N_G(v)$. (Note that then, by (2), $S \cap N_G(v) = \emptyset$ or $S \cap N_G(u) = \emptyset$.)

Let $P = efQ$ be a shortest path $u \xrightarrow{\star} v$ in G, where $e = ux$, $f = xy$, and $Q\colon y \xrightarrow{\star} v$. Notice that, by the assumption (2), $|P| \ge 3$, and so $y \ne v$. In the smaller graph $G{-}f$, let S' be a minimum set that separates u and v.

If $|S'| \ge k$, then, by the induction hypothesis, there are k independent paths $u \xrightarrow{\star} v$ in $G{-}f$. But these are paths of G, and the claim is clear in this case.

If, on the other hand, $|S'| < k$, then u and v are still connected in $G{-}S'$. Every path $u \xrightarrow{\star} v$ in $G{-}S'$ necessarily travels along the edge $f = xy$, and so $x, y \notin S'$.
Let

$$S_x = S' \cup \{x\} \qquad \text{and} \qquad S_y = S' \cup \{y\}\ .$$

These sets separate u and v in G (by the above fact), and they have size k. By our current assumption, the vertices of S_y are adjacent to v, since the path P is shortest and so $uy \notin E_G$ (meaning that u is not adjacent to all of S_y). The assumption (2) yields that u is adjacent to all of S_x, since $ux \in E_G$. But now both u and v are adjacent to the vertices of S', which contradicts the assumption (2). $\square$

Theorem 2.8 (MENGER (1927)). *A graph G is k-connected if and only if every two vertices are connected by at least k independent paths.*

Proof. If any two vertices are connected by k independent paths, then it is clear that $\kappa(G) \ge k$.

In converse, suppose that $\kappa(G) = k$, but that G has vertices u and v connected by at most $k-1$ independent paths. By Theorem 2.7, it must be that $e = uv \in E_G$. Consider the graph $G{-}e$. Now u and v are connected by at most $k-2$ independent paths in $G{-}e$, and by Theorem 2.7, u and v can be separated in $G{-}e$ by a set S with $|S| = k-2$. Since $\nu_G > k$

(because $\kappa(G) = k$), there exists a $w \in G$ that is not in $S \cup \{u, v\}$. The vertex w is separated in $G{-}e$ by S from u or from v; otherwise there would be a path $u \xrightarrow{\star} v$ in $(G{-}e){-}S$. Say, this vertex is u. The set $S \cup \{v\}$ has $k-1$ elements, and it separates u from w in G, which contradicts the assumption that $\kappa(G) = k$. This proves the claim. □

We state without a proof the corresponding separation property for edge connectivity.

DEFINITION. Let G be a graph. A uv-**disconnecting set** is a set $F \subseteq E_G$ such that every path $u \xrightarrow{\star} v$ contains an edge from F.

Theorem 2.9. *Let $u, v \in G$ with $u \neq v$ in a graph G. Then the maximum number of edge-disjoint paths $u \xrightarrow{\star} v$ equals the minimum number k of edges in a uv-disconnecting set.*

Corollary 2.4. *A graph G is k-edge connected if and only if every two vertices are connected by at least k edge disjoint paths.*

Example 2.9. Recall the definition of the cube Q_k from Example 1.5. We show that $\kappa(Q_k) = k$.

First of all, $\kappa(Q_k) \leq \delta(Q_k) = k$. In converse, we show the claim by induction. Extract from Q_k the disjoint subgraphs: G_0 induced by $\{0u \mid u \in \mathbb{B}^{k-1}\}$ and G_1 induced by $\{1u \mid u \in \mathbb{B}^{k-1}\}$. These are (isomorphic to) Q_{k-1}, and Q_k is obtained from the union of G_0 and G_1 by adding the 2^{k-1} edges $(0u, 1u)$ for all $u \in \mathbb{B}^{k-1}$.

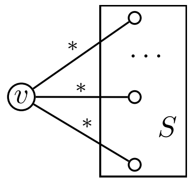
Let S be a separating set of Q_k with $|S| \leq k$. If both $G_0{-}S$ and $G_1{-}S$ were connected, also $Q_k{-}S$ would be connected, since one pair $(0u, 1u)$ necessarily remains in $Q_k{-}S$. So we can assume that $G_0{-}S$ is disconnected. (The case for $G_1{-}S$ is symmetric.) By the induction hypothesis, $\kappa(G_0) = k-1$, and hence S contains at least $k-1$ vertices of G_0 (and so $|S| \geq k-1$). If there were no vertices from G_1 in S, then, of course, $G_1{-}S$ is connected, and the edges $(0u, 1u)$ of Q_k would guarantee that $Q_k{-}S$ is connected; a contradiction. Hence $|S| \geq k$.

Example 2.10. We have $\kappa'(Q_k) = k$ for the k-cube. Indeed, by Whitney's theorem, $\kappa(G) \leq \kappa'(G) \leq \delta(G)$. Since $\kappa(Q_k) = k = \delta(Q_k)$, also $\kappa'(Q_k) = k$.

Algorithmic Problem. The connectivity problems tend to be algorithmically difficult. In the **disjoint paths problem** we are given a set (u_i, v_i) of pairs of vertices for $i = 1, 2, \ldots, k$, and it is asked whether there exist paths $P_i \colon u_i \xrightarrow{\star} v_i$ that have no vertices in common. This problem was shown to be NP-complete by KNUTH in 1975. (However, for *fixed* k, the problem has a fast algorithm due to ROBERTSON and SEYMOUR (1986).)

Dirac's fans

DEFINITION. Let $v \in G$ and $S \subseteq V_G$ such that $v \notin S$ in a graph G. A set of paths from v to a vertex in S is called a (v, S)-**fan**, if they have only v in common.



Theorem 2.10 (DIRAC **(1960)).** *A graph G is k-connected if and only if $\nu_G > k$ and for every $v \in G$ and $S \subseteq V_G$ with $|S| \geq k$ and $v \notin S$, there exists a (v, S)-fan of k paths.*

Proof. Exercise. □

Theorem 2.11 (DIRAC **(1960)).** *Let G be a k-connected graph for $k \geq 2$. Then for any k vertices, there exists a cycle of G containing them.*

Proof. First of all, since $\kappa(G) \geq 2$, G has no cut vertices, and thus no bridges. It follows that every edge, and thus every vertex of G belongs to a cycle.

Let $S \subseteq V_G$ be such that $|S| = k$, and let C be a cycle of G that contains the maximum number of vertices of S. Let the vertices of $S \cap V_C$ be $v_1, \dots, v_r$ listed in order around C so that each pair (v_i, v_{i+1}) (with indices modulo r) defines a path along C (except in the special case where $r = 1$). Such a path is referred to as a *segment* of C. If C contains all vertices of S, then we are done; otherwise, suppose $v \in S$ is not on C.

It follows from Theorem 2.10 that there is a (v, V_C)-fan of at least $\min\{k, |V_C|\}$ paths. Therefore there are two paths $P\colon v \xrightarrow{\star} u$ and $Q\colon v \xrightarrow{\star} w$ in such a fan that end in the same segment (v_i, v_{i+1}) of C. Then the path $W\colon u \xrightarrow{\star} w$ (or $w \xrightarrow{\star} u$) along C contains all vertices of $S \cap V_C$. But now PWQ^{-1} is a cycle of G that contains v and all v_i for $i \in [1, r]$. This contradicts the choice of C, and proves the claim. □

3

Tours and Matchings

3.1 Eulerian graphs

The first proper problem in graph theory was the Königsberg bridge problem. In general, this problem concerns about travels around a graph such that one tries to avoid using the same edge twice. In practice these eulerian problems occur, for instance, in optimizing distribution networks – such as delivering mail, where in order to save time each street should be travelled only once. The same problem occurs in mechanical graph plotting, where one avoids lifting the pen off the paper while drawing the lines.

Euler tours

DEFINITION. A walk $W = e_1e_2 \dots e_n$ is a **trail**, if $e_i \neq e_j$ for all $i \neq j$. An **Euler trail** of a graph G is a trail that visits every edge once. A connected graph G is **eulerian**, if it has a closed trail containing every edge of G. Such a trail is called an **Euler tour**.

Notice that if $W = e_1e_2 \dots e_n$ is an Euler tour (and so $E_G = \{e_1, e_2, \dots, e_n\}$), also $e_ie_{i+1} \dots e_ne_1 \dots e_{i-1}$ is an Euler tour for all $i \in [1, n]$. A complete proof of the following **Euler's Theorem** was first given by HIERHOLZER in 1873.

Theorem 3.1 (EULER (1736), HIERHOLZER (1873)). *A connected graph G is eulerian if and only if every vertex has an even degree.*

Proof. ($\Rightarrow$) Suppose $W\colon u \xrightarrow{\star} u$ is an Euler tour. Let v ($\neq u$) be a vertex that occurs k times in W. Every time an edge arrives at v, another edge departs from v, and therefore $d_G(v) = 2k$. Also, $d_G(u)$ is even, since W starts and ends at u.

($\Leftarrow$) Assume G is a nontrivial connected graph such that $d_G(v)$ is even for all $v \in G$. Let

$$W = e_1e_2 \dots e_n\colon v_0 \xrightarrow{\star} v_n \quad \text{with} \quad e_i = v_{i-1}v_i$$

be a longest trail in G. It follows that all $e = v_nw \in E_G$ are among the edges of W, for, otherwise, W could be prolonged to We. In particular, $v_0 = v_n$, that is, W is a closed trail. (Indeed, if it were $v_n \neq v_0$ and v_n occurs k times in W, then $d_G(v_n) = 2(k-1)+1$ and that would be odd.)

If W is not an Euler tour, then, since G is connected, there exists an edge $f = v_iu \in E_G$ for some i, which is not in W. However, now

$$e_{i+1} \dots e_ne_1 \dots e_if$$

is a trail in G, and it is longer than W. This contradiction to the choice of W proves the claim. □

Example 3.1. The k-cube Q_k is eulerian for even integers k, because Q_k is k-regular.

Theorem 3.2. *A connected graph has an Euler trail if and only if it has at most two vertices of odd degree.*

Proof. If G has an Euler trail $u \xrightarrow{\star} v$, then, as in the proof of Theorem 3.1, each vertex $w \notin \{u, v\}$ has an even degree.

Assume then that G is connected and has at most two vertices of odd degree. If G has no vertices of odd degree then, by Theorem 3.1, G has an Euler trail. Otherwise, by the handshaking lemma, every graph has an even number of vertices with odd degree, and therefore G has exactly two such vertices, say u and v. Let H be a graph obtained from G by adding a vertex w, and the edges uw and vw. In H every vertex has an even degree, and hence it has an Euler tour, say $u \xrightarrow{\star} v \longrightarrow w \longrightarrow u$. Here the beginning part $u \xrightarrow{\star} v$ is an Euler trail of G. □

The Chinese postman

The following problem is due to GUAN MEIGU (1962). Consider a village, where a postman wishes to plan his route to save the legs, but still every street has to be walked through. This problem is akin to Euler's problem and to the shortest path problem.

Let G be a graph with a weight function $\alpha\colon E_G \to \mathbb{R}^+$. The **Chinese postman problem** is to find a minimum weighted tour in G (starting from a given vertex, the post office).

If G is *eulerian*, then any Euler tour will do as a solution, because such a tour traverses each edge exactly once and this is the best one can do. In this case the weight of the optimal tour is the total weight of the graph G, and there is a good algorithm for finding such a tour:

Fleury's algorithm:

- Let $v_0 \in G$ be a chosen vertex, and let W_0 be the trivial path on v_0.
- Repeat the following procedure for $i = 1, 2, \ldots$ as long as possible: suppose a trail $W_i = e_1 e_2 \ldots e_i$ has been constructed, where $e_j = v_{j-1} v_j$.
  Choose an edge e_{i+1} ($\neq e_j$ for $j \in [1, i]$) so that

(i) e_{i+1} has an end v_i, and

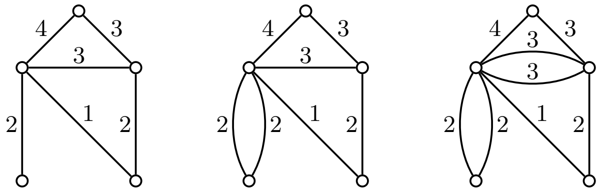
(ii) e_{i+1} is not a bridge of $G_i = G - \{e_1, \ldots, e_i\}$, unless there is no alternative.

Notice that, as is natural, the weights $\alpha(e)$ play no role in the eulerian case.

Theorem 3.3. *If G is eulerian, then any trail of G constructed by Fleury's algorithm is an Euler tour of G.*

Proof. Exercise. □

If G is *not eulerian*, the poor postman has to walk at least one street twice. This happens, *e.g.*, if one of the streets is a dead end, and in general if there is a street corner of an odd number of streets. We can attack this case by reducing it to the eulerian case as follows. An edge $e = uv$ will be **duplicated**, if it is added to G parallel to an existing edge $e' = uv$ with the same weight, $\alpha(e') = \alpha(e)$.



Above we have duplicated two edges. The rightmost multigraph is eulerian.

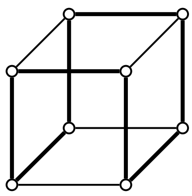
There is a good algorithm by EDMONDS AND JOHNSON (1973) for the construction of an optimal eulerian supergraph by duplications. Unfortunately, this algorithm is somewhat complicated, and we shall skip it.

3.2 Hamiltonian graphs

In the connector problem we reduced the cost of a spanning graph to its minimum. There are different problems, where the cost is measured by an active user of the graph. For instance, in the **travelling salesman problem** a person is supposed to visit each town in his district, and this he should do in such a way that saves time and money. Obviously, he should plan the travel so as to visit each town once, and so that the overall flight time is as short as possible. In terms of graphs, he is looking for a minimum weighted Hamilton cycle of a graph, the vertices of which are the towns and the weights on the edges are the flight times. Unlike for the shortest path and the connector problems no efficient reliable algorithm is known for the travelling salesman problem. Indeed, it is widely believed that no practical algorithm exists for this problem.

Hamilton cycles

DEFINITION. A path P of a graph G is a **Hamilton path**, if P visits every vertex of G once. Similarly, a cycle C is a **Hamilton cycle**, if it visits each vertex once. A graph is **hamiltonian**, if it has a Hamilton cycle.



Note that if $C : u_1 \to u_2 \to \cdots \to u_n$ is a Hamilton cycle, then so is $u_i \to \dots u_n \to u_1 \to \dots u_{i-1}$ for each $i \in [1, n]$, and thus we can choose where to start the cycle.

Example 3.2. It is obvious that each K_n is hamiltonian whenever $n \geq 3$. Also, as is easily seen, $K_{n,m}$ is hamiltonian if and only if $n = m \geq 2$. Indeed, let $K_{n,m}$ have a bipartition

(X,Y), where $|X| = n$ and $|Y| = m$. Now, each cycle in $K_{n,m}$ has even length as the graph is bipartite, and thus the cycle visits the sets X,Y equally many times, since X and Y are stable subsets. But then necessarily $|X| = |Y|$.

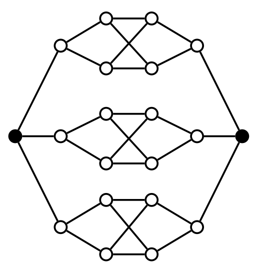
Unlike for eulerian graphs (Theorem 3.1) no good characterization is known for hamiltonian graphs. Indeed, the problem to determine if G is hamiltonian is NP-complete. There are, however, some interesting general conditions.

Lemma 3.1. *If G is hamiltonian, then for every nonempty subset $S \subseteq V_G$,*

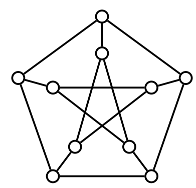
$$c(G{-}S) \le |S| \ .$$

Proof. Let $\emptyset \neq S \subseteq V_G$, $u \in S$, and let $C\colon u \xrightarrow{\star} u$ be a Hamilton cycle of G. Assume $G{-}S$ has k connected components, G_i, $i \in [1,k]$. The case $k = 1$ is trivial, and hence suppose that $k > 1$. Let u_i be the last vertex of C that belongs to G_i, and let v_i be the vertex that follows u_i in C. Now $v_i \in S$ for each i by the choice of u_i, and $v_j \neq v_t$ for all $j \neq t$, because C is a cycle and $u_i v_i \in E_G$ for all i. Thus $|S| \ge k$ as required. □

Example 3.3. Consider the graph on the right. In G, $c(G{-}S) = 3 > 2 = |S|$ for the set S of black vertices. Therefore G does not satisfy the condition of Lemma 3.1, and hence it is not hamiltonian. Interestingly this graph is (X,Y)-bipartite of even order with $|X| = |Y|$. It is also 3-regular.



Example 3.4. Consider the **Petersen graph** on the right, which appears in many places in graph theory as a counter example for various conditions. This graph is not hamiltonian, but it does satisfy the condition $c(G{-}S) \le |S|$ for all $S \neq \emptyset$. Therefore the conclusion of Lemma 3.1 is *not sufficient* to ensure that a graph is hamiltonian.



The following theorem, due to ORE, generalizes an earlier result by DIRAC (1952).

Theorem 3.4 (ORE (1962)). *Let G be a graph of order $\nu_G \ge 3$, and let $u,v \in G$ be such that*

$$d_G(u) + d_G(v) \ge \nu_G \ .$$

Then G is hamiltonian if and only if $G + uv$ is hamiltonian.

Proof. Denote $n = \nu_G$. Let $u,v \in G$ be such that $d_G(u) + d_G(v) \ge n$. If $uv \in E_G$, then there is nothing to prove. Assume thus that $uv \notin E_G$.

($\Rightarrow$) This is trivial since if G has a Hamilton cycle C, then C is also a Hamilton cycle of $G + uv$.

($\Leftarrow$) Denote $e = uv$ and suppose that $G + e$ has a Hamilton cycle C. If C does not use the edge e, then it is a Hamilton cycle of G. Suppose thus that e is on C. We may then assume that $C\colon u \xrightarrow{\star} v \longrightarrow u$. Now $u = v_1 \longrightarrow v_2 \longrightarrow \dots \longrightarrow v_n = v$ is a Hamilton *path* of G. There exists an i with $1 < i < n$ such that $uv_i \in E_G$ and $v_{i-1}v \in E_G$. For, otherwise, $d_G(v) < n - d_G(u)$ would contradict the assumption.

$$v_1 — v_2 — \circ — \circ — v_{i-1} — v_i — \circ — \circ — v_n$$

But now $u = v_1 \xrightarrow{\star} v_{i-1} \longrightarrow v_n \longrightarrow v_{n-1} \xrightarrow{\star} v_{i+1} \longrightarrow v_i \longrightarrow v_1 = u$ is a Hamilton cycle in G. □

Closure

DEFINITION. For a graph G, define inductively a sequence $G_0, G_1, \dots, G_k$ of graphs such that

$$G_0 = G \text{ and } G_{i+1} = G_i + uv\ ,$$

where u and v are any vertices such that $uv \notin E_{G_i}$ and $d_{G_i}(u) + d_{G_i}(v) \geq \nu_G$. This procedure stops when no new edges can be added to G_k for some k, that is, in G_k, for all $u, v \in G$ either $uv \in E_{G_k}$ or $d_{G_k}(u) + d_{G_k}(v) < \nu_G$. The result of this procedure is the **closure** of G, and it is denoted by $cl(G)$ $(= G_k)$.

In each step of the construction of $cl(G)$ there are usually alternatives which edge uv is to be added to the graph, and therefore the above procedure is not deterministic. However, the *final result* $cl(G)$ is independent of the choices.

Lemma 3.2. *The closure $cl(G)$ is uniquely defined for all graphs G of order $\nu_G \geq 3$.*

Proof. Denote $n = \nu_G$. Suppose there are two ways to close G, say

$$H = G + \{e_1, \dots, e_r\} \text{ and } H' = G + \{f_1, \dots, f_s\}\ ,$$

where the edges are added in the given orders. Let $H_i = G + \{e_1, \dots, e_i\}$ and $H'_i = G + \{f_1, \dots, f_i\}$. For the initial values, we have $G = H_0 = H'_0$. Let $e_k = uv$ be the first edge such that $e_k \neq f_i$ for all i. Then $d_{H_{k-1}}(u) + d_{H_{k-1}}(v) \geq n$, since $e_k \in E_{H_k}$, but $e_k \notin E_{H_{k-1}}$. By the choice of e_k, we have $H_{k-1} \subseteq H'$, and thus also $d_{H'}(u) + d_{H'}(v) \geq n$, which means that $e = uv$ must be in H'; a contradiction. Therefore $H \subseteq H'$. Symmetrically, we deduce that $H' \subseteq H$, and hence $H' = H$. □

Theorem 3.5. *Let G be a graph of order $\nu_G \geq 3$.*

(i) G is hamiltonian if and only if its closure $cl(G)$ is hamiltonian.
(ii) If $cl(G)$ is a complete graph, then G is hamiltonian.

Proof. First, $G \subseteq cl(G)$ and G spans $cl(G)$, and thus if G is hamiltonian, so is $cl(G)$.

In the other direction, let $G = G_0, G_1, \dots, G_k = cl(G)$ be a construction sequence of the closure of G. If $cl(G)$ is hamiltonian, then so are $G_{k-1}, \dots, G_1$ and G_0 by Theorem 3.4.

The Claim (ii) follows from (i), since each complete graph is hamiltonian. □

Theorem 3.6. *Let G be a graph of order $\nu_G \geq 3$. Suppose that for all nonadjacent vertices u and v, $d_G(u) + d_G(v) \geq \nu_G$. Then G is hamiltonian. In particular, if $\delta(G) \geq \frac{1}{2}\nu_G$, then G is hamiltonian.*

Proof. Since $d_G(u)+d_G(v) \geq \nu_G$ for all nonadjacent vertices, we have $cl(G) = K_n$ for $n = \nu_G$, and thus G is hamiltonian. The second claim is immediate, since now $d_G(u)+d_G(v) \geq \nu_G$ for all $u, v \in G$ whether adjacent or not. □

Chvátal's condition

The hamiltonian problem of graphs has attracted much attention, at least partly because the problem has practical significance. (Indeed, the first example where DNA computing was applied, was the hamiltonian problem.)

There are some general improvements of the previous results of this chapter, and quite many improvements in various special cases, where the graphs are somehow restricted. We become satisfied by two general results.

Theorem 3.7 (Chvátal **(1972)).** *Let G be a graph with $V_G = \{v_1, v_2, \dots, v_n\}$, for $n \geq 3$, ordered so that $d_1 \leq d_2 \leq \cdots \leq d_n$, for $d_i = d_G(v_i)$. If for every $i < n/2$,*

$$d_i \leq i \implies d_{n-i} \geq n-i\ , \tag{3.1}$$

then G is hamiltonian.

Proof. First of all, we may suppose that G is closed, $G = cl(G)$, because G is hamiltonian if and only if $cl(G)$ is hamiltonian, and adding edges to G does not decrease any of its degrees, that is, if G satisfies (3.1), so does $G + e$ for every e. We show that, in this case, $G = K_n$, and thus G is hamiltonian.

Assume on the contrary that $G \neq K_n$, and let $uv \notin E_G$ with $d_G(u) \leq d_G(v)$ be such that $d_G(u)+d_G(v)$ is as large as possible. Because G is closed, we must have $d_G(u)+d_G(v) < n$, and therefore $d_G(u) = i < n/2$. Let $A = \{w \mid vw \notin E_G, w \neq v\}$. By our choice, $d_G(w) \leq i$ for all $w \in A$, and, moreover,

$$|A| = (n-1) - d_G(v) \geq d_G(u) = i\ .$$

Consequently, there are at least i vertices w with $d_G(w) \leq i$, and so $d_i \leq d_G(u) = i$.

Similarly, for each vertex from $B = \{w \mid uw \notin E_G, w \neq u\}$, $d_G(w) \leq d_G(v) < n - d_G(u) = n - i$, and

$$|B| = (n-1) - d_G(u) = (n-1) - i\ .$$

Also $d_G(u) < n - i$, and thus there are at least $n - i$ vertices w with $d_G(w) < n - i$. Consequently, $d_{n-i} < n - i$. This contradicts the obtained bound $d_i \leq i$ and the condition (3.1). □

Note that the condition (3.1) is easily checkable for any given graph.

3.3 Matchings

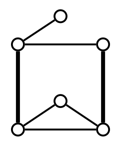
In matching problems we are given an availability relation between the elements of a set. The problem is then to find a pairing of the elements so that each element is paired (matched) uniquely with an available companion.

A special case of the matching problem is the **marriage problem**, which is stated as follows. Given a set X of boys and a set Y of girls, under what condition can each boy marry a girl who cares to marry him? This problem has many variations. One of them is the **job assignment problem**, where we are given n applicants and m jobs, and we should assign each applicant to a job he is qualified. The problem is that an applicant may be qualified for several jobs, and a job may be suited for several applicants.

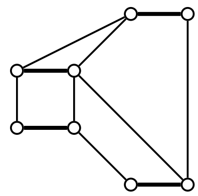
Maximum matchings

DEFINITION. For a graph G, a subset $M \subseteq E_G$ is a **matching** of G, if M contains no adjacent edges. The two ends of an edge $e \in M$ are **matched under** M. A matching M is a **maximum matching**, if for no matching M', $|M| < |M'|$.

The two vertical edges on the right constitute a matching M that is *not a maximum matching*, although you cannot add any edges to M to form a larger matching. This matching is not maximum because the graph has a matching of three edges.



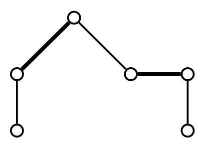
DEFINITION. A matching M **saturates** $v \in G$, if v is an end of an edge in M. Also, M **saturates** $A \subseteq V_G$, if it saturates every $v \in A$. If M saturates V_G, then M is a **perfect matching**.



It is clear that every perfect matching is maximum.
On the right the horizontal edges form a perfect matching.

DEFINITION. Let M be a matching of G. An odd path $P = e_1e_2 \dots e_{2k+1}$ is M**-augmented**, if

- P alternates between $E_G \setminus M$ and M (that is, $e_{2i+1} \in E_{G-M}$ and $e_{2i} \in M$), and
- the ends of P are not saturated.



Lemma 3.3. *If G is connected with $\Delta(G) \leq 2$, then G is a path or a cycle.*

Proof. Exercise. □

We start with a result that gives a necessary and sufficient condition for a matching to be maximum. One can use the first part of the proof to construct a maximum matching in an iterative manner starting from any matching M and from any M-augmented path.

Theorem 3.8 (BERGE **(1957)).** *A matching M of G is a maximum matching if and only if there are no M-augmented paths in G.*

Proof. ($\Rightarrow$) Let a matching M have an M-augmented path $P = e_1 e_2 \dots e_{2k+1}$ in G. Here $e_2, e_4, \dots, e_{2k} \in M$, $e_1, e_3, \dots, e_{2k+1} \notin M$. Define $N \subseteq E_G$ by

$$N = (M \setminus \{e_{2i} \mid i \in [1,k]\}) \cup \{e_{2i+1} \mid i \in [0,k]\} .$$

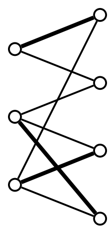
Now, N is a matching of G, and $|N| = |M| + 1$. Therefore M is not a maximum matching.

($\Leftarrow$) Assume N is a maximum matching, but M is not. Hence $|N| > |M|$. Consider the subgraph $H = G[M \triangle N]$ for the symmetric difference $M \triangle N$. We have $d_H(v) \le 2$ for each $v \in H$, because v is an end of at most one edge in M and N. By Lemma 3.3, each connected component A of H is either a path or a cycle.

Since no $v \in A$ can be an end of two edges from N or from M, each connected component (path or a cycle) A alternates between N and M. Now, since $|N| > |M|$, there is a connected component A of H, which has more edges from N than from M. This A cannot be a cycle, because an alternating cycle has even length, and it thus contains equally many edges from N and M. Hence $A\colon u \xrightarrow{\star} v$ is a path (of odd length), which starts and ends with an edge from N. Because A is a connected component of H, the ends u and v are not saturated by M, and, consequently, A is an M-augmented path. This proves the theorem. $\square$

Example 3.5. Consider the k-cube Q_k for $k \ge 1$. Each maximum matching of Q_k has 2^{k-1} edges. Indeed, the matching $M = \{(0u, 1u) \mid u \in \mathbb{B}^{k-1}\}$, has 2^{k-1} edges, and it is clearly perfect.

Hall's theorem



For a subset $S \subseteq V_G$ of a graph G, denote

$$N_G(S) = \{v \mid uv \in E_G \text{ for some } u \in S\} .$$

If G is (X,Y)-bipartite, and $S \subseteq X$, then $N_G(S) \subseteq Y$.

The following result, known as the

Theorem 3.9 (HALL **(1935)).** *Let G be a (X,Y)-bipartite graph. Then G contains a matching M saturating X if and only if*

$$|S| \le |N_G(S)| \quad \textit{for all } S \subseteq X. \tag{3.2}$$

Proof. $(\Rightarrow)$ Let M be a matching that saturates X. If $|S| > |N_G(S)|$ for some $S \subseteq X$, then not all $x \in S$ can be matched with different $y \in N_G(S)$.

$(\Leftarrow)$ Let G satisfy Hall's condition (3.2). We prove the claim by induction on $|X|$.

If $|X| = 1$, then the claim is clear. Let then $|X| \geq 2$, and assume (3.2) implies the existence of a matching that saturates every proper subset of X.

If $|N_G(S)| \geq |S| + 1$ for every nonempty $S \subseteq X$ with $S \neq X$, then choose an edge $uv \in E_G$ with $u \in X$, and consider the induced subgraph $H = G - \{u, v\}$. For all $S \subseteq X \setminus \{u\}$,

$$|N_H(S)| \geq |N_G(S)| - 1 \geq |S| \ ,$$

and hence, by the induction hypothesis, H contains a matching M saturating $X \setminus \{u\}$. Now $M \cup \{uv\}$ is a matching saturating X in G, as was required.

Suppose then that there exists a nonempty subset $R \subseteq X$ with $R \neq X$ such that $|N_G(R)| = |R|$. The induced subgraph $H_1 = G[R \cup N_G(R)]$ satisfies (3.2) (since G does), and hence, by the induction hypothesis, H_1 contains a matching M_1 that saturates R (with the other ends in $N_G(R)$).

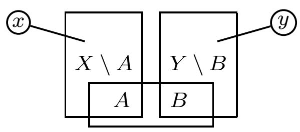
Also, the induced subgraph $H_2 = G[V_G \setminus A]$, for $A = R \cup N_G(R)$, satisfies (3.2). Indeed, if there were a subset $S \subseteq X \setminus R$ such that $|N_{H_2}(S)| < |S|$, then we would have

$$|N_G(S \cup R)| = |N_{H_2}(S)| + |N_{H_1}(R)| < |S| + |N_G(R)| = |S| + |R| = |S \cup R|$$

(since $S \cap R = \emptyset$), which contradicts (3.2) for G. By the induction hypothesis, H_2 has a matching M_2 that saturates $X \setminus R$ (with the other ends in $Y \setminus N_G(R)$). Combining the matchings for H_1 and H_2, we get a matching $M_1 \cup M_2$ saturating X in G. $\square$

Second proof. This proof of the direction $(\Leftarrow)$ uses Menger's theorem. Let H be the graph obtained from G by adding two new vertices x, y such that x is adjacent to each $v \in X$ and y is adjacent to each $v \in Y$. There exists a matching saturating X if (and only if) the number of independent paths $x \xrightarrow{\star} y$ is equal to $|X|$. For this, by Menger's theorem, it suffices to show that every set S that separates x and y in H has at least $|X|$ vertices.

Let $S = A \cup B$, where $A \subseteq X$ and $B \subseteq Y$. Now, vertices in $X \setminus A$ are not adjacent to vertices of $Y \setminus B$, and hence we have $N_G(X \setminus A) \subseteq B$, and thus that $|X \setminus A| \leq |N_G(X \setminus A)| \leq |B|$ using the condition (3.2).



We conclude that $|S| = |A| + |B| \geq |X|$. $\square$

Corollary 3.1 (FROBENIUS **(1917)).** *If G is a k-regular bipartite graph with $k > 0$, then G has a perfect matching.*

Proof. Let G be k-regular (X, Y)-bipartite graph. By regularity, $k \cdot |X| = \varepsilon_G = k \cdot |Y|$, and hence $|X| = |Y|$. Let $S \subseteq X$. Denote by E_1 the set of the edges with an end in S, and by E_2 the set of the edges with an end in $N_G(S)$. Clearly, $E_1 \subseteq E_2$. Therefore, $k \cdot |N_G(S)| = |E_2| \geq |E_1| = k \cdot |S|$, and so $|N_G(S)| \geq |S|$. By Theorem 3.9, G has a matching that saturates X. Since $|X| = |Y|$, this matching is necessarily perfect. $\square$

Applications of Hall's theorem

DEFINITION. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ be a family of finite nonempty subsets of a set S. (S_i need not be distinct.) A **transversal** (or a **system of distinct representatives**) of $\mathcal{S}$ is a subset $T \subseteq S$ of m distinct elements one from each S_i.

As an example, let $S = [1, 6]$, and let $S_1 = S_2 = \{1, 2\}$, $S_3 = \{2, 3\}$ and $S_4 = \{1, 4, 5, 6\}$. For $\mathcal{S} = \{S_1, S_2, S_3, S_4\}$, the set $T = \{1, 2, 3, 4\}$ is a transversal. If we add the set $S_5 = \{2, 3\}$ to $\mathcal{S}$, then it is impossible to find a transversal for this new family.

The connection of transversals to the Marriage Theorem is as follows. Let $S = Y$ and $X = [1, m]$. Form an (X, Y)-bipartite graph G such that there is an edge (i, s) if and only if $s \in S_i$. The possible transversals T of $\mathcal{S}$ are then obtained from the matchings M saturating X in G by taking the ends in Y of the edges of M.

Corollary 3.2. *Let $\mathcal{S}$ be a family of finite nonempty sets. Then $\mathcal{S}$ has a transversal if and only if the union of any k of the subsets S_i of $\mathcal{S}$ contains at least k elements.*

Example 3.6. An $m \times n$ **latin rectangle** is an $m \times n$ integer matrix M with entries $M_{ij} \in [1, n]$ such that the entries in the same row and in the same column are different. Moreover, if $m = n$, then M is a **latin square**. Note that in a $m \times n$ latin rectangle M, we always have that $m \leq n$.

We show the following: *Let M be an $m \times n$ latin rectangle (with $m < n$). Then M can be extended to a latin square by the addition of $n - m$ new rows.*

The claim follows when we show that M can be extended to an $(m+1) \times n$ latin rectangle. Let $A_i \subseteq [1, n]$ be the set of those elements that do not occur in the i-th column of M. Clearly, $|A_i| = n - m$ for each i, and hence $\sum_{i \in I} |A_i| = |I|(n - m)$ for all subsets $I \subseteq [1, n]$. Now $|\cup_{i \in I} A_i| \geq |I|$, since otherwise at least one element from the union would be in more than $n - m$ of the sets A_i with $i \in I$. However, each row has all the n elements, and therefore each i is missing from exactly $n - m$ columns. By Marriage Theorem, the family $\{A_1, A_2, \ldots, A_n\}$ has a transversal, and this transversal can be added as a new row to M. This proves the claim.

Tutte's theorem

The next theorem is a classic characterization of perfect matchings.

DEFINITION. A connected component of a graph G is said to be **odd** (**even**), if it has an odd (even) number of vertices. Denote by $c_{\text{odd}}(G)$ the number of odd connected components in G.

Denote by $m(G)$ be the number of edges in a maximum matching of a graph G.

Theorem 3.10 (Tutte-Berge Formula). *Each maximum matching of a graph G has*

$$m(G) = \min_{S \subseteq V_G} \frac{\nu_G + |S| - c_{\text{odd}}(G - S)}{2} \tag{3.3}$$

elements.

Note that the condition in (ii) includes the case, where $S = \emptyset$.

Proof. We prove the result for connected graphs. The result then follows for disconnected graphs by adding the formulas for the connected components.

We observe first that $\leq$ holds in (3.3), since, for all $S \subseteq V_G$,

$$m(G) \leq |S| + m(G{-}S) \leq |S| + \frac{|V_G \setminus S| - c_{\mathrm{odd}}(G{-}S)}{2} = \frac{\nu_G + |S| - c_{\mathrm{odd}}(G{-}S)}{2}.$$

Indeed, each odd component of $G{-}S$ must have at least one unsaturated vertex.

The proof proceeds by induction on ν_G. If $\nu_G = 1$, then the claim is trivial. Suppose that $\nu_G \geq 2$.

Assume first that there exists a vertex $v \in G$ such that v is saturated by all maximum matchings. Then $m(G{-}v) = m(G) - 1$. For a subset $S' \subseteq G{-}v$, denote $S = S' \cup \{v\}$. By the induction hypothesis, for all $S' \subseteq G{-}v$,

$$m(G){-}1 \geq \frac{1}{2}\big((\nu_G - 1) + |S'| - c_{\mathrm{odd}}(G{-}(S' \cup \{v\}))\big) = \frac{1}{2}\left((\nu_G + |S| - c_{\mathrm{odd}}(G{-}S)\right)){-}1.$$

The claim follows from this.

Suppose then that for each vertex v, there is a maximum matching that does not saturate v. We claim that $m(G) = (\nu_G - 1)/2$. Suppose to the contrary, and let M be a maximum matching having two different unsaturated vertices u and v, and choose M so that the distance $d_G(u, v)$ is as small as possible. Now $d_G(u, v) \geq 2$, since otherwise $uv \in E_G$ could be added to M, contradicting the maximality of M. Let w be an intermediate vertex on a shortest path $u \xrightarrow{\star} v$. By assumption, there exists a maximum matching N that does not saturate w. We can choose N such that the intersection $M \cap N$ is maximal. Since $d_G(u, w) < d_G(u, v)$ and $d_G(w, v) < d_G(u, v)$, N saturates both u and v. The (maximum) matchings N and M leave equally many vertices unsaturated, and hence there exists another vertex $x \neq w$ saturated by M but which is unsaturated by N. Let $e = xy \in M$. If y is also unsaturated by N, then $N \cup \{e\}$ is a matching, contradicting maximality of N. It also follows that $y \neq w$. Therefore there exists an edge $e' = yz$ in N, where $z \neq x$. But now $N' = N \cup \{e\} \setminus \{e'\}$ is a maximum matching that does not saturate w. However, $N \cap M \subset N' \cap M$ contradicts the choice of N. Therefore, every maximum matching leaves exactly one vertex unsaturated, i.e., $m(G) = (\nu_G - 1)/2$.

In this case, for $S = \emptyset$, the right hand side of (3.3) gets value $(\nu_G - 1)/2$, and hence, by the beginning of the proof, this must be the minimum of the right hand side. □

For perfect matchings we have the following corollary, since for a perfect matching we have $m(G) = (1/2)\nu_G$.

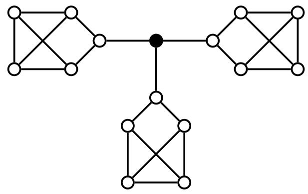
Theorem 3.11 (TUTTE **(1947)).** *Let G be a nontrivial graph. The following are equivalent.*

(i) G has a perfect matching.
(ii) For every proper subset $S \subset V_G$, $c_{\mathrm{odd}}(G{-}S) \leq |S|$.

Tutte's theorem does not provide a good algorithm for constructing a perfect matching, because the theorem requires 'too many cases'. Its applications are mainly in the proofs of other results that are related to matchings. There is a good algorithm due to EDMONDS (1965), which uses 'blossom shrinkings', but this algorithm is somewhat involved.

Example 3.7. The simplest connected graph that has no perfect matching is the path P_3. Here removing the middle vertex creates two odd components.

The next 3-regular graph (known as the **Sylvester graph**) does not have a perfect matching, because removing the black vertex results in a graph with three odd connected components. This graph is the smallest regular graph with an odd degree that has no perfect matching.



Using Theorem 3.11 we can give a short proof of PETERSEN's result for 3-regular graphs (1891).

Theorem 3.12 (PETERSEN **(1891)).** *If G is a bridgeless 3-regular graph, then it has a perfect matching.*

Proof. Let S be a proper subset of V_G, and let G_i, $i \in [1,t]$, be the odd connected components of $G-S$. Denote by m_i the number of edges with one end in G_i and the other in S. Since G is 3-regular,

$$\sum_{v \in G_i} d_G(v) = 3 \cdot \nu_{G_i} \quad \text{and} \quad \sum_{v \in S} d_G(v) = 3 \cdot |S| \ .$$

The first of these implies that

$$m_i = \sum_{v \in G_i} d_G(v) - 2 \cdot \varepsilon_{G_i}$$

is odd. Furthermore, $m_i \neq 1$, because G has no bridges, and therefore $m_i \geq 3$. Hence the number of odd connected components of $G-S$ satisfies

$$t \leq \frac{1}{3} \sum_{i=1}^{t} m_i \leq \frac{1}{3} \sum_{v \in S} d_G(v) = |S| \ ,$$

and so, by Theorem 3.11, G has a perfect matching. □

Stable Marriages

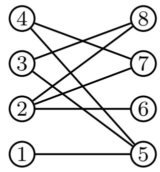
DEFINITION. Consider a bipartite graph G with a bipartition (X,Y) of the vertex set. In addition, each vertex $x \in G$ supplies an order of preferences of the vertices of $N_G(x)$. We write $u <_x v$, if x prefers v to u. (Here $u,v \in Y$, if $x \in X$, and $u,v \in X$, if $x \in Y$.) A matching M of G is said to be **stable**, if for each unmatched pair $xy \notin M$ (with $x \in X$ and $y \in Y$), it is not the case that x and y prefer each other better than their matched companions:

$$xv \in M \text{ and } y <_x v, \quad \text{or} \quad uy \in M \text{ and } x <_y u.$$

We omit the proof of the next theorem.

Theorem 3.13. *For bipartite graphs G, a stable matching exists for all lists of preferences.*

Example 3.8. That was the good news. There is a catch, of course. A stable matching need not saturate X and Y. For instance, the graph on the right does have a perfect matching (of 4 edges).



Suppose the preferences are the following:

$1\colon 5$	$2\colon 6 < 8 < 7$	$3\colon 8 < 5$	$4\colon 7 < 5$
$5\colon 4 < 1 < 3$	$6\colon 2$	$7\colon 2 < 4$	$8\colon 3 < 2$

Then there is no stable matchings of four edges. A stable matching of G is the following: $M = \{28, 35, 47\}$, which leaves 1 and 6 unmatched. (You should check that there is no stable matching containing the edges 15 and 26.)

Theorem 3.14. *Let $G = K_{n,n}$ be a complete bipartite graph. Then G has a perfect and stable matching for all lists of preferences.*

Proof. Let the bipartition be (X, Y). The algorithm by GALE AND SHAPLEY (1962) works as follows.

Procedure.
Set $M_0 = \emptyset$, and $P(x) = \emptyset$ for all $x \in X$.
Then iterate the following process until all vertices are saturated:
Choose a vertex $x \in X$ that is unsaturated in M_{i-1}. Let $y \in Y$ be the most preferred vertex for x such that $y \notin P(x)$.

(1) Add y to $P(x)$.
(2) If y is not saturated, then set $M_i = M_{i-1} \cup \{xy\}$.
(3) If $zy \in M_{i-1}$ and $z <_y x$, then set $M_i = (M_{i-1} \setminus \{zy\}) \cup \{xy\}$.

First of all, the procedure terminates, since a vertex $x \in X$ takes part in the iteration at most n times (once for each $y \in Y$). The final outcome, say $M = M_t$, is a perfect matching, since the iteration continues until there are no unsaturated vertices $x \in X$.

Also, the matching $M = M_t$ is stable. Note first that, by (3), if $xy \in M_i$ and $zy \in M_j$ for some $x \neq z$ and $i < j$, then $x <_y z$. Assume the that $xy \in M$, but $y <_x z$ for some $z \in Y$. Then xy is added to the matching at some step, $xy \in M_i$, which means that $z \in P(x)$ at this step (otherwise x would have 'proposed' z). Hence x took part in the iteration at an earlier step M_k, $k < i$ (where z was put to the list $P(x)$, but xz was not added). Thus, for some $u \in X$, $uz \in M_{k-1}$ and $x <_z u$, and so in M the vertex z is matched to some w with $x <_z w$.

Similarly, if $x <_y v$ for some $v \in X$, then $y <_v z$ for the vertex $z \in Y$ such that $vz \in M$.

□

4

Colourings

4.1 Edge colourings

Colourings of edges and vertices of a graph G are useful, when one is interested in classifying relations between objects.

There are two sides of colourings. In the general case, a graph G with a colouring α is given, and we study the properties of this pair $G^\alpha = (G, \alpha)$. This is the situation, *e.g.*, in transportation networks with bus and train links, where the colour (*buss, train*) of an edge tells the nature of a link.

In the chromatic theory, G is first given and then we search for a colouring that the satisfies required properties. One of the important properties of colourings is 'properness'. In a proper colouring adjacent edges or vertices are coloured differently.

Edge chromatic number

DEFINITION. A k-**edge colouring** $\alpha\colon E_G \to [1, k]$ of a graph G is an assignment of k colours to its edges. We write G^α to indicate that G has the edge colouring α.

A vertex $v \in G$ and a colour $i \in [1, k]$ are **incident** with each other, if $\alpha(vu) = i$ for some $vu \in E_G$. If $v \in G$ is not incident with a colour i, then i is **available** for v.

The colouring α is **proper**, if no two adjacent edges obtain the same colour: $\alpha(e_1) \neq \alpha(e_2)$ for adjacent e_1 and e_2.

The **edge chromatic number** $\chi'(G)$ of G is defined as

$$\chi'(G) = \min\{k \mid \text{ there exists a proper } k\text{-edge colouring of } G\}\ .$$

A k-edge colouring α can be thought of as a partition $\{E_1, E_2, \dots, E_k\}$ of E_G, where $E_i = \{e \mid \alpha(e) = i\}$. Note that it is possible that $E_i = \emptyset$ for some i. We adopt a simplified notation

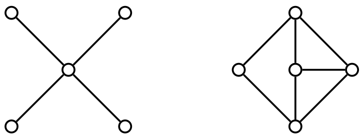
$$G^\alpha[i_1, i_2, \dots, i_t] = G[E_{i_1} \cup E_{i_2} \cup \dots \cup E_{i_t}]$$

for the subgraph of G consisting of those edges that have a colour i_1, i_2, ..., or i_t. That is, the edges having other colours are removed.

Lemma 4.1. *Each colour set E_i in a proper k-edge colouring is a matching. Moreover, for each graph G, $\Delta(G) \leq \chi'(G) \leq \varepsilon_G$.*

Proof. This is clear. $\square$

Example 4.1. The three numbers in Lemma 4.1 can be equal. This happens, for instance, when $G = K_{1,n}$ is a star. But often the inequalities are strict.



A star, and a graph with $\chi'(G) = 4$.

Optimal colourings

We show that for bipartite graphs the lower bound is always optimal: $\chi'(G) = \Delta(G)$.

Lemma 4.2. *Let G be a connected graph that is not an odd cycle. Then there exists a 2-edge colouring (that need not be proper), in which both colours are incident with each vertex v with $d_G(v) \geq 2$.*

Proof. Assume that G is nontrivial; otherwise, the claim is trivial.

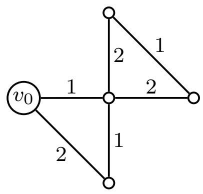
(1) Suppose first that G is eulerian. If G is an even cycle, then a 2-edge colouring exists as required. Otherwise, since now $d_G(v)$ is even for all v, G has a vertex v_1 with $d_G(v_1) \geq 4$. Let $e_1 e_2 \dots e_t$ be an Euler tour of G, where $e_i = v_i v_{i+1}$ (and $v_{t+1} = v_1$). Define

$$\alpha(e_i) = \begin{cases} 1, & \text{if } i \text{ is odd },\\ 2, & \text{if } i \text{ is even }. \end{cases}$$

Hence the ends of the edges e_i for $i \in [2, t-1]$ are incident with both colours. All vertices are among these ends. The condition $d_G(v_1) \geq 4$ guarantees this for v_1. Hence the claim holds in the eulerian case.

(2) Suppose then that G is not eulerian. We define a new graph G_0 by adding a vertex v_0 to G and connecting v_0 to each $v \in G$ of odd degree.

In G_0 every vertex has even degree including v_0 (by the handshaking lemma), and hence G_0 is eulerian. Let $e_0 e_1 \dots e_t$ be an eulerian tour of G_0, where $e_i = v_i v_{i+1}$. By the previous case, there is a required colouring α of G_0 as above. Now, α restricted to E_G is a colouring of G as required by the claim, since each vertex v_i with odd degree $d_G(v_i) \geq 3$ is entered and departed at least once in the tour by an edge of the original graph G: $e_{i-1} e_i$.



$\square$

DEFINITION. For a k-edge colouring α of G, let

$$c_\alpha(v) = |\{i \mid v \text{ is incident with } i \in [1, k]\}| \ .$$

A k-edge colouring β is an **improvement** of α, if

$$\sum_{v \in G} c_\beta(v) > \sum_{v \in G} c_\alpha(v) .$$

Also, α is **optimal**, if it cannot be improved.

Notice that we always have $c_\alpha(v) \le d_G(v)$, and if α is proper, then $c_\alpha(v) = d_G(v)$, and in this case α is optimal. Thus an improvement of a colouring is a change towards a proper colouring. Note also that a graph G always has an optimal k-edge colouring, but it need not have any proper k-edge colourings.

The next lemma is obvious.

Lemma 4.3. *An edge colouring α of G is proper if and only if $c_\alpha(v) = d_G(v)$ for all vertices $v \in G$.*

Lemma 4.4. *Let α be an optimal k-edge colouring of G, and let $v \in G$. Suppose that the colour i is available for v, and the colour j is incident with v at least twice. Then the connected component H of $G^\alpha[i,j]$ that contains v, is an odd cycle.*

Proof. Suppose the connected component H is not an odd cycle. By Lemma 4.2, H has a 2-edge colouring $\gamma\colon E_H \to \{i,j\}$, in which both i and j are incident with each vertex x with $d_H(x) \ge 2$. (We have renamed the colours 1 and 2 to i and j.) We obtain a recolouring β of G as follows:

$$\beta(e) = \begin{cases} \gamma(e), & \text{if } e \in E_H , \\ \alpha(e), & \text{if } e \notin E_H . \end{cases}$$

Since $d_H(v) \ge 2$ (by the assumption on the colour j) and in β both colours i and j are now incident with v, $c_\beta(v) = c_\alpha(v) + 1$. Furthermore, by the construction of β, we have $c_\beta(u) \ge c_\alpha(u)$ for all $u \ne v$. Therefore $\sum_{u \in G} c_\beta(u) > \sum_{u \in G} c_\alpha(u)$, which contradicts the optimality of α. Hence H is an odd cycle. □

Theorem 4.1 (KÖNIG (1916)). *If G is bipartite, then $\chi'(G) = \Delta(G)$.*

Proof. Let α be an optimal Δ-edge colouring of a bipartite G, where $\Delta = \Delta(G)$. If there were a $v \in G$ with $c_\alpha(v) < d_G(v)$, then by Lemma 4.4, G would contain an odd cycle. But a bipartite graph does not contain such cycles. Therefore, for all vertices v, $c_\alpha(v) = d_G(v)$. By Lemma 4.3, α is a proper colouring, and $\Delta = \chi'(G)$ as required. □

Vizing's theorem

In general we can have $\chi'(G) > \Delta(G)$ as one of our examples did show. The following important theorem, due to VIZING, shows that the edge chromatic number of a graph G misses $\Delta(G)$ by at most one colour.

Theorem 4.2 (VIZING (1964)). *For any graph G, $\Delta(G) \le \chi'(G) \le \Delta(G) + 1$.*

Proof. Let $\Delta = \Delta(G)$. We need only to show that $\chi'(G) \le \Delta + 1$. Suppose on the contrary that $\chi'(G) > \Delta + 1$, and let α be an optimal $(\Delta + 1)$-edge colouring of G.

We have (trivially) $d_G(u) < \Delta + 1 < \chi'(G)$ for all $u \in G$, and so

Claim 1. *For each $u \in G$, there exists an available colour $b(u)$ for u.*

Moreover, by the counter hypothesis, α is not a proper colouring, and hence there exists a $v \in G$ with $c_\alpha(v) < d_G(v)$, and hence a colour i_1 that is incident with v at least twice, say

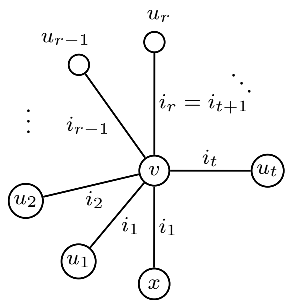
$$\alpha(vu_1) = i_1 = \alpha(vx) \ . \tag{4.1}$$

Claim 2. *There is a sequence of vertices $u_1, u_2, \dots$ such that*

$$\alpha(vu_j) = i_j \ \textit{and} \ i_{j+1} = b(u_j) \ .$$

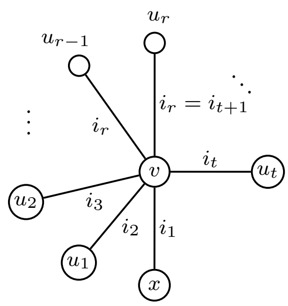
Indeed, let u_1 be as in (4.1). Assume we have already found the vertices $u_1, \dots, u_j$, with $j \geq 1$, such that the claim holds for these. Suppose, contrary to the claim, that v is not incident with $b(u_j) = i_{j+1}$.

We can recolour the edges vu_ℓ by $i_{\ell+1}$ for $\ell \in [1, j]$, and obtain in this way an improvement of α. Here v gains a new colour i_{j+1}. Also, each u_ℓ gains a new colour $i_{\ell+1}$ (and may loose the colour i_ℓ). Therefore, for each u_ℓ either its number of colours remains the same or it increases by one. This contradicts the optimality of α, and proves Claim 2.



Now, let t be the smallest index such that for some $r < t$, $i_{t+1} = i_r$. Such an index t exists, because $d_G(v)$ is finite.

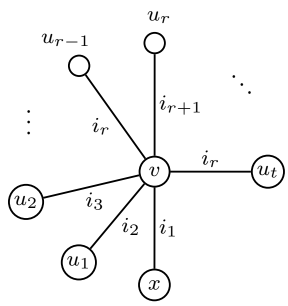
Let β be a recolouring of G such that for $1 \leq j \leq r-1$, $\beta(vu_j) = i_{j+1}$, and for all other edges e, $\beta(e) = \alpha(e)$.



Claim 3. *β is an optimal $(\Delta+1)$-edge colouring of G.*

Indeed, $c_\beta(v) = c_\alpha(v)$ and $c_\beta(u) \geq c_\alpha(u)$ for all u, since each u_j ($1 \leq j \leq r-1$) gains a new colour j_{i+1} although it may loose one of its old colours.

Let then the colouring γ be obtained from β by recolouring the edges vu_j by i_{j+1} for $r \leq j \leq t$. Now, vu_t is recoloured by $i_r = i_{t+1}$.



Claim 4. *γ is an optimal $(\Delta+1)$-edge colouring of G.*

Indeed, the fact $i_r = i_{t+1}$ ensures that i_r is a new colour incident with u_t, and thus that $c_\gamma(u_t) \geq c_\beta(u_t)$. For all other vertices, $c_\gamma(u) \geq c_\beta(u)$ follows as for β.

By Claim 1, there is a colour $i_0 = b(v)$ that is available for v. By Lemma 4.4, the connected components H_1 of $G^\beta[i_0, i_r]$ and H_2 of $G^\gamma[i_0, i_r]$ containing the vertex v are cycles, that is, H_1 is a cycle $(vu_{r-1})P_1(u_r v)$ and H_2 is a cycle $(vu_{r-1})P_2(u_t v)$, where both $P_1 \colon u_{r-1} \xrightarrow{\star} u_r$ and $P_2 \colon u_{r-1} \xrightarrow{\star} u_t$ are paths. However, the edges of P_1 and P_2 have the same colours with respect to β and γ (either i_0 or i_r). This is not possible, since P_1 ends in u_r while P_2 ends in a different vertex u_t. This contradiction proves the theorem. □

Example 4.2. We show that $\chi'(G) = 4$ for the Petersen graph. Indeed, by Vizing' theorem, $\chi'(G) = 3$ or 4. Suppose 3 colours suffice. Let $C \colon v_1 \to \ldots \to v_5 \to v_1$ be the outer cycle and $C' \colon u_1 \to \ldots \to u_5 \to u_1$ the inner cycle of G such that $v_i u_i \in E_G$ for all i.

Observe that every vertex is adjacent to all colours $1, 2, 3$. Now C uses one colour (say 1) once and the other two twice. This can be done uniquely (up to permutations):

$$v_1 \xrightarrow{1} v_2 \xrightarrow{2} v_3 \xrightarrow{3} v_4 \xrightarrow{2} v_5 \xrightarrow{3} v_1.$$

Hence $v_1 \xrightarrow{2} u_1$, $v_2 \xrightarrow{3} u_2$, $v_3 \xrightarrow{1} u_3$, $v_4 \xrightarrow{1} u_4$, $v_5 \xrightarrow{1} u_5$. However, this means that 1 cannot be a colour of any edge in C'. Since C' needs three colours, the claim follows.

Edge Colouring Problem. Vizing's theorem (nor its present proof) does not offer any characterization for the graphs, for which $\chi'(G) = \Delta(G) + 1$. In fact, it is one of the famous open problems of graph theory to find such a characterization. The answer is known (only) for some special classes of graphs. By HOLYER (1981), the problem whether $\chi'(G)$ is $\Delta(G)$ or $\Delta(G) + 1$ is NP-complete.

The proof of Vizing's theorem can be used to obtain a proper colouring of G with at most $\Delta(G) + 1$ colours, when the word 'optimal' is forgotten: colour first the edges as well as you can (if nothing better, then arbitrarily in two colours), and use the proof iteratively to improve the colouring until no improvement is possible – then the proof says that the result is a proper colouring.

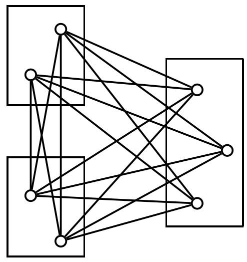
4.2 Ramsey Theory

In general, Ramsey theory studies unavoidable patterns in combinatorics. We consider an instance of this theory mainly for edge colourings (that need not be proper). A typical example of a Ramsey property is the following: given 6 persons each pair of whom are either friends or enemies, there are then 3 persons who are mutual friends or mutual enemies. In graph theoretic terms this means that each colouring of the edges of K_6 with 2 colours results in a monochromatic triangle.

Turan's theorem for complete graphs

We shall first consider the problem of finding a general condition for K_p to appear in a graph. It is clear that every graph contains K_1, and that every nondiscrete graph contains K_2.

DEFINITION. A **complete p-partite graph** G consists of p discrete and disjoint induced subgraphs $G_1, G_2, \ldots, G_p \subseteq G$, where $uv \in E_G$ if and only if u and v belong to different parts, G_i and G_j with $i \neq j$.



Note that a complete p-partite graph is completely determined by its discrete parts G_i, $i \in [1, p]$.

Let $p \geq 3$, and let $H = H_{n,p}$ be the complete $(p-1)$-partite graph of order $n = t(p-1)+r$, where $r \in [1, p-1]$ and $t \geq 0$, such that there are r parts $H_1, \ldots, H_r$ of order $t+1$ and $p-1-r$ parts $H_{r+1}, \ldots, H_{p-1}$ of order t (when $t > 0$). (Here r is the positive residue of n modulo $(p-1)$, and is thus determined by n and p.)

By its definition, $K_p \nsubseteq H$. One can compute that the number ε_H of edges of H is equal to

$$T(n,p) = \frac{p-2}{2(p-1)}n^2 - \frac{r}{2}\left(1 - \frac{r}{p-1}\right) . \tag{4.2}$$

The next result shows that the above bound $T(n,p)$ is optimal.

Theorem 4.3 (TURÁN **(1941)).** *If a graph G of order n has $\varepsilon_G > T(n,p)$ edges, then G contains a complete subgraph K_p.*

Proof. Let $n = (p-1)t + r$ for $1 \leq r \leq p-1$ and $t \geq 0$. We prove the claim by induction on t. If $t = 0$, then $T(n,p) = n(n-1)/2$, and there is nothing to prove.

Suppose then that $t \geq 1$, and let G be a graph of order n such that ε_G is maximum subject to the condition $K_p \nsubseteq G$.

Now G contains a complete subgraph $G[A] = K_{p-1}$, since adding any one edge to G results in a K_p, and $p-1$ vertices of this K_p induce a subgraph $K_{p-1} \subseteq G$.

Each $v \notin A$ is adjacent to at most $p-2$ vertices of A; otherwise $G[A \cup \{v\}] = K_p$. Furthermore, $K_p \nsubseteq G-A$, and $\nu_{G-A} = n-p+1$. Because $n-p+1 = (t-1)(p-1)+r$, we can apply the induction hypothesis to obtain $\varepsilon_{G-A} \leq T(n-p+1, p)$. Now

$$\varepsilon_G \leq T(n-p+1,p) + (n-p+1)(p-2) + \frac{(p-1)(p-2)}{2} = T(n,p) ,$$

which proves the claim. □

When Theorem 4.3 is applied to triangles K_3, we have the following interesting case.

Corollary 4.1 (MANTEL **(1907)).** *If a graph G has $\varepsilon_G > \frac{1}{4}\nu_G^2$ edges, then G contains a triangle K_3.*

Ramsey's theorem

DEFINITION. Let α be an edge colouring of G. A subgraph $H \subseteq G$ is said to be (i-) **monochromatic**, if all edges of H have the same colour i.

The following theorem is one of the jewels of combinatorics.

Theorem 4.4 (RAMSEY (1930)). *Let $p, q \geq 2$ be any integers. Then there exists a (smallest) integer $R(p,q)$ such that for all $n \geq R(p,q)$, any 2-edge colouring of $K_n \to [1,2]$ contains a 1-monochromatic K_p or a 2-monochromatic K_q.*

Before proving this, we give an equivalent statement. Recall that a subset $X \subseteq V_G$ is stable, if $G[X]$ is a discrete graph.

Theorem 4.5. *Let $p, q \geq 2$ be any integers. Then there exists a (smallest) integer $R(p,q)$ such that for all $n \geq R(p,q)$, any graph G of order n contains a complete subgraph of order p or a stable set of order q.*

Be patient, this will follow from Theorem 4.6. The number $R(p,q)$ is known as the **Ramsey number** for p and q.

It is clear that $R(p,2) = p$ and $R(2,q) = q$.

Theorems 4.4 and 4.5 follow from the next result which shows (inductively) that an upper bound exists for the Ramsey numbers $R(p,q)$.

Theorem 4.6 (ERDÖS and SZEKERES (1935)). *The Ramsey number $R(p,q)$ exists for all $p, q \geq 2$, and*

$$R(p,q) \leq R(p,q-1) + R(p-1,q) \ .$$

Proof. We use induction on $p+q$. It is clear that $R(p,q)$ exists for $p=2$ or $q=2$, and it is thus exists for $p+q \leq 5$.

It is now sufficient to show that if G is a graph of order $R(p,q-1)+R(p-1,q)$, then it has a complete subgraph of order p or a stable subset of order q.

Let $v \in G$, and denote by $A = V_G \setminus (N_G(v) \cup \{v\})$ the set of vertices that are not adjacent to v. Since G has $R(p,q-1)+R(p-1,q)-1$ vertices different from v, either $|N_G(v)| \geq R(p-1,q)$ or $|A| \geq R(p,q-1)$ (or both).

Assume first that $|N_G(v)| \geq R(p-1,q)$. By the definition of Ramsey numbers, $G[N_G(v)]$ contains a complete subgraph B of order $p-1$ or a stable subset S of order q. In the first case, $B \cup \{v\}$ induces a complete subgraph K_p in G, and in the second case the same stable set of order q is good for G.

If $|A| \geq R(p,q-1)$, then $G[A]$ contains a complete subgraph of order p or a stable subset S of order $q-1$. In the first case, the same complete subgraph of order p is good for G, and in the second case, $S \cup \{v\}$ is a stable subset of G of q vertices. This proves the claim. $\square$

A concrete upper bound is given in the following result.

Theorem 4.7 (ERDÖS **and** SZEKERES **(1935)).** *For all $p, q \geq 2$,*

$$R(p,q) \leq \binom{p+q-2}{p-1} .$$

Proof. For $p = 2$ or $q = 2$, the claim is clear. We use induction on $p + q$ for the general statement. Assume that $p, q \geq 3$. By Theorem 4.6 and the induction hypothesis,

$$\begin{aligned} R(p,q) &\leq R(p,q-1) + R(p-1,q) \\ &\leq \binom{p+q-3}{p-1} + \binom{p+q-3}{p-2} = \binom{p+q-2}{p-1} , \end{aligned}$$

which is what we wanted. □

In the table below we give some known values and estimates for the Ramsey numbers $R(p, q)$. As can be read from the table[1], not so much is known about these numbers.

$p \backslash q$	3	4	5	6	7	8	9	10
3	6	9	14	18	23	28	36	40-43
4	9	18	25	35-41	49-61	55-84	69-115	80-149
5	14	25	43-49	58-87	80-143	95-216	121-316	141-442

The first unknown $R(p, p)$ (where $p = q$) is for $p = 5$. It has been verified that $43 \leq R(5, 5) \leq 49$, but to determine the exact value is an open problem.

Generalizations

Theorem 4.4 can be generalized as follows.

Theorem 4.8. *Let $q_i \geq 2$ be integers for $i \in [1, k]$ with $k \geq 2$. Then there exists an integer $R = R(q_1, q_2, \ldots, q_k)$ such that for all $n \geq R$, any k-edge colouring of K_n has an i-monochromatic K_{q_i} for some i.*

Proof. The proof is by induction on k. The case $k = 2$ is treated in Theorem 4.4. For $k > 2$, we show that $R(q_1, \ldots, q_k) \leq R(q_1, \ldots, q_{k-2}, p)$, where $p = R(q_{k-1}, q_k)$.

Let $n = R(q_1, \ldots, q_{k-2}, p)$, and let $\alpha \colon E_{K_n} \to [1, k]$ be an edge colouring. Let $\beta \colon E_{K_n} \to [1, k-1]$ be obtained from α by identifying the colours $k - 1$ and k:

$$\beta(e) = \begin{cases} \alpha(e) & \text{if } \alpha(e) < k-1 , \\ k-1 & \text{if } \alpha(e) = k-1 \text{ or } k . \end{cases}$$

By the induction hypothesis, K_n^β has an i-monochromatic K_{q_i} for some $1 \leq i \leq k-2$ (and we are done, since this subgraph is monochromatic in K_n^α) or K_n^β has a $(k-1)$-monochromatic subgraph $H^\beta = K_p$. In the latter case, by Theorem 4.4, H^α and thus K_n^α has a $(k-1)$-monochromatic or a k-monochromatic subgraph, and this proves the claim. □

[1] S.P. RADZISZOWSKI, Small Ramsey numbers, Electronic J. of Combin., 2000 on the Web

Since for each graph H, $H \subseteq K_m$ for $m = \nu_H$, we have

Corollary 4.2. *Let $k \geq 2$ and $H_1, H_2, \dots, H_k$ be arbitrary graphs. Then there exists an integer $R(H_1, H_2, \dots, H_k)$ such that for all complete graphs K_n with $n \geq R(H_1, H_2, \dots, H_k)$ and for all k-edge colourings α of K_n, K_n^α contains an i-monochromatic subgraph H_i for some i.*

This generalization is trivial from Theorem 4.8. However, the generalized Ramsey numbers $R(H_1, H_2, \dots, H_k)$ can be much smaller than their counter parts (for complete graphs) in Theorem 4.8.

Example 4.3. We leave the following statement as an exercise: If T is a tree of order m, then

$$R(T, K_n) = (m-1)(n-1) + 1 \ ,$$

that is, any graph G of order at least $R(T, K_n)$ contains a subgraph isomorphic to T, or the complement of G contains a complete subgraph K_n.

Examples of Ramsey numbers*

Some exact values are known in Corollary 4.2, even in more general cases, for some dear graphs (see RADZISZOWSKI's survey). Below we list some of these results for cases, where the graphs are equal. To this end, let

$$R_k(G) = R(G, G, \dots, G) \qquad (k \text{ times } G).$$

The best known lower bound of $R_2(G)$ for connected graphs was obtained by BURR AND ERDÖS (1976),

$$R_2(G) \geq \left\lfloor \frac{4\nu_G - 1}{3} \right\rfloor \qquad (G \text{ connected}).$$

Here is a list of some special cases:

$$R_2(P_n) = n + \left\lfloor \frac{n}{2} \right\rfloor - 1,$$

$$R_2(C_n) = \begin{cases} 6 & \text{if } n = 3 \text{ or } n = 4, \\ 2n - 1 & \text{if } n \geq 5 \text{ and } n \text{ odd} , \\ 3n/2 - 1 & \text{if } n \geq 6 \text{ and } n \text{ even}, \end{cases}$$

$$R_2(K_{1,n}) = \begin{cases} 2n - 1 & \text{if } n \text{ is even}, \\ 2n & \text{if } n \text{ is odd}, \end{cases}$$

$$R_2(K_{2,3}) = 10, \qquad R_2(K_{3,3}) = 18.$$

The values $R_2(K_{2,n})$ are known for $n \leq 16$, and in general, $R_2(K_{2,n}) \leq 4n - 2$. The value $R_2(K_{2,17})$ is either 65 or 66.

Let W_n denote the **wheel** on n vertices. It is a cycle C_{n-1}, where a vertex v with degree $n - 1$ is attached. Note that $W_4 = K_4$. Then $R_2(W_5) = 15$ and $R_2(W_6) = 17$.

For three colours, much less is known. In fact, the only nontrivial result for complete graphs is: $R_3(K_3) = 17$. Also, $128 \le R_3(K_4) \le 235$, and $385 \le R_3(K_5)$, but no nontrivial upper bound is known for $R_3(K_5)$. For the square C_4, we know that $R_3(C_4) = 11$.

Needless to say that no exact values are known for $R_k(K_n)$ for $k \ge 4$ and $n \ge 3$.

It follows from Theorem 4.4 that for any complete K_n, there *exists* a graph G (well, any sufficiently large complete graph) such that any 2-edge colouring of G has a monochromatic (induced) subgraph K_n. Note, however, that in Corollary 4.2 the monochromatic subgraph H_i is not required to be induced.

The following impressive theorem improves the results we have mentioned in this chapter and it has a difficult proof.

Theorem 4.9 (DEUBER**,** ERDÖS**,** HAJNAL**,** PÓSA**, and** RÖDL **(around 1973)).** *Let H be any graph. Then there exists a graph G such that any 2-edge colouring of G has an monochromatic induced subgraph H.*

Example 4.4. As an application of Ramsey's theorem, we shortly describe Schur's theorem. For this, consider the partition $\{1, 4, 10, 13\},\ \{2, 3, 11, 12\},\ \{5, 6, 7, 8, 9\}$ of the set $\mathbb{N}_{13} = [1, 13]$. We observe that in no partition class there are three integers such that $x + y = z$. However, if you try to partition $\mathbb{N}_{14}$ into three classes, then you are bound to find a class, where $x + y = z$ has a solution.

SCHUR (1916) solved this problem in a general setting. The following gives a short proof using Ramsey's theorem.

For each $n \ge 1$, there exists an integer $S(n)$ such that any partition $S_1, \dots, S_n$ of $\mathbb{N}_{S(n)}$ has a class S_i containing two integers x, y such that $x + y \in S_i$.

Indeed, let $S(n) = R(3, 3, \dots, 3)$, where 3 occurs n times, and let K be a complete on $\mathbb{N}_{S(n)}$. For a partition $S_1, \dots, S_n$ of $\mathbb{N}_{S(n)}$, define an edge colouring α of K by

$$\alpha(ij) = k, \text{ if } |i - j| \in S_k \ .$$

By Theorem 4.8, K^α has a monochromatic triangle, that is, there are three vertices $1 \le i < j < t \le S(n)$ such that $t - j, j - i, t - i \in S_k$ for some k. But $(t - j) + (j - i) = t - i$ proves the claim.

There are quite many interesting corollaries to Ramsey's theorem in various parts of mathematics including not only graph theory, but also, *e.g.*, geometry and algebra, see

R.L. GRAHAM, B.L. ROTHSCHILD AND J.L. SPENCER, "Ramsey Theory", Wiley, (2nd ed.) 1990.

4.3 Vertex colourings

The vertices of a graph G can also be classified using colourings. These colourings tell that certain vertices have a common property (or that they are similar in some respect), if they share the same colour. In this chapter, we shall concentrate on proper vertex colourings, where adjacent vertices get different colours.

The chromatic number

DEFINITION. A k-**colouring** (or a k-**vertex colouring**) of a graph G is a mapping $\alpha\colon V_G \to [1,k]$. The colouring α is **proper**, if adjacent vertices obtain a different colour: for all $uv \in E_G$, we have $\alpha(u) \neq \alpha(v)$. A colour $i \in [1,k]$ is said to be **available** for a vertex v, if no neighbour of v is coloured by i.

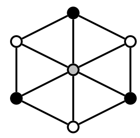
A graph G is k-**colourable**, if there is a proper k-colouring for G. The (vertex) **chromatic number** $\chi(G)$ of G is defines as

$$\chi(G) = \min\{k \mid \text{ there exists a proper } k\text{-colouring of } G\}\ .$$

If $\chi(G) = k$, then G is k-**chromatic**.

Each proper vertex colouring $\alpha\colon V_G \to [1,k]$ provides a partition $\{V_1, V_2, \ldots, V_k\}$ of the vertex set V_G, where $V_i = \{v \mid \alpha(v) = i\}$.

Example 4.5. The graph on the right, which is often called a wheel (of order 7), is 3-chromatic.



By the definitions, a graph G is 2-colourable if and only if it is bipartite.

Again, the 'names' of the colours are immaterial:

Lemma 4.5. *Let α be a proper k-colouring of G, and let π be any permutation of the colours. Then the colouring $\beta = \pi\alpha$ is a proper k-colouring of G.*

Proof. Indeed, if $\alpha\colon V_G \to [1,k]$ is proper, and if $\pi\colon [1,k] \to [1,k]$ is a bijection, then $uv \in E_G$ implies that $\alpha(u) \neq \alpha(v)$, and hence also that $\pi\alpha(u) \neq \pi\alpha(v)$. It follows that $\pi\alpha$ is a proper colouring. $\square$

Example 4.6. A graph is **triangle-free**, if it has no subgraphs isomorphic to K_3. We show that *there are triangle-free graphs with arbitrarily large chromatic numbers.*

The following construction is due to GRÖTZEL: Let G be any triangle-free graph with $V_G = \{v_1, v_2, \ldots, v_n\}$. Let G^t be a new graph obtained by adding $n+1$ new vertices v and $u_1, u_2, \ldots, u_n$ such that G^t has all the edges of G plus the edges $u_i v$ and $u_i x$ for all $x \in N(v_i)$ and for all $i \in [1,n]$.

Claim. G^t is triangle-free and it is $k+1$-chromatic

Indeed, let $U = \{u_1, \ldots, u_n\}$. We show first that G^t is triangle-free. Now, U is stable, and so a triangle contains at most (and thus exactly) one vertex $u_i \in U$. If $\{u_i, v_j, v_k\}$ induces a triangle, so does $\{v_i, v_j, v_k\}$ by the definition of G^t, but the latter triangle is already in G; a contradiction.

For the chromatic number we notice first that $\chi(G^t) \leq (k+1)$. If α is a proper k-colouring of G, extend it by setting $\alpha(u_i) = \alpha(v_i)$ and $\alpha(v) = k+1$.

Secondly, $\chi(G^t) > k$. Assume that α is a proper k-colouring of G^t, say with $\alpha(v) = k$. Then $\alpha(u_i) \neq k$. Recolour each v_i by $\alpha(u_i)$. This gives a proper $(k-1)$-colouring to G; a contradiction. Therefore $\chi(G^t) = k+1$.

Now using inductively the above construction starting from the triangle-free graph K_2, we obtain larger triangle -free graphs with high chromatic numbers.

Critical graphs

DEFINITION. A k-chromatic graph G is said to be k-**critical**, if $\chi(H) < k$ for all $H \subseteq G$ with $H \neq G$.

In a critical graph an elimination of any edge and of any vertex will reduce the chromatic number: $\chi(G-e) < \chi(G)$ and $\chi(G-v) < \chi(G)$ for $e \in E_G$ and $v \in G$. Each K_n is n-critical, since in $K_n-(uv)$ the vertices u and v can gain the same colour.

Example 4.7. The graph $K_2 = P_2$ is the only 2-critical graph. The 3-critical graphs are exactly the odd cycles C_{2n+1} for $n \geq 1$, since a 3-chromatic G is not bipartite, and thus must have a cycle of odd length.

Theorem 4.10. *If G is k-critical for $k \geq 2$, then it is connected, and $\delta(G) \geq k-1$.*

Proof. Note that for any graph G with the connected components $G_1, G_2, \ldots, G_m$, $\chi(G) = \max\{\chi(G_i) \mid i \in [1,m]\}$. Connectivity claim follows from this observation.

Let then G be k-critical, but $\delta(G) = d_G(v) \leq k-2$ for $v \in G$. Since G is critical, there is a proper $(k-1)$-colouring of $G-v$. Now v is adjacent to only $\delta(G) < k-1$ vertices. But there are k colours, and hence there is an available colour i for v. If we recolour v by i, then a proper $(k-1)$-colouring is obtained for G; a contradiction. □

The case (iii) of the next theorem is due to SZEKERES AND WILF (1968).

Theorem 4.11. *Let G be any graph with $k = \chi(G)$.*

(i) G has a k-critical subgraph H.
(ii) G has at least k vertices of degree $\geq k-1$.
(iii) $k \leq 1 + \max_{H \subseteq G}\ \delta(H)$.

Proof. For (i), we observe that a k-critical subgraph $H \subseteq G$ is obtained by removing vertices and edges from G as long as the chromatic number remains k.

For (ii), let $H \subseteq G$ be k-critical. By Theorem 4.10, $d_H(v) \geq k-1$ for every $v \in H$. Of course, also $d_G(v) \geq k-1$ for every $v \in H$. The claim follows, because, clearly, every k-critical graph H must have at least k vertices.

For (iii), let $H \subseteq G$ be k-critical. By Theorem 4.10, $\chi(G) - 1 \leq \delta(H)$, which proves this claim. □

Lemma 4.6. *Let v be a cut vertex of a connected graph G, and let A_i, for $i \in [1,m]$, be the connected components of $G{-}v$. Denote $G_i = G[A_i \cup \{v\}]$. Then $\chi(G) = \max\{\chi(G_i) \mid i \in [1,m]\}$. In particular, a critical graph does not have cut vertices.*

Proof. Suppose each G_i has a proper k-colouring α_i. By Lemma 4.5, we may take $\alpha_i(v) = 1$ for all i. These k-colourings give a k-colouring of G. □

Brooks' theorem

For *edge* colourings we have Vizing's theorem, but no such strong results are known for vertex colouring.

Lemma 4.7. *For all graphs G, $\chi(G) \leq \Delta(G) + 1$. In fact, there exists a proper colouring $\alpha\colon V_G \to [1, \Delta(G)+1]$ such that $\alpha(v) \leq d_G(v) + 1$ for all vertices $v \in G$.*

Proof. We use **greedy colouring** to prove the claim. Let $V_G = \{v_1, \dots, v_n\}$ be ordered in some way, and define $\alpha\colon V_G \to \mathbb{N}$ inductively as follows: $\alpha(v_1) = 1$, and

$$\alpha(v_i) = \min\{j \mid \alpha(v_t) \neq j \text{ for all } t < i \text{ with } v_i v_t \in E_G\} .$$

Then α is proper, and $\alpha(v_i) \leq d_G(v_i) + 1$ for all i. The claim follows from this. □

Although, we always have $\chi(G) \leq \Delta(G) + 1$, the chromatic number $\chi(G)$ usually takes much lower values – as seen in the bipartite case. Moreover, the maximum value $\Delta(G) + 1$ is obtained only in two special cases as was shown by BROOKS in 1941.

The next proof of Brook's theorem is by LOVÁSZ (1975) as modified by BRYANT (1996).

Lemma 4.8. *Let G be a 2-connected graph. Then the following are equivalent:*

(i) G is a complete graph or a cycle.
(ii) For all $u, v \in G$, if $uv \notin E_G$, then $\{u, v\}$ is a separating set.
(iii) For all $u, v \in G$, if $d_G(u, v) = 2$, then $\{u, v\}$ is a separating set.

Proof. It is clear that (i) implies (ii), and that (ii) implies (iii). We need only to show that (iii) implies (i). Assume then that (iii) holds.

We shall show that either G is a complete graph or $d_G(v) = 2$ for all $v \in G$, from which the theorem follows.

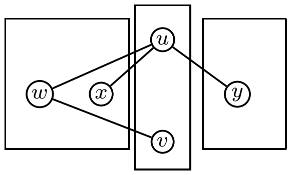
First of all, $d_G(v) \geq 2$ for all v, since G is 2-connected. Let w be a vertex of maximum degree, $d_G(w) = \Delta(G)$.

If the neighbourhood $N_G(w)$ induces a complete subgraph, then G is complete. Indeed, otherwise, since G is connected, there exists a vertex $u \notin N_G(w) \cup \{w\}$ such that u is adjacent to a vertex $v \in N_G(w)$. But then $d_G(v) > d_G(w)$, and this contradicts the choice of w.

Assume then that there are different vertices $u, v \in N_G(w)$ such that $uv \notin E_G$. This means that $d_G(u, v) = 2$ (the shortest path is $u \to w \to v$), and by (iii), $\{u, v\}$ is a separating set of G. Consequently, there is a partition $V_G = W \cup \{u, v\} \cup U$, where $w \in W$, and all paths from a vertex of W to a vertex of U go through either u or v.

We claim that $W = \{w\}$, and thus that $\Delta(G) = 2$ as required. Suppose on the contrary that $|W| \geq 2$. Since w is not a cut vertex (since G has no cut vertices), there exists an $x \in W$ with $x \neq w$ such that $xu \in E_G$ or $xv \in E_G$, say $xu \in E_G$.

Since v is not a cut vertex, there exists a $y \in U$ such that $uy \in E_G$. Hence $d_G(x, y) = 2$, and by (iii), $\{x, y\}$ is a separating set. Accordingly, $V_G = W_1 \cup \{x, y\} \cup U_1$, where all paths from W_1 to U_1 pass through x or y. Assume that $w \in W_1$, and hence that also $u, v \in W_1$. (Since $uw, vw \in E_{V_G - \{x,y\}}$).



There exists a vertex $z \in U_1$. Note that $U_1 \subseteq W \cup U$. If $z \in W$ (or $z \in U$, respectively), then all paths from z to u must pass through x (or y, respectively), and x (or y, respectively) would be a cut vertex of G. This contradiction, proves the claim. □

Theorem 4.12 (BROOKS **(1941)).** *Let G be connected. Then $\chi(G) = \Delta(G) + 1$ if and only if either G is an odd cycle or a complete graph.*

Proof. ($\Longleftarrow$) Indeed, $\chi(C_{2k+1}) = 3$, $\Delta(C_{2k+1}) = 2$, and $\chi(K_n) = n$, $\Delta(K_n) = n - 1$.

($\Longrightarrow$) Assume that $k = \chi(G)$. We may suppose that G is k-critical. Indeed, assume the claim holds for k-critical graphs. Let $k = \Delta(G) + 1$, and let $H \subset G$ be a k-critical proper subgraph. Since $\chi(H) = k = \Delta(G) + 1 > \Delta(H)$, we must have $\chi(H) = \Delta(H) + 1$, and thus H is a complete graph or an odd cycle. Now G is connected, and therefore there exists an edge $uv \in E_G$ with $u \in H$ and $v \notin H$. But then $d_G(u) > d_H(u)$, and $\Delta(G) > \Delta(H)$, since $H = K_n$ or $H = C_n$.

Let then G be any k-critical graph for $k \geq 2$. By Lemma 4.6, it is 2-connected. If G is an *even* cycle, then $k = 2 = \Delta(G)$. Suppose now that G is neither complete nor a cycle (odd or even). We show that $\chi(G) \leq \Delta(G)$.

By Lemma 4.8, there exist $v_1, v_2 \in G$ with $d_G(v_1, v_2) = 2$, say $v_1 w, w v_2 \in E_G$ with $v_1 v_2 \notin E_G$, such that $H = G - \{v_1, v_2\}$ is connected. Order $V_H = \{v_3, v_4, \dots, v_n\}$ such that $v_n = w$, and for all $i \geq 3$,

$$d_H(v_i, w) \geq d_H(v_{i+1}, w) .$$

Therefore for each $i \in [1, n - 1]$, we find at least one $j > i$ such that $v_i v_j \in E_G$ (possibly $v_j = w$). In particular, for all $1 \leq i < n$,

$$|N_G(v_i) \cap \{v_1, \dots, v_{i-1}\}| < d_G(v_i) \leq \Delta(G) . \tag{4.3}$$

Then colour $v_1, v_2, \dots, v_n$ in this order as follows: $\alpha(v_1) = 1 = \alpha(v_2)$ and

$$\alpha(v_i) = \min\{r \mid r \neq \alpha(v_j) \text{ for all } v_j \in N_G(v_i) \text{ with } j < i\} .$$

The colouring α is proper.

By (4.3), $\alpha(v_i) \leq \Delta(G)$ for all $i \in [1, n-1]$. Also, $w = v_n$ has two neighbours, v_1 and v_2, of the same colour 1, and since v_n has at most $\Delta(G)$ neighbours, there is an available colour for v_n, and so $\alpha(v_n) \leq \Delta(G)$. This shows that G has a proper $\Delta(G)$-colouring, and, consequently, $\chi(G) \leq \Delta(G)$. □

Example 4.8. Suppose we have n objects $V = \{v_1, \dots, v_n\}$, some of which are not compatible (like chemicals that react with each other, or worse, graph theorists who will fight during a conference). In the **storage problem** we would like to find a partition of the set V with as few classes as possible such that no class contains two incompatible elements. In graph theoretical terminology we consider the graph $G = (V, E)$, where $v_i v_j \in E$ just in case v_i and v_j are incompatible, and we would like to colour the vertices of G properly using as few colours as possible. This problem requires that we find $\chi(G)$.

Unfortunately, no good algorithms are known for determining $\chi(G)$, and, indeed, the chromatic number problem is NP-complete. Already the problem if $\chi(G) = 3$ is NP-complete. (However, as we have seen, the problem whether $\chi(G) = 2$ has a fast algorithm.)

The chromatic polynomial

A given graph G has many different proper vertex colourings $\alpha\colon V_G \to [1, k]$ for sufficiently large natural numbers k. Indeed, see Lemma 4.5 to be certain on this point.

DEFINITION. The **chromatic polynomial** of G is the function $\chi_G\colon \mathbb{N} \to \mathbb{N}$, where

$$\chi_G(k) = |\{\alpha \mid \alpha\colon V_G \to [1, k] \text{ a proper colouring}\}| \ .$$

This notion was introduced by BIRKHOFF (1912), BIRKHOFF AND LEWIS (1946), to attack the famous 4-Colour Theorem, but its applications have turned out to be elsewhere.

If $k < \chi(G)$, then clearly $\chi_G(k) = 0$, and, indeed,

$$\chi(G) = \min\{k \mid \chi_G(k) \neq 0\} \ .$$

Therefore, if we can find the chromatic polynomial of G, then we easily compute the chromatic number $\chi(G)$ just by evaluating $\chi_G(k)$ for $k = 1, 2, \dots$ until we hit a nonzero value. Theorem 4.13 will give the tools for constructing χ_G.

Example 4.9. Consider the complete graph K_4 on $\{v_1, v_2, v_3, v_4\}$. Let $k \geq \chi(K_4) = 4$. The vertex v_1 can be first given any of the k colours, after which $k-1$ colours are available for v_2. Then v_3 has $k-2$ and finally v_4 has $k-3$ available colours. Therefore there are $k(k-1)(k-2)(k-3)$ different ways to properly colour K_4 with k colours, and so

$$\chi_{K_4}(k) = k(k-1)(k-2)(k-3) \ .$$

On the other hand, in the discrete graph $\overline{K}_4$ has no edges, and thus any k-colouring is a proper colouring. Therefore

$$\chi_{\overline{K}_4}(k) = k^4 \ .$$

Remark. The considered method for checking the number of possibilities to colour a ‘next vertex’ is exceptional, and for more nonregular graphs it should be avoided.

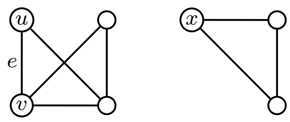
DEFINITION. Let G be a graph, $e = uv \in E_G$, and let $x = x(uv)$ be a new **contracted vertex**. The graph $G * e$ on

$$V_{G*e} = (V_G \setminus \{u, v\}) \cup \{x\}$$

is obtained from G by **contracting** the edge e, when

$$E_{G*e} = \{f \mid f \in E_G,\ f \text{ has no end } u \text{ or } v\} \cup \{wx \mid wu \in E_G \text{ or } wv \in E_G\} \ .$$

Hence $G * e$ is obtained by introducing a new vertex x, and by replacing all edges wu and wv by wx, and the vertices u and v are deleted. (Of course, no loops or parallel edges are allowed in the new graph $G * e$.)



Theorem 4.13. *Let G be a graph, and let $e \in E_G$. Then*

$$\chi_G(k) = \chi_{G-e}(k) - \chi_{G*e}(k).$$

Proof. Let $e = uv$. The proper k-colourings $\alpha \colon V_G \to [1, k]$ of $G-e$ can be divided into two disjoint cases, which together show that $\chi_{G-e}(k) = \chi_G(k) + \chi_{G*e}(k)$:

(1) If $\alpha(u) \neq \alpha(v)$, then α corresponds to a unique proper k-colouring of G, namely α. Hence the number of such colourings is $\chi_G(k)$.

(2) If $\alpha(u) = \alpha(v)$, then α corresponds to a unique proper k-colouring of $G * e$, namely α, when we set $\alpha(x) = \alpha(u)$ for the contracted vertex $x = x(uv)$. Hence the number of such colourings is $\chi_{G*e}(k)$. □

Theorem 4.14. *The chromatic polynomial is a polynomial.*

Proof. The proof is by induction on ε_G. Indeed, $\chi_{\overline{K}_n}(k) = k^n$ for the discrete graph, and for two polynomials P_1 and P_2, also $P_1 - P_2$ is a polynomial. The claim follows from Theorem 4.13, since there $G-e$ and $G * e$ have less edges than G. □

The connected components of a graph can be coloured independently, and so

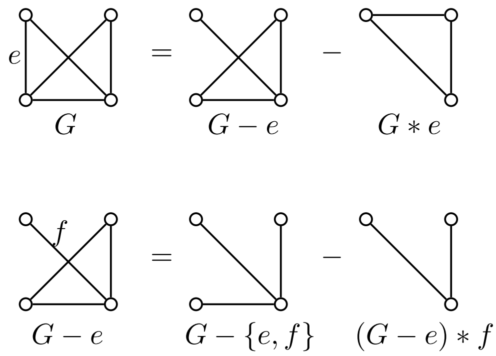
Lemma 4.9. *Let the graph G have the connected components $G_1, G_2, \dots, G_m$. Then*

$$\chi_G(k) = \chi_{G_1}(k)\chi_{G_2}(k) \dots \chi_{G_m}(k) \ .$$

Theorem 4.15. *Let T be a tree of order n. Then $\chi_T(k) = k(k-1)^{n-1}$.*

Proof. We use induction on n. For $n \leq 2$, the claim is obvious. Suppose that $n \geq 3$, and let $e = vu \in E_T$, where v is a leaf. By Theorem 4.13, $\chi_T(k) = \chi_{T-e}(k) - \chi_{T*e}(k)$. Here $T * e$ is a tree of order $n-1$, and thus, by the induction hypothesis, $\chi_{T*e}(k) = k(k-1)^{n-2}$. The graph $T-e$ consists of the isolated v and a tree of order $n-1$. By Lemma 4.9, and the induction hypothesis, $\chi_{T-e}(k) = k \cdot k(k-1)^{n-2}$. Therefore $\chi_T(k) = k(k-1)^{n-1}$. □

Example 4.10. Consider the graph G of order 4 from the above. Then we have the following reductions.



Theorem 4.13 reduces the computation of χ_G to the discrete graphs. However, we know the chromatic polynomials for trees (and complete graphs, as an exercise), and so there is no need to prolong the reductions beyond these. In our example, we have obtained

$$\begin{aligned}\chi_{G-e}(k) &= \chi_{G-\{e,f\}}(k) - \chi_{(G-e)*f}(k) \\ &= k(k-1)^3 - k(k-1)^2 = k(k-1)^2(k-2)\ ,\end{aligned}$$

and so

$$\begin{aligned}\chi_G(k) &= \chi_{G-e}(k) - \chi_{G*e}(k) = k(k-1)^2(k-2) - k(k-1)(k-2) \\ &= k(k-1)(k-2)^2 = k^4 - 5k^3 + 8k^2 - 4k\ .\end{aligned}$$

For instance, for 3 colours, there are 6 proper colourings of the given graph.

Chromatic Polynomial Problems. It is difficult to determine χ_G of a given graph, since the reduction method provided by Theorem 4.13 is time consuming. Also, there is known no characterization, which would tell from any polynomial $P(k)$ whether it is a chromatic polynomial of some graph. For instance, the polynomial $k^4 - 3k^3 + 3k^2$ is not a chromatic polynomial of any graph, but it seems to satisfy the general properties (that are known or conjectured) of these polynomials. REED (1968) conjectured that the coefficients of a chromatic polynomial should first increase and then decrease in absolute value. REED (1968) and TUTTE (1974) proved that for each G of order $\nu_G = n$:

- The degree of $\chi_G(k)$ equals n.
- The coefficient of k^n equals 1.
- The coefficient of k^{n-1} equals $-\varepsilon_G$.
- The constant term is 0.
- The coefficients alternate in sign.
- $\chi_G(m) \leq m(m-1)^n - 1$ for all positive integers m, when G is connected.
- $\chi_G(x) \neq 0$ for all real numbers $0 < x < 1$.

5

Graphs on Surfaces

5.1 Planar graphs

The plane representations of graphs are by no means unique. Indeed, a graph G can be drawn in arbitrarily many different ways. Also, the properties of a graph are not necessarily immediate from one representation, but may be apparent from another. There are, however, important families of graphs, the **surface graphs**, that rely on the (topological or geometrical) properties of the drawings of graphs. We restrict ourselves in this chapter to the most natural of these, the planar graphs. The geometry of the plane will be treated intuitively.

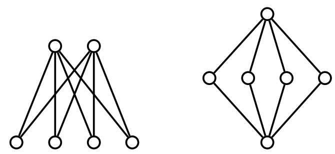
A planar graph will be a graph that can be drawn in the plane so that no two edges intersect with each other. Such graphs are used, *e.g.*, in the design of electrical (or similar) circuits, where one tries to (or has to) avoid crossing the wires or laser beams. Planar graphs come into use also in some parts of mathematics, especially in group theory and topology.

There are fast algorithms (linear time algorithms) for testing whether a graph is planar or not. However, the algorithms are all rather difficult to implement. Most of them are based on an algorithm designed by AUSLANDER AND PARTER (1961) see Section 6.5 of

S. SKIENA, "Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica", Addison-Wesley, 1990.

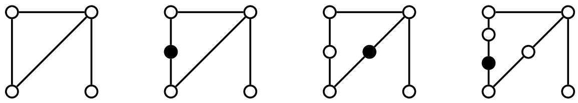
Definition

DEFINITION. A graph G is a **planar graph**, if it has a plane figure $P(G)$, called the **plane embedding** of G, where the lines (or continuous curves) corresponding to the edges do not intersect each other except at their ends.



The complete bipartite graph $K_{2,4}$ is a planar graph.

DEFINITION. An edge $e = uv \in E_G$ is **subdivided**, when it is replaced by a path $u \longrightarrow x \longrightarrow v$ of length two by introducing a *new* vertex x. A **subdivision** H of a graph G is obtained from G by a sequence of subdivisions.



The following result is clear.

Lemma 5.1. *A graph is planar if and only if its subdivisions are planar.*

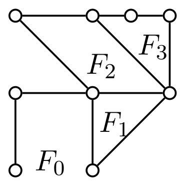
Geometric properties

It is clear that the graph theoretical properties of G are inherited by all of its plane embeddings. For instance, the way we draw a graph G in the plane does not change its maximum degree or its chromatic number. More importantly, there are – as we shall see – some nontrivial topological (or geometric) properties that are shared by the plane embeddings.

We recall first some elements of the plane geometry. Let F be an **open set** of the plane $\mathbb{R} \times \mathbb{R}$, that is, every point $x \in F$ has a disk centred at x and contained in F. Then F is a **region**, if any two points $x, y \in F$ can be joined by a continuous curve the points of which are all in F. The **boundary** $\partial(F)$ of a region F consists of those points for which every neighbourhood contains points from F and its complement.

Let G be a planar graph, and $P(G)$ one of its plane embeddings. Regard now each edge $e = uv \in E_G$ as a line from u to v. The set $(\mathbb{R} \times \mathbb{R}) \setminus E_G$ is open, and it is divided into a finite number of disjoint regions, called the **faces** of $P(G)$.

DEFINITION. A face of $P(G)$ is an **interior face**, if it is bounded. The (unique) face that is unbounded is called the **exterior face** of $P(G)$. The edges that surround a face F constitute the boundary $\partial(F)$ of F. The **exterior boundary** is the boundary of the exterior face. The vertices (edges, resp.) on the exterior boundary are called **exterior vertices exterior edges**, resp.). Vertices (edges, resp.) that are not on the exterior boundary are **interior vertices interior edges**, resp.).



Embeddings $P(G)$ satisfy some properties that we accepts at face value.

Lemma 5.2. *Let $P(G)$ be a plane embedding of a planar graph G.*

(i) Two different faces F_1 and F_2 are disjoint, and their boundaries can intersect only on edges.
(ii) $P(G)$ has a unique exterior face.
(iii) Each edge e belongs to the boundary of at most two faces.
(iv) Each cycle of G surrounds (that is, its interior contains) at least one internal face of $P(G)$.
(v) A bridge of G belongs to the boundary of only one face.
(vi) An edge that is not a bridge belongs to the boundary of exactly two faces.

If $P(G)$ is a plane embedding of a graph G, then so is any drawing $P'(G)$ which is obtained from $P(G)$ by an injective mapping of the plane that preserves continuous curves. This means, in particular, that *every planar graph has a plane embedding inside any geometric circle of arbitrarily small radius, or inside any geometric triangle*.

Euler's formula

Lemma 5.3. *A plane embedding $P(G)$ of a planar graph G has no interior faces if and only if G is acyclic, that is, if and only if the connected components of G are trees.*

Proof. This is clear from Lemma 5.2. □

The next general form of **Euler's formula** was proved by LEGENDRE (1794).

Theorem 5.1 (Euler's formula). *Let G be a connected planar graph, and let $P(G)$ be any of its plane embeddings. Then*

$$\nu_G - \varepsilon_G + \varphi = 2 \; ,$$

where φ is the number of faces of $P(G)$.

Proof. We shall prove the claim by induction on the number of faces φ of a plane embedding $P(G)$. First, notice that $\varphi \geq 1$, since each $P(G)$ has an exterior face.

If $\varphi = 1$, then, by Lemma 5.3, there are no cycles in G, and since G is connected, it is a tree. In this case, by Theorem 2.4, we have $\varepsilon_G = \nu_G - 1$, and the claim holds.

Suppose then that the claim is true for all plane embeddings with less than φ faces for $\varphi \geq 2$. Let $P(G)$ be a plane embedding of a connected planar graph such that $P(G)$ has φ faces.

Let $e \in E_G$ be an edge that is not a bridge. The subgraph $G{-}e$ is planar with a plane embedding $P(G{-}e) = P(G){-}e$ obtained by simply erasing the edge e. Now $P(G{-}e)$ has $\varphi - 1$ faces, since the two faces of $P(G)$ that are separated by e are merged into one face of $P(G{-}e)$. By the induction hypothesis, $\nu_{G-e} - \varepsilon_{G-e} + (\varphi - 1) = 2$, and hence $\nu_G - (\varepsilon_G - 1) + (\varphi - 1) = 2$, and the claim follows. □

In particular, we have the following invariant property of planar graphs.

Corollary 5.1. *Let G be a planar graph. Then every plane embedding of G has the same number of faces:*

$$\varphi_G = \varepsilon_G - \nu_G + 2$$

Maximal planar graphs

Lemma 5.4. *If G is a planar graph of order $\nu_G \geq 3$, then $\varepsilon_G \leq 3\nu_G - 6$. Moreover, if G has no triangles C_3, then $\varepsilon_G \leq 2\nu_G - 4$.*

Proof. If G is disconnected with connected components G_i, for $i \in [1, k]$, and if the claim holds for these smaller (necessarily planar) graphs G_i, then it holds for G, since

$$\varepsilon_G = \sum_{i=1}^{\nu_G} \varepsilon_{G_i} \leq 3 \sum_{i=1}^{\nu_G} \nu_{G_i} - 6k = 3\nu_G - 6k \leq 3\nu_G - 6 \ .$$

It is thus sufficient to prove the claim for connected planar graphs.

Also, the case where $\varepsilon_G \leq 2$ is clear. Suppose thus that $\varepsilon_G \geq 3$.

Each face F of an embedding $P(G)$ contains at least three edges on its boundary $\partial(F)$. Hence $3\varphi \leq 2\varepsilon_G$, since each edge lies on at most two faces. The first claim follows from Euler's formula.

The second claim is proved similarly except that, in this case, each face F of $P(G)$ contains at least four edges on its boundary (when G is connected and $\varepsilon_G \geq 4$). $\square$

An upper bound for $\delta(G)$ for planar graphs was achieved by HEAWOOD.

Theorem 5.2 (HEAWOOD **(1890)).** *If G is a planar graph, then $\delta(G) \leq 5$.*

Proof. If $\nu_G \leq 2$, then there is nothing to prove. Suppose $\nu_G \geq 3$. By the handshaking lemma and the previous lemma,

$$\delta(G) \cdot \nu_G \leq \sum_{v \in G} d_G(v) = 2\varepsilon_G \leq 6\nu_G - 12 \ .$$

It follows that $\delta(G) \leq 5$. $\square$

Theorem 5.3. *K_5 and $K_{3,3}$ are not planar graphs.*

Proof. By Lemma 5.4, a planar graph of order 5 has at most 9 edges, but K_5 has 5 vertices and 10 edges. By the second claim of Lemma 5.4, a triangle-free planar graph of order 6 has at most 8 edges, but $K_{3,3}$ has 6 vertices and 9 edges. $\square$

DEFINITION. A planar graph G is **maximal**, if $G + e$ is nonplanar for every $e \notin E_G$.

Example 5.1. Clearly, if we remove one edge from K_5, the result is a maximal planar graph. However, if an edge is removed from $K_{3,3}$, the result is not maximal!

Lemma 5.5. *Let F be a face of a plane embedding $P(G)$ that has at least four edges on its boundary. Then there are two nonadjacent vertices on the boundary of F.*

Proof. Assume that the set of the boundary vertices of F induces a complete subgraph K. The edges of K are either on the boundary or they are not inside F (since F is a face.) Add a new vertex x inside F, and connect the vertices of K to x. The result is a plane embedding of a graph H with $V_H = V_G \cup \{x\}$ (that has G as its induced subgraph). The induced subgraph $H[K \cup \{x\}]$ is complete, and since H is planar, we have $|K| < 4$ as required. $\square$

By the previous lemma, if a face has a boundary of at least four edges, then an edge can be added to the graph (inside the face), and the graph remains to be planar. Hence we have proved

Corollary 5.2. *If G is a maximal planar graph with $\nu_G \geq 3$, then G is triangulated, that is, every face of a plane embedding $P(G)$ has a boundary of exactly three edges.*

Theorem 5.4. *For a maximal planar graph G of order $\nu_G \geq 3$, $\varepsilon_G = 3\nu_G - 6$.*

Proof. Each face F of an embedding $P(G)$ is a triangle having three edges on its boundary. Hence $3\varphi = 2\varepsilon_G$, since there are now no bridges. The claim follows from Euler's formula. □

Kuratowski's theorem

Theorem 5.5 will give a simple criterion for planarity of graphs. This theorem (due to KURATOWSKI in 1930) is one of the jewels of graph theory. In fact, the theorem was proven earlier by PONTRYAGIN (1927-1928), and also independently by FRINK AND SMITH (1930). For history of the result, see

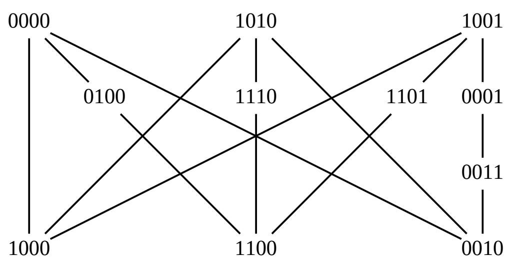
J.W. KENNEDY, L.V. QUINTAS, AND M.M. SYSLO, The theorem on planar graphs. *Historia Math.* **12** (1985), 356 – 368.

The graphs K_5 and $K_{3,3}$ are the smallest nonplanar graphs, and, by Lemma 5.1, if G contains a subdivision of K_5 or $K_{3,3}$ as a subgraph, then G is not planar. We prove the converse of this result in what follows. Therefore

Theorem 5.5 (KURATOWSKI (1930)). *A graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ as a subgraph.*

We prove this result along the lines of THOMASSEN (1981) using 3-connectivity.

Example 5.2. The cube Q_k is planar only for $k = 1, 2, 3$. Indeed, the graph Q_4 contains a subdivision of $K_{3,3}$, and thus by Theorem 5.5 it is not planar. On the other hand, each Q_k with $k \geq 4$ has Q_4 as a subgraph, and therefore they are nonplanar. The subgraph of Q_4 that is a subdivision of $K_{3,3}$ is given below.



DEFINITION. A graph G is called a **Kuratowski graph**, if it is a subdivision of K_5 or $K_{3,3}$.

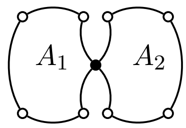
Lemma 5.6. *Let $E \subseteq E_G$ be the set of the boundary edges of a face F in a plane embedding of G. Then there exists a plane embedding $P(G)$, where the edges of E are exterior edges.*

Proof. This is a geometric proof. Choose a circle that contains every point of the plane embedding (including all points of the edges) such that the centre of the circle is inside the given face. Then use geometric inversion with respect to this circle. This will map the given face as the exterior face of the image plane embedding. □

Lemma 5.7. *Let G be a nonplanar graph without Kuratowski graphs such that ε_G is minimal in this respect. Then G is 3-connected.*

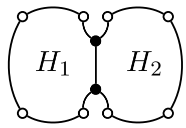
Proof. We show first that G is 2-connected. On the contrary, assume that v is a cut vertex of G, and let $A_1, \dots, A_k$ be the connected components of $G-v$.

Since G is minimal nonplanar with respect to ε_G, the subgraphs $G_i = G[A_i \cup \{v\}]$ have plane embeddings $P(G_i)$, where v is an exterior vertex. We can glue these plane embeddings together at v to obtain a plane embedding of G, and this will contradict the choice of G.



Assume then that G has a separating set $S = \{u, v\}$. Let G_1 and G_2 be any subgraphs of G such that $E_G = E_{G_1} \cup E_{G_2}$, $S = V_{G_1} \cap V_{G_2}$, and both G_1 and G_2 contain a connected component of $G-S$. Since G is 2-connected (by the above), there are paths $u \xrightarrow{\star} v$ in G_1 and G_2. Indeed, both u and v are adjacent to a vertex of each connected component of $G-S$. Let $H_i = G_i + uv$. (Maybe $uv \in E_G$.)

If both H_1 and H_2 are planar, then, by Lemma 5.6, they have plane embeddings, where uv is an exterior edge. It is now easy to glue H_1 and H_2 together on the edge uv to obtain a plane embedding of $G + uv$, and thus of G.

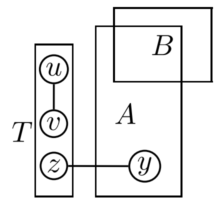


We conclude that H_1 or H_2 is nonplanar, say H_1. Now $\varepsilon_{H_1} < \varepsilon_G$, and so, by the minimality of G, H_1 contains a Kuratowski graph H. However, there is a path $u \xrightarrow{\star} v$ in H_2, since $G_2 \subseteq H_2$. This path can be regarded as a subdivision of uv, and thus G contains a Kuratowski graph. This contradiction shows that G is 3-connected. □

Lemma 5.8. *Let G be a 3-connected graph of order $\nu_G \geq 5$. Then there exists an edge $e \in E_G$ such that the contraction $G * e$ is 3-connected.*

Proof. On the contrary suppose that for any $e \in E_G$, the graph $G * e$ has a separating set S with $|S| = 2$. Let $e = uv$, and let $x = x(uv)$ be the contracted vertex. Necessarily $x \in S$, say $S = \{x, z\}$ (for, otherwise, S would separate G already). Therefore $T = \{u, v, z\}$ separates G. Assume that e and S are chosen such that $G-T$ has a connected component A with the least possible number of vertices.

There exists a vertex $y \in A$ with $zy \in E_G$. (Otherwise $\{u, v\}$ would separate G.) The graph $G * (zy)$ is not 3-connected by assumption, and hence, as in the above, there exists a vertex w such that $R = \{z, y, w\}$ separates G. It can be that $w \in \{u, v\}$, but by symmetry we can suppose that $w \neq u$.



Since $uv \in E_G$, $G-R$ has a connected component B such that $u, v \notin B$. For each $y' \in B$, there exists a path $P\colon u \xrightarrow{\star} y'$ in $G-\{z, w\}$, since G is 3-connected, and hence this P goes through y. Therefore y' is connected to y also in $G-T$, that is, $y' \in A$, and so $B \subseteq A$. The inclusion is proper, since $y \notin B$. Hence $|B| < |A|$, and this contradicts the choice of A. □

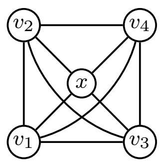
By the next lemma, a Kuratowski graph cannot be created by contractions.

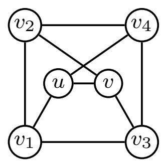
Lemma 5.9. *Let G be a graph. If for some $e \in E_G$ the contraction $G * e$ has a Kuratowski subgraph, then so does G.*

Proof. The proof consists of several cases depending on the Kuratowski graph, and how the subdivision is made. We do not consider the details of these cases.

Let H be a Kuratowski graph of $G * e$, where $x = x(uv)$ is the contracted vertex for $e = uv$. If $d_H(x) = 2$, then the claim is obviously true. Suppose then that $d_H(x) = 3$ or 4. If there exists at most one edge $xy \in E_H$ such that $uy \in E_G$ (or $vy \in E_G$), then one easily sees that G contains a Kuratowski graph.

There remains only one case, where H is a subdivision of K_5, and both u and v have 3 neighbours in the subgraph of G corresponding to H. In this case, G contains a subdivision of $K_{3,3}$. □





Lemma 5.10. *Every 3-connected graph G without Kuratowski subgraphs is planar.*

Proof. The proof is by induction on ν_G. The only 3-connected graph of order 4 is the planar graph K_4. Therefore we can assume that $\nu_G \geq 5$.

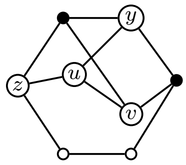
By Lemma 5.8, there exists an edge $e = uv \in E_G$ such that $G * e$ (with a contracted vertex x) is 3-connected. By Lemma 5.9, $G * e$ has no Kuratowski subgraphs, and hence $G * e$ has a plane embedding $P(G * e)$ by the induction hypothesis. Consider the part $P(G * e)-x$, and let C be the boundary of the face of $P(G * e)-x$ containing x (in $P(G * e)$). Here C is a cycle of G (since G is 3-connected).

Now since $G-\{u, v\} = (G * e)-x$, $P(G * e)-x$ is a plane embedding of $G-\{u, v\}$, and $N_G(u) \subseteq V_C \cup \{v\}$ and $N_G(v) \subseteq V_C \cup \{u\}$. Assume, by symmetry, that $d_G(v) \leq d_G(u)$. Let

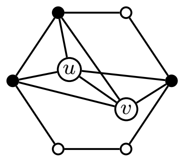
$N_G(v) \setminus \{u\} = \{v_1, v_2, \ldots, v_k\}$ in order along the cycle C. Let $P_{i,j}\colon v_i \xrightarrow{\star} v_j$ be the path along C from v_i to v_j. We obtain a plane embedding of $G{-}u$ by drawing (straight) edges vv_i for $1 \leq i \leq k$.

(1) If $N_G(u) \setminus \{v\} \in P_{i,i+1}$ ($i+1$ is taken modulo k) for some i, then, clearly, G has a plane embedding (obtained from $P(G){-}u$ by putting u inside the triangle (v, v_i, v_{i+1}) and by drawing the edges with an end u inside this triangle).

(2) Assume there are $y, z \in N_G(u) \setminus \{v\}$ such that $y \in P_{ij}$ and $z \notin P_{ij}$ for some i and j, where $y, z \notin \{v_i, v_j\}$. Now, $\{u, v_i, v_{i+1}\} \cup \{v, z, y\}$ form a subdivision of $K_{3,3}$.



By (1) and (2), we can assume that $N_G(u)\setminus\{v\} \subseteq N_G(v)$. Therefore, $N_G(u)\setminus\{v\} = N_G(v)\setminus\{u\}$ by the assumption $d_G(v) \leq d_G(u)$. Also, by (1), $d_G(v) = d_G(u) > 3$. But now u, v, v_1, v_2, v_3 give a subdivision of K_5. □



Proof of Theorem 5.5. By Theorem 5.3 and Lemma 5.1, we need to show that each nonplanar graph G contains a Kuratowski subgraph. On the contrary, suppose that G is a nonplanar graph that has a minimal size ε_G such that G does not contain a Kuratowski subgraph. Then, by Lemma 5.7, G is 3-connected, and by Lemma 5.10, it is planar. This contradiction proves the claim. □

Example 5.3. Any graph G can be drawn in the plane so that three of its edges never intersect at the same point. The **crossing number** $\times(G)$ is the minimum number of intersections of its edges in such plane drawings of G. Therefore G is planar if and only if $\times(G) = 0$, and, for instance, $\times(K_5) = 1$.

We show that $\times(K_6) = 3$. For this we need to show that $\times(K_6) \geq 3$. For the equality, one is invited to design a drawing with exactly 3 crossings.

Let $X(K_6)$ be a drawing of K_6 using c crossings so that two edges cross at most once. Add a new vertex at each crossing. This results in a planar graph G on $c+6$ vertices and $2c+15$ edges. Now $c \geq 3$, since $\varepsilon_G = 2c + 15 \leq 3(c+6) - 6 = 3\nu_G - 6$.

5.2 Colouring planar graphs

The most famous problem in the history of graph theory is that of the chromatic number of planar graphs. The problem was known as the 4-**Colour Conjecture** for more than 120 years, until it was solved by APPEL AND HAKEN in 1976: if G is a planar graph, then $\chi(G) \leq 4$. The 4-Colour Conjecture has had a deep influence on the theory of graphs during the last 150 years. The solution of the 4-Colour Theorem is difficult, and it requires the assistance of a computer.

The 5-colour theorem

We prove HEAWOOD's result (1890) that each planar graph is properly 5-colourable.

Lemma 5.11. *If G is a planar graph, then $\chi(G) \leq 6$.*

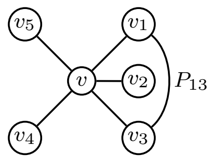
Proof. The proof is by induction on ν_G. Clearly, the claim holds for $\nu_G \leq 6$. By Theorem 5.2, a planar graph G has a vertex v with $d_G(v) \leq 5$. By the induction hypothesis, $\chi(G-v) \leq 6$. Since $d_G(v) \leq 5$, there is a colour i available for v in the 6-colouring of $G-v$, and so $\chi(G) \leq 6$. □

The proof of the following theorem is partly geometric in nature.

Theorem 5.6 (HEAWOOD (1890)). *If G is a planar graph, then $\chi(G) \leq 5$.*

Proof. Suppose the claim does not hold, and let G be a 6-critical planar graph. Recall that for k-critical graphs H, $\delta(H) \geq k-1$, and thus there exists a vertex v with $d_G(v) = \delta(G) \geq 5$. By Theorem 5.2, $d_G(v) = 5$.

Let α be a proper 5-colouring of $G-v$. Such a colouring exists, because G is 6-critical. By assumption, $\chi(G) > 5$, and therefore for each $i \in [1,5]$, there exists a neighbour $v_i \in N_G(v)$ such that $\alpha(v_i) = i$. Suppose these neighbours v_i of v occur in the plane in the geometric order of the figure.



Consider the subgraph $G[i,j] \subseteq G$ made of colours i and j. The vertices v_i and v_j are in the same connected component of $G[i,j]$ (for, otherwise we interchange the colours i and j in the connected component containing v_j to obtain a recolouring of G, where v_i and v_j have the same colour i, and then recolour v with the remaining colour j).

Let $P_{ij}\colon v_i \xrightarrow{\star} v_j$ be a path in $G[i,j]$, and let $C = (vv_1)P_{13}(v_3v)$. By the geometric assumption, exactly one of v_2, v_4 lies inside the region enclosed by the cycle C. Now, the path P_{24} must meet C at some vertex of C, since G is planar. This is a contradiction, since the vertices of P_{24} are coloured by 2 and 4, but C contains no such colours. □

The final word on the chromatic number of planar graphs was proved by APPEL AND HAKEN in 1976.

Theorem 5.7 (4-Colour Theorem). *If G is a planar graph, then $\chi(G) \leq 4$.*

By the following theorem, each planar graph can be decomposed into two bipartite graphs.

Theorem 5.8. *Let $G = (V,E)$ be a 4-chromatic graph, $\chi(G) \leq 4$. Then the edges of G can be partitioned into two subsets E_1 and E_2 such that (V,E_1) and (V,E_2) are both bipartite.*

Proof. Let $V_i = \alpha^{-1}(i)$ be the set of vertices coloured by i in a proper 4-colouring α of G. The define E_1 as the subset of the edges of G that are between the sets V_1 and V_2; V_1 and V_4; V_3 and V_4. Let E_2 be the rest of the edges, that is, they are between the sets V_1 and V_3; V_2 and V_3; V_2 and V_4. It is clear that (V,E_1) and (V,E_2) are bipartite, since the sets V_i are stable. □

Map colouring*

The 4-Colour Conjecture was originally stated for maps. In the **map-colouring problem** we are given several countries with common borders, and we wish to colour each country so that no neighbouring countries obtain the same colour. *How many colours are needed?*

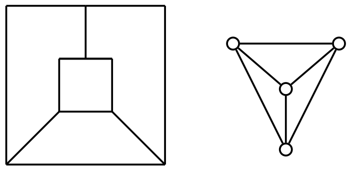
A border between two countries is assumed to have a positive length – in particular, countries that have only one point in common are not allowed in the map colouring.

Formally, we define a **map** as a connected planar (embedding of a) graph with no bridges. The edges of this graph represent the boundaries between countries. Hence a country is a face of the map, and two neighbouring countries share a common edge (not just a single vertex). We deny bridges, because a bridge in such a map would be a boundary inside a country.

The map-colouring problem is restated as follows:

How many colours are needed for the faces of a plane embedding so that no adjacent faces obtain the same colour.

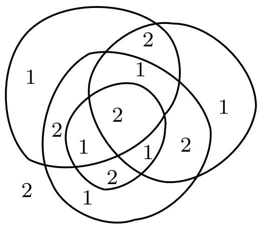
The illustrated map can be 4-coloured, and it cannot be coloured using only 3 colours, because every two faces have a common border.



Let $F_1, F_2, \dots, F_n$ be the countries of a map M, and define a graph G with $V_G = \{v_1, v_2, \dots, v_n\}$ such that $v_iv_j \in E_G$ if and only if the countries F_i and F_j are neighbours. It is easy to see that G is a planar graph. Using this notion of a dual graph, we can state the map-colouring problem in new form: *What is the chromatic number of a planar graph?* By the 4-Colour Theorem it is at most four.

Map-colouring can be used in rather generic topological setting, where the maps are defined by curves in the plane. As an example, consider finitely many simple closed curves in the plane. These curves divide the plane into regions. *The regions are* 2*-colourable*.

That is, the graph where the vertices correspond to the regions, and the edges correspond to the neighbourhood relation, is bipartite. To see this, colour a region by 1, if the region is inside an odd number of curves, and, otherwise, colour it by 2.



History of the 4-Colour Theorem

That four colours suffice planar maps was conjectured around 1850 by FRANCIS GUTHRIE, a student of DE MORGAN at University College of London. During the following 120 years many outstanding mathematicians tried to solve the problem, and some of them even thought that they had been successful.

In 1879 CAYLEY pointed out some difficulties that lie in the conjecture. The same year ALFRED KEMPE published a paper, where he claimed a proof of the 4CC. The basic idea in KEMPE's argument (known later as *Kempe chains*) was the same as later used by HEAWOOD to prove the 5-Colour Theorem, (Theorem 5.6).

For more than 10 years KEMPE's proof was considered to be valid. For instance, TAIT published two papers on the 4CC in the 1880's that contained clever ideas, but also some further errors. In 1890 HEAWOOD showed that KEMPE's proof had serious gaps. As we shall see in the next chapter, HEAWOOD discovered the number of colours needed for all maps on *other* surfaces than the plane. Also, he proved that if the number of edges around each region is divisible by 3, then the map is 4-colourable.

One can *triangulate* any planar graph G (drawn in the plane), by adding edges to divide the faces into triangles. BIRKHOFF introduced one of the basic notions (reducibility) needed in the proof of the 4CC. In a triangulation, a *configuration* is a part that is contained inside a cycle. An *unavoidable set* is a set of configurations such that any triangulation must contain one of the configurations in the set. A configuration is said to be *reducible*, if it is not contained in a triangulation of a minimal counter example to the 4CC.

The search for avoidable sets began in 1904 with work of WEINICKE, and in 1922 FRANKLIN showed that the 4CC holds for maps with at most 25 regions. This number was increased to 27 by REYNOLDS (1926), to 35 by WINN (1940), to 39 by ORE AND STEMPLE (1970), to 95 by MAYER (1976).

The final notion for the solution was due to HEESCH, who in 1969 introduced *discharging*. This consists of assigning to a vertex v the *charge* $6 - d_G(v)$. From Euler's formula we see that for the sum of the charges, we have

$$\sum_v (6 - d_G(v)) = 12.$$

Now, a given set S of configurations can be proved to be unavoidable, if for a triangulation, that does not contain a configuration from S, one can 'redistribute' the charges so that no v comes up with a positive charge.

According to HEESCH one might be satisfied with a set of 8900 configurations to prove the 4CC. There were difficulties with his approach that were solved in 1976 by APPEL AND HAKEN. They based the proof on reducibility using Kempe chains, and ended up with an unavoidable set with over 1900 configurations and some 300 discharging rules. The proof used 1200 hours of computer time. (KOCH assisted with the computer calculations.) A simplified proof by ROBERTSON, SANDERS, SEYMOUR AND THOMAS (1997) uses 633 configurations and 32 discharging rules. Because of these simplifications also the computer time is much less than in the original proof.

The following book contains the ideas of the proof of the 4-Colour Theorem.

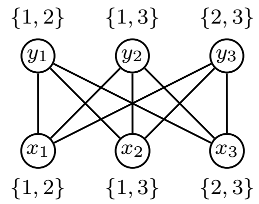
T.L. SAATY AND P.C. KAINEN, "The Four-Color Problem", Dover, 1986.

List colouring

DEFINITION. Let G be a graph so that each of its vertices v is given a list (set) $\Lambda(v)$ of colours. A proper colouring $\alpha\colon V_G \to [1, m]$ of G is a (Λ-)**list colouring**, if each vertex v gets a colour from its list, $\alpha(v) \in \Lambda(v)$.

The **list chromatic number** $\chi_\ell(G)$ is the smallest integer k such that G has a Λ-list colouring for all lists of size k, $|\Lambda(v)| = k\}$. Also, G is k-**choosable**, if $\chi_\ell(G) \le k$.

Example 5.4. The bipartite graph $K_{3,3}$ is not 2-choosable. Indeed, let the bipartition of $K_{3,3}$ be (X, Y), where $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$. The lists for the vertices shown in the figure show that $\chi_\ell(K_{3,3}) > 2$.



Obviously $\chi(G) \le \chi_\ell(G)$, since proper colourings are special cases of list colourings, but equality does not hold in general. However, it was proved by VIZING (1976) and ERDÖS, RUBIN AND TAYLOR (1979) that

$$\chi_\ell(G) \le \Delta(G) + 1 \ .$$

For planar graphs we *do not* have a ‘4-list colour theorem’. Indeed, it was shown by VOIGT (1993) that there exists a planar graph with $\chi_\ell(G) = 5$. At the moment, the smallest such a graph was produced by MIRZAKHANI (1996), and it is of order 63.

Theorem 5.9 (THOMASSEN (1994)). *Let G be a planar graph. Then $\chi_\ell(G) \le 5$.*

In fact, THOMASSEN proved a stronger statement:

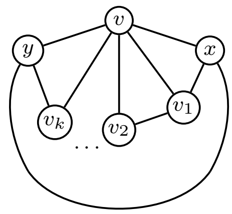
Theorem 5.10. *Let G be a planar graph and let C be the cycle that is the boundary of the exterior face. Let Λ consist of lists such that $|\Lambda(v)| = 3$ for all $v \in C$, and $|\Lambda(v)| = 5$ for all $v \notin C$. Then G has a Λ-list colouring α.*

Proof. We can assume that the planar graph G is connected, and that it is given by a **near-triangulation**; an embedding, where the interior faces are triangles. (If the boundary of a face has more than 3 edges, then we can add an edge inside the face.) This is because adding edges to a graph can only make the list colouring more difficult. Note that the exterior boundary is unchanged by a triangulation of the interior faces.

The proof is by induction on ν_G under the additional constraint that one of the vertices of C has a fixed colour. (Thus we prove a stronger statement than claimed.) For $\nu_G \le 3$, the claim is obvious. Suppose then that $\nu_G \ge 4$.

Let $x \in C$ be a vertex, for which we fix a colour $\alpha(x) \in \Lambda(x)$. Let $v \in C$ be a vertex adjacent to x, that is, $C\colon v \to x \xrightarrow{\star} v$.

Let $N_G(v) = \{x, v_1, \dots, v_k, y\}$, where $y \in C$, and v_i are ordered such that the faces are triangles as in the figure. It can be that $N_G(v) = \{x, y\}$, in which case $xy \in E_G$.



Consider the subgraph $H = G - v$. The exterior boundary of H is the cycle $x \to v_1 \to \dots \to v_k \to y \xrightarrow{\star} x$. Since $|\Lambda(v)| = 3$, there are two colours $r, s \in \Lambda(v)$ that differ from $\alpha(x)$.

We define new lists for H as follows: $\Lambda'(v_i) \subseteq \Lambda(v_i) \setminus \{r, s\}$ such that $|\Lambda'(v_i)| = 3$ for each $i \in [1, k]$, and otherwise $\Lambda'(z) = \Lambda(z)$.
Now $\nu_H = \nu_G - 1$, and by the induction hypothesis (with $\alpha(x)$ still fixed), H has a Λ'-list colouring α. For the vertex v, we choose $\alpha(v) = r$ or s such that $\alpha(v) \neq \alpha(y)$. This gives a Λ'-list colouring for G. Since $\Lambda'(z) \subseteq \Lambda(z)$ for all z, we have that α is a Λ-list colouring of G. □

Straight lines and kissing circles*

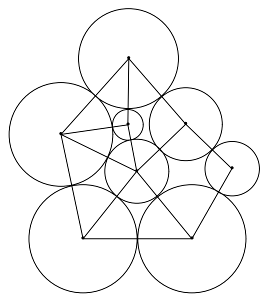
We state an interesting result of WAGNER, the proof of which can be deduced from the above proof of Kuratowski's theorem. The result is known as **Fáry's Theorem**.

Theorem 5.11 (WAGNER **(1936)).** *A planar graph G has a plane embedding, where the edges are straight lines.*

This raises a difficult problem:

Integer Length Problem. Can all planar graphs be drawn in the plane such that the edges are straight lines of integer lengths?

We say that two circles **kiss** in the plane, if they intersect in one point and their interiors do not intersect. For a set of circles, we draw a graph by putting an edge between two midpoints of kissing circles.
The following improvement of the above theorem is due to KOEBE (1936), and it was rediscovered independently by ANDREEV (1970) and THURSTON (1985).



Theorem 5.12 (KOEBE **(1936)).** *A graph is planar if and only if it is a kissing graph of circles.*

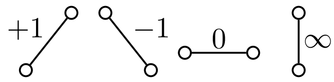
Graphs can be represented as plane figures in many different ways. For this, consider a set S of curves of the plane (that are continuous between their end points). The **string graph** of S is the graph $G = (S, E)$, where $uv \in E$ if and only if the curves u and v intersect. At first it might seem that every graph is a string graph, but this is not the case.

It is known that all planar graphs are string graphs (this is a trivial result).

Line Segment Problem. A graph is a **line segment graph** if it is a string graph for a set L of straight line segments in the plane. *Is every planar graph a line segment graph for some set L of lines?*

Note that there are also nonplanar graphs that are line segment graphs. Indeed, all complete graphs are such graphs.

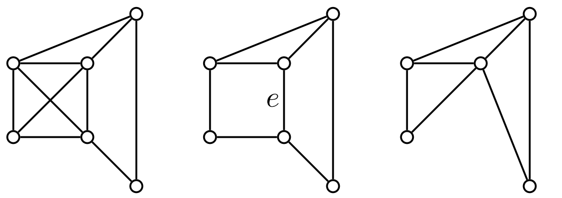
The above question remains open even in the case when the slopes of the lines are $+1$, -1, 0 and ∞. A positive answer to this 4-slope problem for planar graphs would prove the 4-Colour Theorem.



The Minor Theorem*

DEFINITION. A graph H is a **minor** of G, denoted by $H \preccurlyeq G$, if H is isomorphic to a graph obtained from a *subgraph* of G by successively contracting edges.

A recent result of ROBERTSON AND SEYMOUR (1983-2000) on graph minors is (one of) the deepest results of graph theory. The proof goes beyond these lectures. Indeed, the proof of Theorem 5.13 is around 500 pages long.



Note that every subgraph $H \subseteq G$ is a minor, $H \preccurlyeq G$.

The following properties of the minor relation are easily established:

(i) $G \preccurlyeq G$,
(ii) $H \preccurlyeq G$ and $G \preccurlyeq H$ imply $G \cong H$,
(iii) $H \preccurlyeq L$ and $L \preccurlyeq G$ imply $H \preccurlyeq G$.

The conditions (i) and (iii) ensure that the relation $\preccurlyeq$ is a **quasi-order**, that is, it is reflexive and transitive. It turns out to be a **well-quasi-order**, that is, every infinite sequence $G_1, G_2, \dots$ of graphs has two graphs G_i and G_j with $i < j$ such that $G_i \preccurlyeq G_j$.

Theorem 5.13 (Minor Theorem). *The minor order $\preccurlyeq$ is a well-quasi-order on graphs. In particular, in any infinite family $\mathcal{F}$ of graphs, one of the graphs is a (proper) minor of another.*

Each property $\mathcal{P}$ of graphs defines a family of graphs, namely, the family of those graphs that satisfy this property.

DEFINITION. A family $\mathcal{F}$ of graphs is said to be **minor closed**, if every minor H of a graph $G \in \mathcal{F}$ is also in $\mathcal{F}$. A property $\mathcal{P}$ of graphs is said to be **inherited by minors**, if all minors of a graph G satisfy $\mathcal{P}$ whenever G does.

The following families of graphs are minor closed: the family of (1) all graphs, (2) planar graphs (and their generalizations to other surfaces), (3) acyclic graphs.

The acyclic graphs include all trees. However, the family of trees is not closed under taking subgraphs, and thus it is not minor closed. More importantly, the subgraph order of trees ($T_1 \subseteq T_2$) is *not* a well-quasi-order.

WAGNER proved a minor version of Kuratowski's theorem:

Theorem 5.14 (WAGNER **(1937)).** *A graph G is nonplanar if and only if $K_5 \preccurlyeq G$ or $K_{3,3} \preccurlyeq G$.*

Proof. Exercise. □

ROBERTSON AND SEYMOUR (1998) proved the **Wagner's conjecture**:

Theorem 5.15 (Minor Theorem 2). *Let $\mathcal{P}$ be a property of graphs inherited by minors. Then there exists a* finite set *$\mathcal{F}$ of graphs such that G satisfies $\mathcal{P}$ if and only if G does not have a minor from $\mathcal{F}$.*

One of the impressive application of Theorem 5.15 concerns embeddings of graphs on surfaces, see the next chapters. By Theorem 5.15, one can test (with a *fast* algorithm) whether a graph can be embedded onto a surface.

Every graph can be drawn in the 3-dimensional space without crossing edges. An old problem asks if there exists an algorithm that would determine whether a graph can be drawn so that its cycles do not form (nontrivial) knots. This problem is solved by the above results, since the property 'knotless' is inherited by minors: there *exists* a fast algorithm to do the job. However, this algorithm is not known!

Hadwiger's Problem. HADWIGER conjectured in 1943 that for every graph G,

$$K_{\chi(G)} \preccurlyeq G \ ,$$

that is, *if $\chi(G) \geq r$, then G has a complete graph K_r as its minor*. The conjecture is trivial for $r = 2$, and it is known to hold for all $r \leq 6$. The cases for $r = 5$ and 6 follow from the 4-Colour Theorem.

5.3 Genus of a graph

A graph is planar, if it can be drawn in the plane without crossing edges. A plane is an important special case of a surface. In this section we study shortly drawing graphs in other surfaces.

There are quite many interesting surfaces many of which are rather difficult to draw. We shall study the 'easy surfaces' – those that are compact and orientable. These are surfaces that have both an inside and an outside, and can be entirely characterized by the number of holes in them. This number is the *genus* of the surface. There are also non-orientable compact surfaces such as the Klein bottle and the projective plane.

Background on surfaces

We shall first have a quick look at the general surfaces and their classification without going into the details. Consider the space $\mathbb{R}^3$, which has its (usual) distance function $d(x, y) \in \mathbb{R}$ of its points.

Two figures (i.e., sets of points) A and B are **topologically equivalent** (or **homeomorphic**) if there exists a bijection $f \colon A \to B$ such that f and its inverse $f^{-1} \colon B \to A$ are continuous. In particular, two figures are topologically equivalent if one can be deformed to the other by bending, squeezing, stretching, and shrinking without tearing it apart or gluing any of its parts together. All these deformations should be such that they can be undone.

A set of points X is a **surface**, if X is connected (there is a continuous line inside X between any two given points) and every point $x \in X$ has a neighbourhood that is topologically equivalent to an open planar disk $D(a) = \{x \mid dist(a, x) < 1\}$.

We deal with surfaces of the real space, and in this case a surface X is **compact**, if X is closed and bounded. Note that the plane is not compact, since it it not bounded. A subset of a compact surface X is a **triangle** if it is topologically equivalent to a triangle in the plane. A finite set of triangles T_i, $i = 1, 2, \dots, m$, is a **triangulation** of X if $X = \cup_{i=1}^{m} T_i$ and any nonempty intersection $T_i \cap T_j$ with $i \neq j$ is either a vertex or an edge.

The following is due to RADÓ (1925).

Theorem 5.16. *Every compact surface has a triangulation.*

Each triangle of a surface can be oriented by choosing an order for its vertices up to cyclic permutations. Such a permutation induces a direction for the edges of the triangle. A triangulation is said to be **oriented** if the triangles are assigned orientations such that common edges of two triangles are always oriented in reverse directions. A surface is **orientable** if it admits an oriented triangulation.

Equivalently, orientability can be described as follows.

Theorem 5.17. *A compact surface X is orientable if and only if it has no subsets that are topologically equivalent to the Möbius band.*

In the Möbius band (which itself is not a surface according the above definition) one can travel around and return to the starting point with left and right reversed.

A **connected sum** $X \# Y$ of two compact surfaces is obtained by cutting an open disk off from both surfaces and then gluing the surfaces together along the boundary of the disks. (Such a deformation is not allowed by topological equivalence.)

The next result is known as the **classification theorem of compact surfaces**.

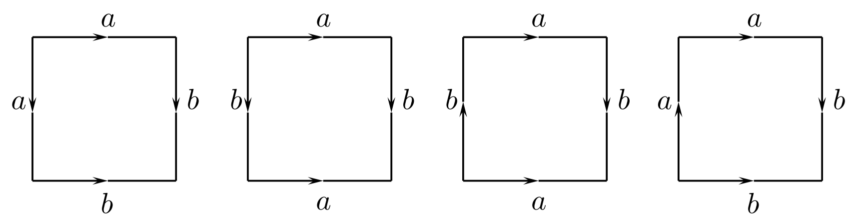
Theorem 5.18 (DEHN AND HEEGAARD **(1907)).** *Let X be a compact surface. Then*

(i) if X is orientable, then it is topologically equivalent to a sphere $S = S_0$ or a connected sum of tori: $S_n = S_1 \# S_1 \# \ldots \# S_1$ for some $n \geq 1$, where S_1 is a torus.
(ii) if X is nonorientable, then X is topologically equivalent to a connected sum of projective planes: $P_n = P \# P \# \ldots \# P$ for some $n \geq 1$, where P is a projective plane.

It is often difficult to imagine how a figure (say, a graph) can be drawn in a surface. There is a helpful, and difficult to prove, result due to RADÓ (1920), stating that every compact surface (orientable or not) has a description by a **plane model**, which consists of a polygon in the plane such that

- each edge of the polygon is labelled by a letter,
- each letter is a label of exactly two edges of the polygon, and
- each edge is given an orientation (clockwise or counter clockwise).

Given a plane model M, a compact surface is obtained by gluing together the edges having the same label in the direction that they have.

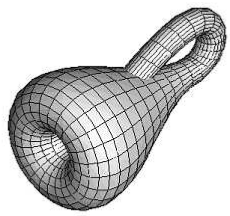


Sphere Torus Klein bottle Projective plane

From a plane model one can easily determine if the surface is oriented or not. It is nonoriented if and only if, for some label a, the edges labelled by a have the same direction when read clockwise. (This corresponds to the Möbius band.)

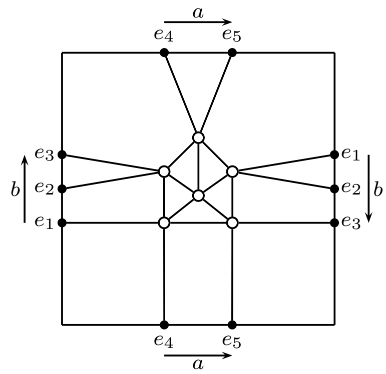
A plane model, and thus a compact surface, can also be represented by a (circular) word by reading the model clockwise, and concatenating the labels with the convention that a^{-1} is chosen if the direction of the edge is counter clockwise. Hence, the sphere is represented by the word $abb^{-1}a^{-1}$, the torus by $aba^{-1}b^{-1}$, the Klein bottle by $aba^{-1}b$ and the projective plane by $abb^{-1}a$.

These surfaces, as do the other surfaces, have many other plane models and representing words as well. A word representing a connected sum of two surfaces, represented by words W_1 and W_2, is obtained by concatenating these words to W_1W_2. By studying the relations of the representing words, Theorem 5.18 can be proved.


Klein bottle

Drawing a graph (or any figure) in a surface can be elaborated compared to drawing in a plane model, where a line that enters an edge of the polygon must continue by the corresponding point of the other edge with the same label (since these points are identified when we glue the edges together).

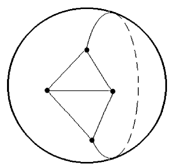
Example 5.5. On the right we have drawn K_6 in the Klein bottle. The black dots indicate, where the lines enter and leave the edges of the plane model. Recall that in the plane model for the Klein bottle the vertical edges of the square have the same direction.



Sphere

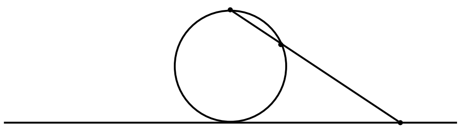
DEFINITION. In general, if S is a surface, then a graph G has an S-**embedding**, if G can be drawn in S without crossing edges.

Let S_0 be (the surface of) a **sphere**. According to the next theorem a sphere has exactly the same embeddings as do the plane. In the one direction the claim is obvious: if G is a planar graph, then it can be drawn in a bounded area of the plane (without crossing edges), and this bounded area can be ironed on the surface of a large enough sphere.



Clearly, if a graph can be embedded in one sphere, then it can be embedded in any sphere – the size of the sphere is of no importance. On the other hand, if G is embeddable in a sphere S_0, then there is a small area of the sphere, where there are no points of the edges. We then puncture the sphere at this area, and stretch it open until it looks like a region of the plane. In this process no crossings of edges can be created, and hence G is planar.

Another way to see this is to use projection of the sphere to a plane:



Theorem 5.19. *A graph G has an S_0-embedding if and only if it is planar.*

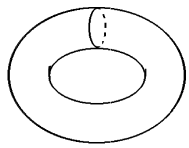
Therefore instead of planar embeddings we can equally well study embeddings of graphs in a sphere. This is sometimes convenient, since the sphere is closed and it has no boundaries.

Most importantly, a planar graph drawn in a sphere has no exterior face – all faces are bounded (by edges).

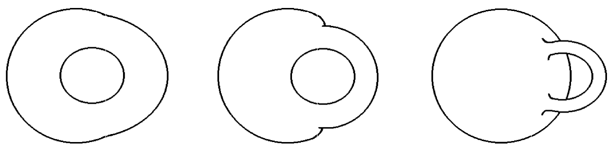
If a sphere is deformed by pressing or stretching, its embeddability properties will remain the same. In topological terms the surface has been distorted by a continuous transformation.

Torus

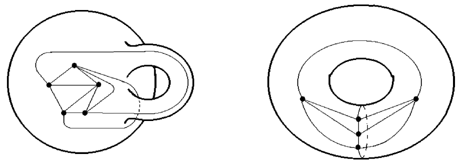
Consider next a surface which is obtained from the sphere S_0 by pressing a hole in it. This is a **torus** S_1 (or an **orientable surface of genus** 1). The S_1-embeddable graphs are said to have **genus** equal to 1.



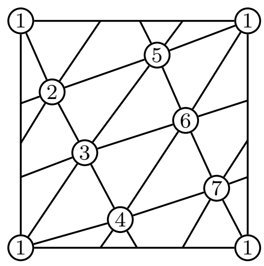
Sometimes it is easier to consider handles than holes: a torus S_1 can be deformed (by a continuous transformation) into a **sphere with a handle**.



If a graph G is S_1-embeddable, then it can be drawn in any one of the above surfaces without crossing edges.

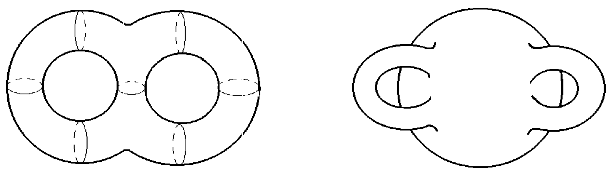


Example 5.6. The smallest nonplanar graphs K_5 and $K_{3,3}$ have genus 1. Also, K_7 has genus 1 as can be seen from the plane model (of the torus) on the right.



Genus

Let S_n ($n \geq 0$) be a sphere with n holes in it. The drawing of an S_4 can already be quite complicated, because we do not put any restrictions on the places of the holes (except that we must not tear the surface into disjoint parts). However, once again an S_n can be transformed (topologically) into a sphere with n handles.



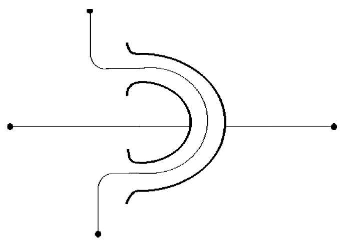
DEFINITION. We define the **genus** $g(G)$ of a graph G as the smallest integer n, for which G is S_n-embeddable.

For planar graphs, we have $g(G) = 0$, and, in particular, $g(K_4) = 0$. For K_5, we have $g(K_5) = 1$, since K_5 is nonplanar, but is embeddable in a torus. Also, $g(K_{3,3}) = 1$.

The next theorem states that any graph G can be embedded in some surface S_n with $n \geq 0$.

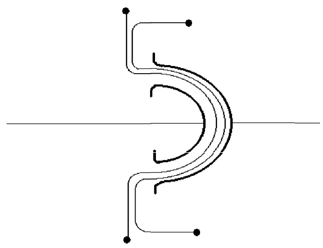
Theorem 5.20. *Every graph has a genus.*

This result has an easy intuitive verification. Indeed, consider a graph G and any of its plane (or sphere) drawing (possibly with many crossing edges) such that no three edges cross each other in the same point (such a drawing can be obtained). At each of these crossing points create a handle so that one of the edges goes below the handle and the other uses the handle to cross over the first one.



We should note that the above argument does not determine $g(G)$, only that G can be embedded in some S_n. However, clearly $g(G) \leq n$, and thus the genus $g(G)$ of G exists.

The same handle can be utilized by several edges.



Euler's formula with genus*

The drawing of a planar graph G in a sphere has the advantage that the faces of the embedding are not divided into internal and external. The external face of G becomes an 'ordinary face' after G has been drawn in S_0.

In general, a *face* of an embedding of G in S_n (with $g(G) = n$) is a region of S_n surrounded by edges of G. Let again φ_G denote the number of faces of an embedding of G in S_n. We omit the proof of the next generalization of Euler's formula.

Theorem 5.21. *If G is a connected graph, then*

$$\nu_G - \varepsilon_G + \varphi_G = 2 - 2g(G)\ .$$

If G is a planar graph, then $g(G) = 0$, and the above formula is the Euler's formula for planar graphs.

DEFINITION. A face of an embedding $P(G)$ in a surface is a 2-**cell**, if every simple closed curve (that does not intersect with itself) can be continuously deformed to a single point.

The complete graph K_4 can be embedded in a torus such that it has a face that is not a 2-cell. But this is because $g(K_4) = 0$, and the genus of the torus is 1. We omit the proof of the general condition discovered by YOUNGS:

Theorem 5.22 (YOUNGS (1963)). *The faces of an embedding of a connected graph G in a surface of genus $g(G)$ are 2-cells.*

Lemma 5.12. *For a connected G with $\nu_G \geq 3$ we have $3\varphi_G \leq 2\varepsilon_G$.*

Proof. If $\nu_G = 3$, then the claim is trivial. Assume thus that $\nu_G \geq 4$. In this case we need the knowledge that φ_G is counted in a surface that determines the genus of G (and in no surface with a larger genus). Now every face has a border of at least three edges, and, as before, every nonbridge is on the boundary of exactly two faces. □

Theorem 5.23. *For a connected G with $\nu_G \geq 3$,*

$$g(G) \geq \frac{1}{6}\varepsilon_G - \frac{1}{2}(\nu_G - 2) \ .$$

Proof. By the previous lemma, $3\varphi_G \leq 2\varepsilon_G$, and by the generalized Euler's formula, $\varphi_G = \varepsilon_G - \nu_G + 2 - 2g(G)$. Combining these we obtain that $3\varepsilon_G - 3\nu_G + 6 - 6g(G) \leq 2\varepsilon_G$, and the claim follows. □

By this theorem, we can compute lower bounds for the genus $g(G)$ without drawing any embeddings. As an example, let $G = K_8$. In this case $\nu_G = 8$, $\varepsilon_G = 28$, and so $g(G) \geq \frac{5}{3}$. Since the genus is always an integer, $g(G) \geq 2$. We deduce that K_8 cannot be embedded in the surface S_1 of the torus.

If $H \subseteq G$, then clearly $g(H) \leq g(G)$, since H is obtained from G by omitting vertices and edges. In particular,

Lemma 5.13. *For a graph G of order n, $g(G) \leq g(K_n)$.*

For the complete graphs K_n a good lower bound was found early.

Theorem 5.24 (HEAWOOD (1890)). *If $n \geq 3$, then*

$$g(K_n) \geq \frac{(n-3)(n-4)}{12}.$$

Proof. The number of edges in K_n is equal to $\varepsilon_G = \frac{1}{2}n(n-1)$. By Theorem 5.23, we obtain $g(K_n) \geq (1/6)\varepsilon_G - (1/2)(n-2) = (1/12)(n-3)(n-4)$. □

This result was dramatically improved to obtain

Theorem 5.25 (RINGEL AND YOUNGS **(1968)).** *If $n \geq 3$, then*

$$g(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil .$$

Therefore $g(K_6) = \lceil 3 \cdot 2/12 \rceil = \lceil 1/2 \rceil = 1$. Also, $g(K_7) = 1$, but $g(K_8) = 2$.

By Theorem 5.25,

Theorem 5.26. *For all graphs G of order $n \geq 3$,*

$$g(G) \leq \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil .$$

Also, we know the exact genus for the complete bipartite graphs:

Theorem 5.27 (RINGEL **(1965)).** *For the complete bipartite graphs,*

$$g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil .$$

Chromatic numbers*

For the planar graphs G, the proof of the 4-Colour Theorem, $\chi(G) \leq 4$, is extremely long and difficult. This in mind, it is surprising that the generalization of the 4-Colour Theorem for genus ≥ 1 is much easier. HEAWOOD proved a hundred years ago:

Theorem 5.28 (HEAWOOD). *If $g(G) = g \geq 1$, then*

$$\chi(G) \leq \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor .$$

Notice that for $g = 0$ this theorem would be the 4-colour theorem. HEAWOOD proved it ‘only’ for $g \geq 1$.

Using the result of RINGEL AND YOUNGS and some elementary computations we can prove that the above theorem is the best possible.

Theorem 5.29. *For each $g \geq 1$, there exists a graph G with genus $g(G) = g$ so that*

$$\chi(G) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor .$$

If a nonplanar graph G can be embedded in a torus, then $g(G) = 1$, and $\chi(G) \leq \lfloor (7 + \sqrt{1 + 48g})/2 \rfloor = 7$. Moreover, for $G = K_7$ we have that $\chi(K_7) = 7$ and $g(K_7) = 1$.

Three dimensions*

Every graph can be drawn without crossing edges in the 3-dimensional space. Such a drawing is called **spatial embedding** of the graph. Indeed, such an embedding can be achieved by putting all vertices of G on a line, and then drawing the edges in different planes that contain the line. Alternatively, the vertices of G can be put in a sphere, and drawing the edges as straight lines crossing the sphere inside.

A spatial embedding of a graph G is said to have **linked cycles**, if two cycles of G form a link (they cannot be separated in the space). By CONWAY and GORDON in 1983 every spatial embedding of K_6 contains linked cycles.

It was shown by ROBERTSON, SEYMOUR AND THOMAS (1993) that there is a set of 7 graphs such that a graph G has a spatial embedding without linked cycles if and only if G does not have a minor belonging to this set.

This family of forbidden graphs was originally found by SACHS (without proof), and it contains K_6 and the Petersen graph. Every graph in the set has 15 edges, which is curious.

For further results and proofs concerning graphs in surfaces, see

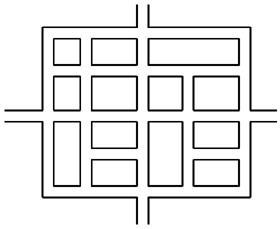
B. MOHAR AND C. THOMASSEN, “Graphs on Surfaces”, Johns Hopkins, 2001.

6

Directed Graphs

6.1 Digraphs

In some problems the relation between the objects is not symmetric. For these cases we need directed graphs, where the edges are oriented from one vertex to another.
As an example consider a map of a small town. Can you make the streets one-way, and still be able to drive from one house to another (or exit the town)?



Definitions

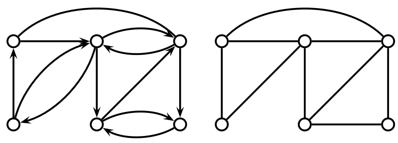
DEFINITION. A **digraph** (or a **directed graph**) $D = (V_D, E_D)$ consists of the vertices V_D and (directed) edges $E_D \subseteq V_D \times V_D$ (without loops vv). We still write uv for (u, v), but note that now $uv \neq vu$. For each pair $e = uv$ define the **inverse** of e as $e^{-1} = vu$ $(= (v, u))$.

Note that $e \in E_D$ does *not* imply $e^{-1} \in E_D$.

DEFINITION. Let D be a digraph. Then A is its

- **subdigraph**, if $V_A \subseteq V_D$ and $E_A \subseteq E_D$,
- **induced subdigraph**, $A = D[X]$, if $V_A = X$ and $E_A = E_D \cap (X \times X)$.

The **underlying graph** $U(D)$ of a digraph D is the graph on V_D such that if $e \in E_D$, then the undirected edge with the same ends is in $U(D)$.



A digraph D is an **orientation** of a graph G, if $G = U(D)$ and $e \in E_D$ implies $e^{-1} \notin E_D$. In this case, D is said to be an **oriented graph**.

DEFINITION. Let D be a digraph. A walk $W = e_1 e_2 \dots e_k \colon u \xrightarrow{\star} v$ of $U(D)$ is a **directed walk**, if $e_i \in E_D$ for all $i \in [1, k]$. Similarly, we define **directed paths** and **directed cycles** as directed walks and closed directed walks without repetitions of vertices.

The digraph D is **di-connected**, if, for all $u \neq v$, there exist directed paths $u \xrightarrow{\star} v$ and $v \xrightarrow{\star} u$. The maximal induced di-connected subdigraphs are the **di-components** of D.

Note that a graph $G = U(D)$ might be connected, although the digraph D is not di-connected.

DEFINITION. The **indegree** and the **outdegree** of a vertex are defined as follows

$$d^I_D(v) = |\{e \in E_D \mid e = xv\}|, \qquad d^O_D(v) = |\{e \in E_D \mid e = vx\}| \,.$$

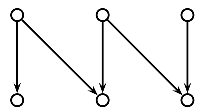
We have the following **handshaking lemma**. (You offer and accept a handshake.)

Lemma 6.1. *Let D be a digraph. Then*

$$\sum_{v \in D} d^I_D(v) = |E_D| = \sum_{v \in D} d^O_D(v).$$

Directed paths

The relationship between paths and directed paths is in general rather complicated. This digraph has a path of length five, but its directed paths are of length one.



There is a nice connection between the lengths of directed paths and the chromatic number $\chi(D) = \chi(U(D))$.

Theorem 6.1 (ROY **(1967),**GALLAI **(1968)).** *A digraph D has a directed path of length $\chi(D) - 1$.*

Proof. Let $A \subseteq E_D$ be a minimal set of edges such that the subdigraph $D{-}A$ contains no directed cycles. Let k be the length of the longest directed path in $D{-}A$.

For each vertex $v \in D$, assign a colour $\alpha(v) = i$, if a longest directed path from v has length $i - 1$ in $D{-}A$. Here $1 \le i \le k + 1$.

First we observe that if $P = e_1e_2 \dots e_r$ ($r \ge 1$) is any directed path $u \xrightarrow{\star} v$ in $D{-}A$, then $\alpha(u) \ne \alpha(v)$. Indeed, if $\alpha(v) = i$, then there exists a directed path $Q\colon v \xrightarrow{\star} w$ of length $i - 1$, and PQ is a directed path, since $D{-}A$ does not contain directed cycles. Since $PQ\colon u \xrightarrow{\star} w$, $\alpha(u) \ne i = \alpha(v)$. In particular, if $e = uv \in E_{D-A}$, then $\alpha(u) \ne \alpha(v)$.

Consider then an edge $e = vu \in A$. By the minimality of A, $(D{-}A) + e$ contains a directed cycle $C\colon u \xrightarrow{\star} v \longrightarrow u$, where the part $u \xrightarrow{\star} v$ is a directed path in $D{-}A$, and hence $\alpha(u) \ne \alpha(v)$. This shows that α is a proper colouring of $U(D)$, and therefore $\chi(D) \le k + 1$, that is, $k \ge \chi(D) - 1$. □

The bound $\chi(D) - 1$ is the best possible in the following sense:

Theorem 6.2. *Every graph G has an orientation D, where the longest directed paths have lengths $\chi(G) - 1$.*

Proof. Let $k = \chi(G)$ and let α be a proper k-colouring of G. As usual the set of colours is $[1, k]$. We orient each edge $uv \in E_G$ by setting $uv \in E_D$, if $\alpha(u) < \alpha(v)$. Clearly, the so obtained orientation D has no directed paths of length $\geq k - 1$. □

DEFINITION. An orientation D of an undirected graph G is **acyclic**, if it has no directed cycles. Let $a(G)$ be the number of acyclic orientations of G.

The next result is charming, since $\chi_G(-1)$ measures the number of proper colourings of G using -1 colours!

Theorem 6.3 (STANLEY (1973)). *Let G be a graph of order n. Then the number of the acyclic orientations of G is*

$$a(G) = (-1)^n \chi_G(-1) \ ,$$

where χ_G is the chromatic polynomial of G.

Proof. The proof is by induction on ε_G. First, if G is discrete, then $\chi_G(k) = k^n$, and $a(G) = 1 = (-1)^n(-1)^n = (-1)^n \chi_G(-1)$ as required.

Now $\chi_G(k)$ is a polynomial that satisfies the recurrence $\chi_G(k) = \chi_{G-e}(k) - \chi_{G*e}(k)$. To prove the claim, we show that $a(G)$ satisfies the same recurrence.

Indeed, if

$$a(G) = a(G-e) + a(G * e) \tag{6.1}$$

then, by the induction hypothesis,

$$a(G) = (-1)^n \chi_{G-e}(-1) + (-1)^{n-1} \chi_{G*e}(-1) = (-1)^n \chi_G(-1) \ .$$

For (6.1), we observe that every acyclic orientation of G gives an acyclic orientation of $G-e$. On the other hand, if D is an acyclic orientation of $G-e$ for $e = uv$, it extends to an acyclic orientation of G by putting $e_1 \colon u \to v$ or $e_2 \colon v \to u$. Indeed, if D has no directed path $u \xrightarrow{\star} v$, we choose e_2, and if D has no directed path $v \xrightarrow{\star} u$, we choose e_1. Note that since D is acyclic, it cannot have both ways $u \xrightarrow{\star} v$ and $v \xrightarrow{\star} u$.

We conclude that $a(G) = a(G-e) + b$, where b is the number of acyclic orientations D of $G-e$ that extend in both ways e_1 and e_2. The acyclic orientations D that extend in both ways are exactly those that contain

$$\text{neither } u \xrightarrow{\star} v \text{ nor } v \xrightarrow{\star} u \text{ as a directed path.} \tag{6.2}$$

Each acyclic orientation of $G * e$ corresponds in a natural way to an acyclic orientation D of $G-e$ that satisfies (6.2). Therefore $b = a(G * e)$, and the proof is completed. □

One-way traffic

Every graph can be oriented, but the result may not be di-connected. In the **one-way traffic problem** the resulting orientation should be di-connected, for otherwise someone is not able to drive home. ROBBINS' theorem solves this problem.

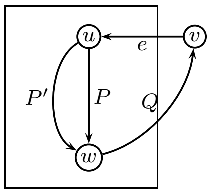
DEFINITION. A graph G is **di-orientable**, if there is a di-connected oriented graph D such that $G = U(D)$.

Theorem 6.4 (ROBBINS **(1939)).** *A connected graph G is di-orientable if and only if G has no bridges.*

Proof. If G has a bridge e, then any orientation of G has at least two di-components (both sides of the bridge).

Suppose then that G has no bridges. Hence G has a cycle C, and a cycle is always di-orientable. Let then $H \subseteq G$ be maximal such that it has a di-orientation D_H. If $H = G$, then we are done.

Otherwise, there exists an edge $e = vu \in E_G$ such that $u \in H$ but $v \notin H$ (because G is connected). The edge e is not a bridge and thus there exists a cycle



$$C' = ePQ \colon v \longrightarrow u \xrightarrow{\star} w \xrightarrow{\star} v$$

in G, where w is the last vertex inside H.

In the di-orientation D_H of H there is a directed path $P' \colon u \xrightarrow{\star} w$. Now, we orient $e \colon v \longrightarrow u$ and the edges of Q in the direction $Q \colon w \xrightarrow{\star} v$ to obtain a directed cycle $eP'Q \colon v \longrightarrow u \xrightarrow{\star} w \xrightarrow{\star} v$. In conclusion, $G[V_H \cup V_C]$ has a di-orientation, which contradicts the maximality assumption on H. This proves the claim. □

Example 6.1. Let D be a digraph. A **directed Euler tour** of D is a directed closed walk that uses each edge exactly once. A **directed Euler trail** of D is a directed walk that uses each edge exactly once.

The following two results are left as exercises.

(1) *Let D be a digraph such that $U(D)$ is connected. Then D has a directed Euler tour if and only if $d_D^I(v) = d_D^O(v)$ for all vertices v.*

(2) *Let D be a digraph such that $U(D)$ is connected. Then D has a directed Euler trail if and only if $d_D^I(v) = d_D^O(v)$ for all vertices v with possibly excepting two vertices x, y for which* $|d_D^I(v) - d_D^O(v)| = 1$.

The above results hold equally well for *multidigraphs*, that is, for directed graphs, where we allow parallel directed edges between the vertices.

Example 6.2. The following problem was first studied by HUTCHINSON AND WILF (1975) with a motivation from DNA sequencing. Consider words over an alphabet $A = \{a_1, a_2, \dots, a_n\}$ of n letters, that is, each word w is a sequence of letters. In the case of DNA, the letters are A, T, C, G. In a problem instance, we are given nonnegative integers s_i and r_{ij} for $1 \le i, j \le n$, and the question is: does there exist a word w in which each letter a_i occurs exactly s_i times, and a_i is followed by a_j exactly r_{ij} times.

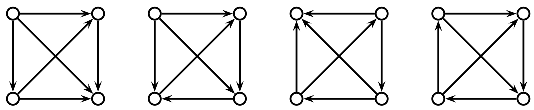
For instance, if $n = 2$, $s_1 = 3$, and $r_{11} = 1$, $r_{12} = 2$, $r_{21} = 1$, $r_{22} = 0$, then the word $a_1a_2a_1a_1a_2$ is a solution to the problem.

Consider a multidigraph D with $V_D = A$ for which there are r_{ij} edges $a_i a_j$. It is rather obvious that a directed Euler trail of D gives a solution to the sequencing problem.

Tournaments

DEFINITION. A **tournament** T is an orientation of a complete graph.

Example 6.3. There are four tournaments of four vertices that are not isomorphic with each other. (Isomorphism of directed graphs is defined in the obvious way.)



Theorem 6.5 (RÉDEI (1934)). *Every tournament has a directed Hamilton path.*

Proof. The chromatic number of K_n is $\chi(K_n) = n$, and hence by Theorem 6.1, a tournament T of order n has a directed path of length $n-1$. This is then a directed Hamilton path visiting each vertex once. □

The vertices of a tournament can be easily reached from one vertex (sometimes called the **king**).

Theorem 6.6 (LAUDAU (1953)). *Let v be a vertex of a tournament T of maximum outdegree. Then for all u, there is a directed path $v \xrightarrow{\star} u$ of length at most two.*

Proof. Let T be an orientation of K_n, and let $d_T^O(v) = d$ be the maximum outdegree in T. Suppose that there exists an x, for which the directed distance from v to x is at least three. It follows that $xv \in E_T$ and $xu \in E_T$ for all u with $vu \in E_T$. But there are d vertices in $A = \{y \mid vy \in E_T\}$, and thus $d+1$ vertices in $\{y \mid xy \in E_T\} = A \cup \{v\}$. It follows that the outdegree of x is $d+1$, which contradicts the maximality assumption made for v. □

Problem. Ádám's conjecture states that *in every digraph D with a directed cycle there exists an edge uv the reversal of which decreases the number of directed cycles*. Here the new digraph has the edge vu instead of uv.

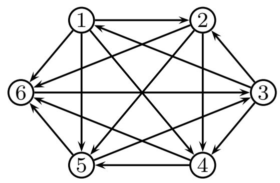
Example 6.4. Consider a tournament of n teams that play once against each other, and suppose that each game has a winner. The situation can be presented as a tournament, where the vertices correspond to the teams v_i, and there is an edge $v_i v_j$, if v_i won v_j in their mutual game.

DEFINITION. A team v is *a* **winner** (there may be more than one winner), if v comes out with the most victories in the tournament.

Theorem 6.6 states that a winner v either defeated a team u or v defeated a team that defeated u.

A **ranking** of a tournament is a linear ordering of the teams $v_{i_1} > v_{i_2} > \cdots > v_{i_n}$ that should reflect the scoring of the teams. One way of ranking a tournament could be by a Hamilton path: the ordering can be obtained from a directed Hamilton path $P\colon v_{i_1} \longrightarrow v_{i_2} \longrightarrow \ldots \longrightarrow v_{i_n}$. However, a tournament may have several directed Hamilton paths, and some of these may do unjust for the 'real' winner.

Example 6.5. Consider a tournament of six teams $1, 2, \ldots, 6$, and let T be the scoring digraph as in the figure. Here $1 \longrightarrow 2 \longrightarrow 4 \longrightarrow 5 \longrightarrow 6 \longrightarrow 3$ is a directed Hamilton path, but this extends to a directed Hamilton cycle (by adding $3 \longrightarrow 1$)! So for every team there is a Hamilton path, where it is a winner, and in another, it is a looser.



Let $s_1(j) = d^O_T(j)$ be the **winning number** of the team j (the number of teams beaten by j). In the above tournament,

$$s_1(1) = 4,\ s_1(2) = 3,\ s_1(3) = 3,\ s_1(4) = 2,\ s_1(5) = 2,\ s_1(6) = 1\ .$$

So, is team 1 the winner? If so, is 2 or 3 next? Define the **second-level scoring** for each team by

$$s_2(j) = \sum_{ji \in E_T} s_1(i)\ .$$

This tells us how good teams j beat. In our example, we have

$$s_2(1) = 8,\ s_2(2) = 5,\ s_2(3) = 9,\ s_2(4) = 3,\ s_2(5) = 4,\ s_2(6) = 3\ .$$

Now, it seems that 3 is the winner,but 4 and 6 have the same score. We continue by defining inductively the m**th-level scoring** by

$$s_m(j) = \sum_{ji \in E_T} s_{m-1}(i)\ .$$

It can be proved (using matrix methods) that for a di-connected tournament with at least four teams, *the level scorings will eventually stabilize in a ranking of the tournament*: there exits an m for which the mth-level scoring gives the same ordering as do the $(m+k)$th-level scorings for all $k \geq 1$. If T is not di-connected, then the level scoring should be carried out with respect to the di-components.

In our example the level scoring gives $1 \longrightarrow 3 \longrightarrow 2 \longrightarrow 5 \longrightarrow 4 \longrightarrow 6$ as the ranking of the tournament.

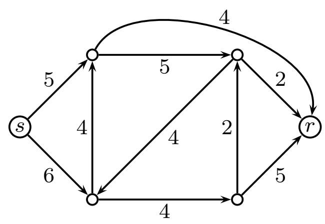
6.2 Network Flows

Various transportation networks or water pipelines are conveniently represented by weighted directed graphs. These networks usually possess also some additional requirements. Goods are transported from specific places (warehouses) to final locations (marketing places) through a network of roads. In modeling a transportation network by a digraph, we must make sure that the number of goods remains the same at each crossing of the roads. The problem setting for such networks was proposed by T.E. Harris in the 1950s. The connection to *Kirchhoff's Current Law* (1847) is immediate. According to this law, in every electrical network the amount of current flowing in a vertex equals the amount flowing out that vertex.

Flows

DEFINITION. A **network** N consists of

- an **underlying digraph** $D = (V, E)$,
- two distinct vertices s and r, called the **source** and the **sink** of N, and
- a **capacity function** $\alpha\colon V \times V \to \mathbb{R}_+$ (nonnegative real numbers), for which $\alpha(e) = 0$, if $e \notin E$.



Denote $V_N = V$ and $E_N = E$.

Let $A \subseteq V_N$ be a set of vertices, and $f\colon V_N \times V_N \to \mathbb{R}$ any function such that $f(e) = 0$, if $e \notin E_N$. We adopt the following notations:

$$[A, \overline{A}] = \{e \in E_D \mid e = uv,\ u \in A,\ v \notin A\}\ ,$$
$$f^+(A) = \sum_{e \in [A,\overline{A}]} f(e) \quad \text{and} \quad f^-(A) = \sum_{e \in [\overline{A},A]} f(e)\ .$$

In particular,

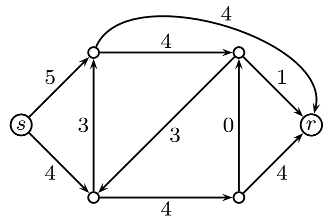
$$f^+(u) = \sum_{v \in N} f(uv) \quad \text{and} \quad f^-(u) = \sum_{v \in N} f(vu)\ .$$

DEFINITION. A **flow** in a network N is a function $f\colon V_N \times V_N \to \mathbb{R}_+$ such that

$$0 \le f(e) \le \alpha(e) \ \text{ for all } \ e, \quad \text{and} \quad f^-(v) = f^+(v) \ \text{ for all } \ v \notin \{s, r\}\ .$$

Example 6.6. The value $f(e)$ can be taught of as the rate at which transportation actually happens along the channel e which has the maximum capacity $\alpha(e)$. The second condition states that there should be no loss.

If $N = (D, s, r, \alpha)$ is a network of water pipes, then the value $\alpha(e)$ gives the capacity ($x\ m^3/\mathrm{min}$) of the pipe e.
The previous network has a flow that is indicated on the right.



A flow f in N is something that the network can handle. *E.g.*, in the above figure the source should not try to feed the network the full capacity ($11\ m^3/\mathrm{min}$) of its pipes, because the junctions cannot handle this much water.

DEFINITION. Every network N has a **zero flow** defined by $f(e) = 0$ for all e. For a flow f and each subset $A \subseteq V_N$, define the **resultant flow from** A and the **value** of f as the numbers

$$\mathrm{val}(f_A) = f^+(A) - f^-(A) \qquad \text{and} \qquad \mathrm{val}(f) = \mathrm{val}(f_s)\ (= f^+(s) - f^-(s))\ .$$

A flow f of a network N is a **maximum flow**, if there does not exist any flow f' such that $\mathrm{val}(f) < \mathrm{val}(f')$.

The value $\mathrm{val}(f)$ of a flow is the overall number of goods that are (to be) transported through the network from the source to the sink. In the above example, $\mathrm{val}(f) = 9$.

Lemma 6.2. *Let $N = (D, s, r, \alpha)$ be a network with a flow f.*

(i) If $A \subseteq N \setminus \{s, r\}$, then $\mathrm{val}(f_A) = 0$.
(ii) $\mathrm{val}(f) = -\mathrm{val}(f_r)$.

Proof. Let $A \subseteq N \setminus \{s, r\}$. Then

$$0 = \sum_{v \in A} (f^+(v) - f^-(v)) = \sum_{v \in A} f^+(v) - \sum_{v \in A} f^-(v) = f^+(A) - f^-(A) = \mathrm{val}(f_A),$$

where the third equality holds since the values of the edges uv with $u, v \in A$ cancel each out.

The second claim is also clear. □

Improvable flows

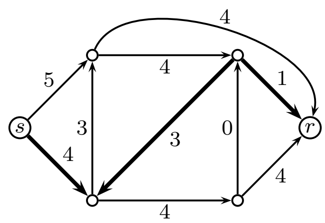
Let f be a flow in a network N, and let $P = e_1 e_2 \dots e_n$ be an *undirected* path in N where an edge e_i is **along** P, if $e_i = v_i v_{i+1} \in E_N$, and **against** P, if $e_i = v_{i+1} v_i \in E_N$.

We define a nonnegative number $\iota(P)$ for P as follows:

$$\iota(P) = \min_{e_i} \iota(e), \qquad \text{where } \iota(e) = \begin{cases} \alpha(e) - f(e) & \text{if } e \text{ is along } P\ , \\ f(e) & \text{if } e \text{ is against } P. \end{cases}$$

DEFINITION. Let f be a flow in a network N. A path $P\colon s \xrightarrow{\star} r$ is (f-)**improvable**, if $\iota(P) > 0$.

On the right, the bold path has value $\iota(P) = 1$, and therefore this path is improvable.

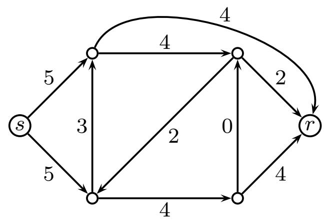


Lemma 6.3. *Let N be a network. If f is a maximum flow of N, then it has no improvable paths.*

Proof. Define

$$f'(e) = \begin{cases} f(e) + \iota(P) & \text{if } e \text{ is along P ,} \\ f(e) - \iota(P) & \text{if } e \text{ is against P ,} \\ f(e) & \text{if } e \text{ is not in P .} \end{cases}$$

Then f' is a flow, since at each intermediate vertex $v \notin \{s, r\}$, we have $(f')^-(v) = (f')^+(v)$, and the capacities of the edges are not exceeded. Now $\text{val}(f') = \text{val}(f) + \iota(P)$, since P has exactly one edge $sv \in E_N$ for the source s. Hence, if $\iota(P) > 0$, then we can improve the flow.



□

Max-Flow Min-Cut Theorem

DEFINITION. Let $N = (D, s, r, \alpha)$ be a network. For a subset $S \subset V_N$ with $s \in S$ and $r \notin S$, let the **cut** by S be

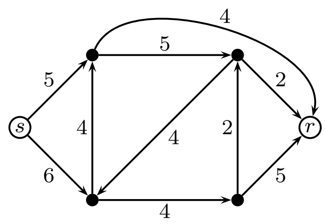
$$[S] = [S, \overline{S}] \quad (= \{uv \in E_N \mid u \in S, v \notin S\}) .$$

The **capacity** of the cut $[S]$ is the sum

$$\alpha[S] = \alpha^+(S) = \sum_{e \in [S]} \alpha(e) .$$

A cut $[S]$ is a **minimum cut**, if there is no cut $[R]$ with $\alpha[R] < \alpha[S]$.

Example 6.7. In our original network the capacity of the cut for the indicated vertices is equal to 10.



Lemma 6.4. *For a flow f and a cut $[S]$ of N,*

$$\text{val}(f) = \text{val}(f_S) = f^+(S) - f^-(S) .$$

Proof. Let $S_I = S \setminus \{s\}$. Now $\text{val}(S_I) = 0$ (since $S_I \subseteq N \setminus \{s, r\}$), and $\text{val}(f) = \text{val}(f_s)$. Hence

$$\begin{aligned} \text{val}(f_S) = & \text{val}(f_s) - \sum_{v \in S_I} f(sv) + \sum_{v \in S_I} f(vs) \\ & + \text{val}(f_{S_I}) + \sum_{v \in S_I} f(sv) - \sum_{v \in S_I} f(vs) \\ = & \text{val}(f_s) = \text{val}(f) . \end{aligned}$$

□

Theorem 6.7. *For a flow f and any cut $[S]$ of N, $\mathrm{val}(f) \le \alpha[S]$. Furthermore, equality holds if and only if for each $u \in S$ and $v \notin S$,*

(i) if $e = uv \in E_N$, then $f(e) = \alpha(e)$,
(ii) if $e = vu \in E_N$, then $f(e) = 0$.

Proof. By the definition of a flow,

$$f^+(S) = \sum_{e \in [S]} f(e) \le \sum_{e \in [S]} \alpha(e) = \alpha[S] \ ,$$

and $f^-(S) \ge 0$. By Lemma 6.4, $\mathrm{val}(f) = \mathrm{val}(f_S) = f^+(S) - f^-(S)$, and hence $\mathrm{val}(f) \le \alpha[S]$, as required. Also, the equality $\mathrm{val}(f) = \alpha[S]$ holds if and only if

(1) $f^+(S) = \alpha[S]$ and (2) $f^-(S) = 0$. This holds if and only if $f(e) = \alpha(e)$ for all $e \in [S]$ (since $f(e) \le \alpha(e)$), and
(2) $f(e) = 0$ for all $e = vu$ with $u \in S$, $v \notin S$.

This proves the claim. □

In particular, if f is a maximum flow and $[S]$ a minimum cut, then

$$\mathrm{val}(f) \le \alpha[S] \ .$$

Corollary 6.1. *If f is a flow and $[S]$ a cut such that $\mathrm{val}(f) = \alpha[S]$, then f is a maximum flow and $[S]$ a minimum cut.*

The following main result of network flows was proved independently by ELIAS, FEINSTEIN, SHANNON, by FORD AND FULKERSON, and by ROBACKER in 1955 – 56. The present approach is due to Ford and Fulkerson.

Theorem 6.8. *A flow f of a network N is maximum if and only if there are no f-improvable paths in N.*

Proof. By Lemma 6.3, a maximum flow cannot have improvable paths.

Conversely, assume that N contains no f-improvable paths, and let

$$S_I = \{u \in N \mid \text{for some path } P\colon s \xrightarrow{\star} u,\ \iota(P) > 0\} \ .$$

Set $S = S_I \cup \{s\}$.

Consider an edge $e = uv \in E_N$, where $u \in S$ and $v \notin S$. Since $u \in S$, there exists a path $P\colon s \xrightarrow{\star} u$ with $\iota(P) > 0$. Moreover, since $v \notin S$, $\iota(Pe) = 0$ for the path $Pe\colon s \xrightarrow{\star} v$. Therefore $\iota(e) = 0$, and so $f(e) = \alpha(e)$.

By the same argument, for an edge $e = vu \in E_N$ with $v \notin S$ and $u \in S$, $f(e) = 0$.

By Theorem 6.7, we have $\mathrm{val}(f) = \alpha[S]$. Corollary 6.1 implies now that f is a maximum flow (and $[S]$ is a minimum cut). □

Theorem 6.9. *Let N be a network, where the capacity function $\alpha\colon V \times V \to \mathbb{N}$ has integer values. Then N has a maximum flow with integer values.*

Proof. Let f_0 be the zero flow, $f_0(e) = 0$ for all $e \in V \times V$. A maximum flow is constructed using Lemma 6.3 by increasing and decreasing the values of the edges by integers only. □

The proof of Theorem 6.8 showed also

Theorem 6.10 (Max-Flow Min-Cut). *In a network N, the value $\mathrm{val}(f)$ of a maximum flow equals the capacity $\alpha[S]$ of a minimum cut.*

Applications to graphs*

The Max-Flow Min-Cut Theorem is a strong result, and many of our previous results follow from it.

We mention a connection to the Marriage Theorem, Theorem 3.9. For this, let G be a bipartite graph with a bipartition (X, Y), and consider a network N with vertices $\{s, r\} \cup X \cup Y$. Let the edges (with their capacities) be $sx \in E_N$ ($\alpha(sx) = 1$), $yr \in E_N$ ($\alpha(yr) = 1$) for all $x \in X$, $y \in Y$ together with the edges $xy \in E_N$ ($\alpha(xy) = |X| + 1$), if $xy \in E_G$ for $x \in X$, $y \in Y$. Then G has a matching that saturates X if and only if N has a maximum flow of value $|X|$. Now Theorem 6.10 gives Theorem 3.9.

Next we apply the theorem to **unit networks**, where the capacities of the edges are equal to one ($\alpha(e) = 1$ for all $e \in E_N$). We obtain results for (directed) graphs.

Lemma 6.5. *Let N be a unit network with source s and sink r.*

(i) The value $\mathrm{val}(f)$ of a maximum flow equals the maximum number of edge-disjoint directed paths $s \xrightarrow{\star} r$.
(ii) The capacity of a minimum cut $[S]$ equals the minimum number of edges whose removal destroys the directed connections $s \xrightarrow{\star} r$ from s to r.

Proof. Exercise. □

Corollary 6.2. *Let u and v be two vertices of a digraph D. The maximum number of edge-disjoint directed paths $u \xrightarrow{\star} v$ equals the minimum number of edges, whose removal destroys all the directed connections $u \xrightarrow{\star} v$ from D.*

Proof. A network N with source s and sink r is obtained by setting the capacities equal to 1. The claim follows from Lemma 6.5 and Corollary 6.10. □

Corollary 6.3. *Let u and v be two vertices of a graph G. The maximum number of edge-disjoint paths $u \xrightarrow{\star} v$ equals the minimum number of edges, whose removal destroys all the connections $u \xrightarrow{\star} v$ from D.*

Proof. Consider the digraph D that is obtained from G by replacing each (undirected) edge $uv \in E_G$ by two directed edges $uv \in E_D$ and $vu \in E_D$. The claim follows then easily from Corollary 6.2. □

The next corollary is **Menger's Theorem** for edge connectivity.

Corollary 6.4. *A graph G is k-edge connected if and only if any two distinct vertices of G are connected by at least k independent paths.*

Proof. The claim follows immediately from Corollary 6.3. □

Seymour's 6-flows*

DEFINITION. A k-**flow** (H, α) of an undirected graph G is an orientation H of G together with an edge colouring $\alpha \colon E_H \to [0, k-1]$ such that for all vertices $v \in V$,

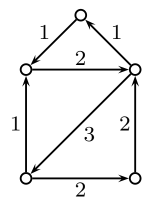
$$\sum_{e=vu \in E_H} \alpha(e) = \sum_{f=uv \in E_H} \alpha(f) \ , \tag{6.3}$$

that is, the sum of the incoming values equals the sum of the outgoing values. A k-flow is **nowhere zero**, if $\alpha(e) \neq 0$ for all $e \in E_H$.

In the k-flows we do not have any source or sink. For convenience, *let* $\alpha(e^{-1}) = -\alpha(e)$ *for all* $e \in E_H$ *in the orientation* H of G so that the condition (6.3) becomes

$$\sum_{e=vu \in E_H} \alpha(e) = 0 \, . \tag{6.4}$$

Example 6.8. A graph with a nowhere zero 4-flow.



The condition (6.4) generalizes to the subsets $A \subseteq V_G$ in a natural way,

$$\sum_{e \in [A, \overline{A}]} \alpha(e) = 0 \ , \tag{6.5}$$

since the values of the edges inside A cancel out each other. In particular,

Lemma 6.6. *If G has a nowhere zero k-flow for some k, then G has no bridges.*

Tutte's Problem. It was conjectured by TUTTE (1954) that *every bridgeless graph has a nowhere zero* 5*-flow*. The Petersen graph has a nowhere zero 5-flow but does not have any nowhere 4-flows, and so 5 is the best one can think of. Tutte's conjecture resembles the 4-Colour Theorem, and indeed, the conjecture is known to hold for the planar graphs. The proof of this uses the 4-Colour Theorem.

In order to fully appreciate Seymour's result, Theorem 6.11, we mention that it was proved as late as 1976 (by JAEGER) that every bridgeless G has a nowhere zero k-flow for *some* integer k.

SEYMOUR's remarkable result reads as follows:

Theorem 6.11 (SEYMOUR's (1981)). *Every bridgeless graph has a nowhere zero* 6*-flow.*

Proof. Omitted. □

DEFINITION. The **flow number** $f(G)$ of a bridgeless graph G is the least integer k for which G has a nowhere zero k-flow.

Theorem 6.12. *A connected graph* G *has a flow number* $f(G) = 2$ *if and only if it is eulerian.*

Proof. Suppose G is eulerian, and consider an Euler tour W of G. Let D be the orientation of G corresponding to the direction of W. If an edge $uv \in E_D$, let $\alpha(e) = 1$. Since W arrives and leaves each vertex equally many times, the function α is a nowhere zero 2-flow.

Conversely, let α be a nowhere zero 2-flow of an orientation D of G. Then necessarily the degrees of the vertices are even, and so G is eulerian. □

Example 6.9. For each 3-regular bipartite graph G, we have $f(G) \leq 3$. Indeed, let G be (X, Y)-biparte. By Corollary 3.1, a 3-regular graph has a perfect matching M. Orient the edges $e \in M$ from X to Y, and set $\alpha(e) = 2$. Orient the edges $e \notin M$ from Y to X, and set $\alpha(e) = 1$. Since each $x \in X$ has exactly one neighbour $y_1 \in Y$ such that $xy_1 \in M$, and two neighbours $y_2, y_3 \in Y$ such that $xy_2, xy_3 \notin M$, we have that $f(G) \leq 3$.

Theorem 6.13. *We have* $f(K_4) = 4$*, and if* $n > 4$*, then*

$$f(K_n) = \begin{cases} 2 & \text{if } n \text{ is odd }, \\ 3 & \text{if } n \text{ is even }. \end{cases}$$

Proof. Exercise. □

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