Lecture 1: Examples, connectedness, paths and cycles

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2011-10-22 lör

Outline

Topic

Literature, teaching format and examination

Literature Grimaldi, Ralph P.; *Discrete and combinatorial mathematics*, 5th edition, Pearson Addison Wesley, 2004. Chapters 11-13 Extra material handed out. (Not too demanding)

Teaching format: We have twenty combined lectures and tutorials. Weight will be put on problem solving.

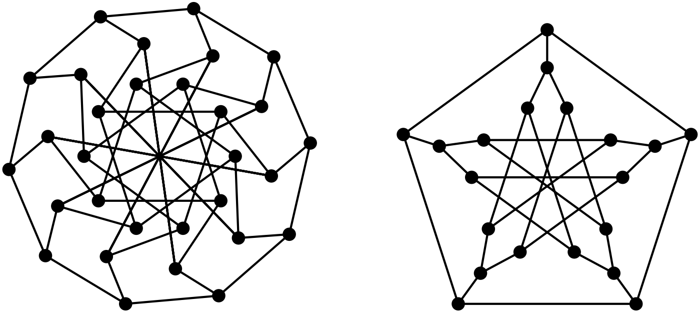
Examination We will have four "problem sessions", where you can present your own solutions to problems. Problem sheets will be handed out and posted at least a week in advance. Details about dates etc., will be posted on the course page later.
Participation in the problem sessions (at least 2/4) will give bonus at the written exam in December; the first exam-question will automatically be graded with full points.

Lesson and exercise planning

No	Section	Content	Proposed exercises
1	11.1	Definitions and examples, paths and cycles	2,4,6,7,8,10,13,14,15,16
2	11.2	Subgraphs and graph-homomorphisms	1,2,4,5,7,8,10,11,13
3	11.3	Degrees, parity and euler tours	1,2,3,5,8,11,13,16,26,27,30,32,33,37
4	11.4 (+)	Planar graphs, cycles and cuts	1,2,11,13,17,19,21,22,23,27,28
5	11.5	Hamiltonian graphs	1,2,4,10,12,18,20,21,25,26
6	11.6	Chromatic properties	1,2,4,7,8,9,14,16,17,18,19
7	11.6 (+)	More about colourings	S11 17,18,21,22, (+)
8	12.1	Acyclic graphs and trees	2,4,5,7,9,10,11,18,19,21,22,23,25
9	12.2	Rooted trees and search	1,2,5,7,9,10,12,14,15,16,19,20
10	12.3	Trees and sorting	1,3,4
11	12.4	Trees and codes	1,3,6,7
12	12.5	Decomposition into blocks	1,4,5,6,7,8,12,13 S12 5,6,7,8,11,14
13	13.1	Shortest path problems	1,2,4,5,6,7,9
14	13.2	Greedy algorithms and minimum spanning tree	1,2,4,5,6,7,9
15	13.3	Transport networks	1,3,6 (+)
16	13.3 (+)	Transport problems	
17	13.4	Matchings in bipartite graphs	1,2,4,5,7,13 S13 5,6,8,9
18	13.4 (+)	Duality principles in linear programming	
19	(+)	Random graphs and random graph statistics	
20		Tutorial (buffer)	

Topic

A simple symmetric relation (undirected graphs)

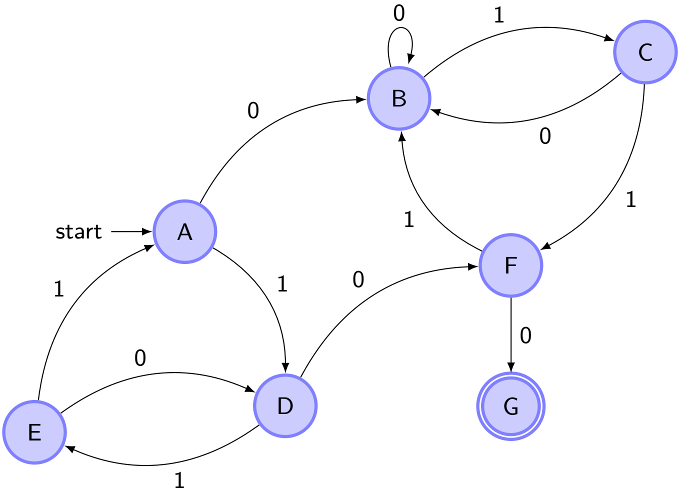


(a) A symmetric graph on 30 vertices (b) Another 3-regular graph

Figure: Undirected simple graphs: Main object

Automata and state transition graphs

Directed graphs, where arrows stand for a transition in time between states are also very common: Markov chains, automata, game theory, ...



Dependency graphs, factor graphs, etc.

Many application of graphs starts from graphs where nodes representing variables are connected if they are *dependent* in some way. Examples are Bayesian networks, graphical models, factor graphs, . . .

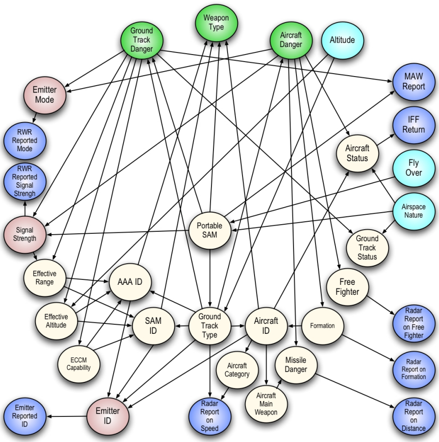
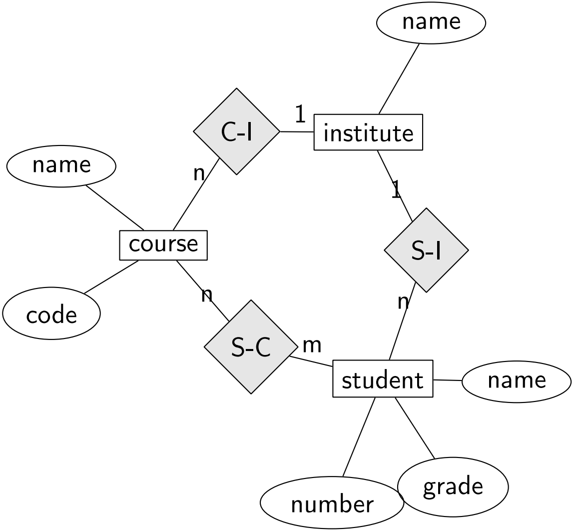


Figure: A Baeysian network

Relationship diagrams, data base models, . . .

In Computer Science the use of graphs has a long tradition. Here is an example of a entity–relation diagram used in database constructions.



Topic

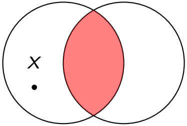
Naive set theory

Examples of sets are $\{1,2\}$, $\{1,1,2\} = \{1,2\}$, $\{3n \in \mathbb{Z} : n \text{ is prime }\}$, $\{\{1,3\},\{2\}\}$, etc.

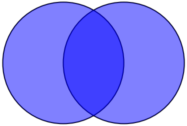
The statement that x is an *element* of A, is written $x \in A$.

Two sets A and B are equal when they have precisely the same elements, that is, $x \in A \iff x \in B$.

A set encode as a "geometric" object a unary relation on objects — or a proposition about outcomes. Logic is obtained by set operations.



$A \cap B$



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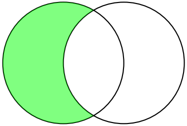


$A \setminus B$

Sets of sets and the power set 2^X

Given a set X, we obtain the *power set* 2^X consisting of all subsets of X.
The set $\binom{X}{k}$ is the set of all *k-subsets* of X.

If $X = \{1, 2, 3\}$ then

$$2^X = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{3,1\}, \{1,2,3\}\} .$$

and

$$\binom{X}{2} = \{\{1,2\}, \{2,3\}, \{3,1\}\} \quad \text{and} \quad \binom{X}{3} = \{\{1,2,3\}\}.$$

Note the cardinalities (=number of elements)

$$|2^X| = 2^{|X|}, \qquad \left|\binom{X}{k}\right| = \binom{|X|}{k}.$$

The Cartesian product of sets

Given two sets A and B, *the Cartesian product*, $A \times B$ is the set of pairs (two-tuples) (a, b) such that $a \in A$ and $b \in B$.
For example, given the set of playing card ranks and suits

$$R = \{A, K, Q, J, 10, 9, \ldots, 3, 2\}$$

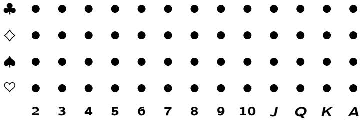
and

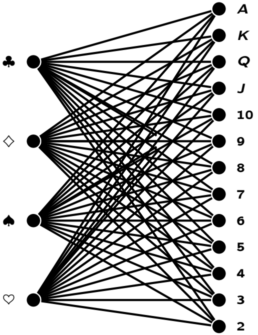
$$S = \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$$

we obtain the 52-element set

$$R \times S = \{(A, \spadesuit), (K, \spadesuit), \ldots, \\ (2, \spadesuit), (A, \heartsuit), \ldots, (3, \clubsuit), (2, \clubsuit)\}.$$

A poker-hand is thus a 5-*subset* of $R \times S$, i.e. an element in $\binom{R \times S}{5}$.





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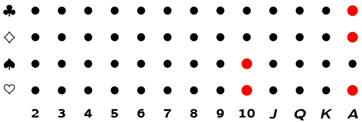
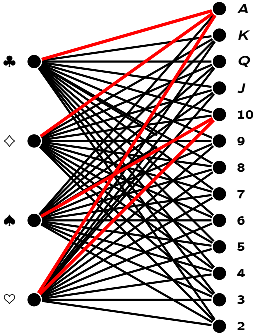


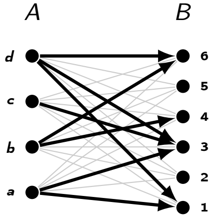


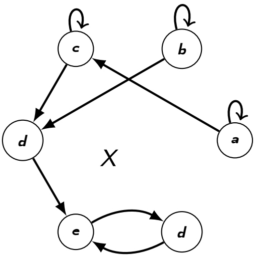
Full house

Relations: Subsets of a cartesian product

A (binary) *relation* R between elements of A and elements of B is (can be interpreted as) a *subset* $R \subset A \times B$, where aRb precisely if $(a, b) \in R$.

If $A = B = X$, we say a relation $Q \subset X \times X$ is a relation *on* X.





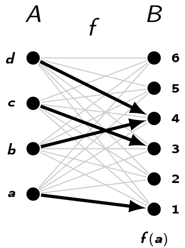
Functions

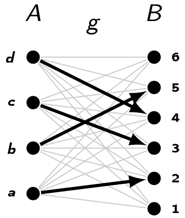
A relation $f \subset A \times B$ is a *function* from A to B if for every $a \in A$ there is exactly one element in B related to a. This element is written $f(a)$.

A function is *surjective* if every element in B is related to at least one element in A.

A function is *injective* if no element in B is related to more than one element in A.

A *bijective* function $f : A \to B$ is one which is both surjective and injective. In this case we have $|B| = |f(A)| = |A|$.





Symmetric, transitive and reflexive relations

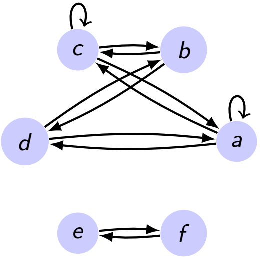
A relation Q on X is *symmetric* if, for all $x, y \in X$, xQy implies yQx.

It is *reflexive* if xQx for all $x \in X$.

A relation Q on X is *transitive* if, for all $x, y, z \in X$,

$$xQy \ \wedge yQz \implies xQz.$$

A symmetric, reflexive and transitive relation is an *equivalence relation*.



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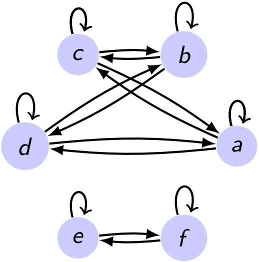

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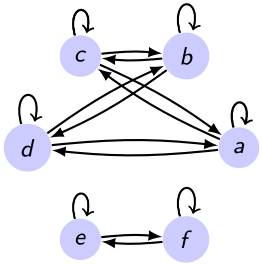

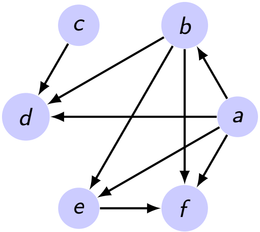

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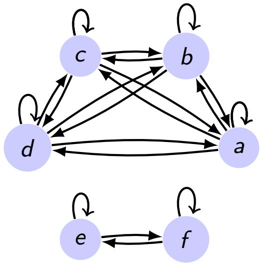
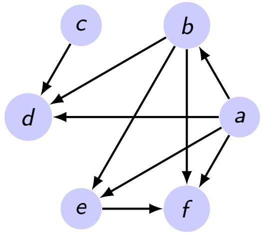




Topic

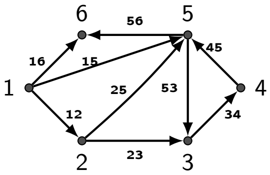
Definition of simple directed graphs

Def: Let V be a set and $E \subset V \times V$ a binary relation on V. A simple *digraph* G is the pair $G = (V, E)$, where V is called the set of *vertices* and E the set of directed *edges* or *arcs*.

We say $i \in V$ is *adjacent to* $j \in V$ if $ij = (i, j) \in E$ and adjacent *from* $k \in V$ if $(k, i) \in E$.

The edge $(i, j) \in E$ is an *out-edge* at i and an *in-edge* at j; j is an out-neighbour of i and i is an in-neighbour of j.

If E does not include edges of the form (i, i), the graph is *without loops*.



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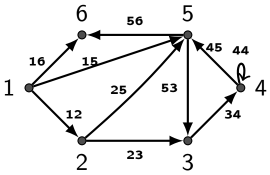


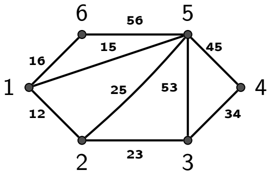
Undirected simple graph

Def: An *undirected simple graph* is a pair $G = (V, E)$, where direction $E \subset \binom{V}{2}$. We often write elements $\{i, j\}$ of E as ij — direction is disregarded.

An undirected graph corresponds to a *symmetric* adjacency relation; i adjacent to j and vice versa precisely if $\{i, j\} \in E$.

This correspond to an underlying symmetric reflexive or anti-reflexive relation.

We say that a vertex v is *incident* with an edge $e \in E$ if $v \in e$.



General *multigraphs*; head and tail functions; incidence relation

Let V and E be two finite sets. A general (multi-) digraph $G = (V, E, \text{head} \times \text{tail})$ is given by two functions

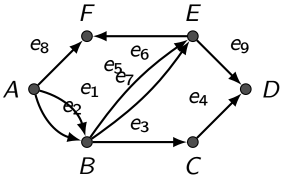
$$\text{head} : E \to V \text{ and } \text{tail} : E \to V.$$

The multiplicity of an edge $(u, v) \in V \times V$ is the number edges e such that $\text{head}\, e = u$ and $\text{tail}\, e = v$.

The *incidence relation*, Inc, from E to V, is defined as follows: e is incident with v if $v = \text{head}\, e$ *or* $v = \text{tail}\, e$.

An *undirected* multi-graph $G = (V, E)$ is an *equivalence class* with a fixed incidence relation. Multiplicity of edge $\{u, v\}$ is the number of edges such that $\text{Inc}(\{e\}) = \{u, v\}$. Every digraph $\vec{G}$ with the same incidence relation is called an *orientation* of G.

E	head	tail
e_1	A	B
e_2	A	B
e_3	B	C
e_4	C	D
e_5	B	E
e_6	E	F
e_7	B	E
e_8	A	F
e_9	E	D



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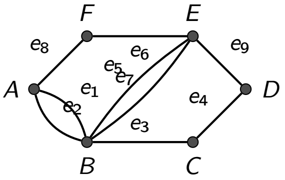
e_9	E	D



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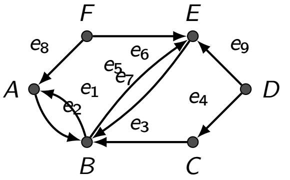


The Cashier/Guard problem

Consider the following (simple, undirected) graph. (The normal case!)

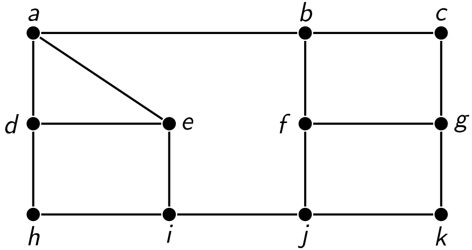


Figure: A graph of cashiers connected by aisles

Place a minimum set of guards so that each cashier has at least one guard in some neighbouring aisle. Find a *minimum* set S such that every vertex is either in S or adjacent to some vertex in S.

What is the minimum size of a set $S \subset V$, such that every edge is incident with at least one vertex in S?

Topic

Walks and reachability

The problem of Reachability

Can we reach vertex v from vertex u by "walking around" in the graph? If so, in how many ways can we do this?

Def: An uv-walk W of *length* n in the (multi-di-) graph $G = (V, E)$ is an alternating sequence of vertices $x_i \in V$, $i = 0, 1, \ldots, n$, and edges $e_j \in E$, $j = 1, \ldots, n$,

$$W : \quad u = x_0, e_1, x_1, e_2, x_2, \ldots, e_{n-1}, x_{n-1}, e_n, x_n = v,$$

such that, for all $i = 1, \ldots, n$, both x_{i-1} and x_i are incident with e_i.

If for all i, $x_{i-1} = \operatorname{head} e_i$ and $x_i = \operatorname{tail} e_i$ it is a *directed* walk otherwise it is an *oriented* walk.

If G is simple then a walk W is specified by the sequence

$$W := (x_0, x_1, \ldots, x_n) \in V^* := V \cup V^2 \cup \cdots \cup V^n \cup \ldots .$$

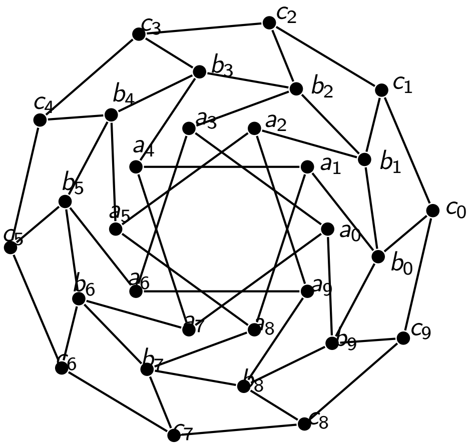
Closed walks, trails, circuits, paths and cycles

An uv-walk

$$u = x_0, x_1, \ldots, x_n = v,$$

is

- closed if $a = b$
- a *trail* if all edges are distinct
- a *circuit* if it is a closed trail
- a *path* if all vertices are distinct
- a *cycle* if it is a closed path



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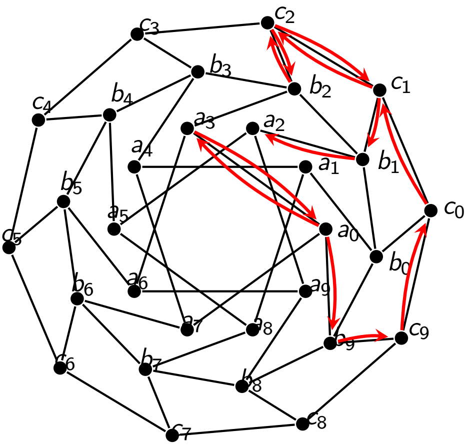

A a_0a_2-walk

Closed walks, trails, circuits, paths and cycles

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$$u = x_0, x_1, \ldots, x_n = v,$$

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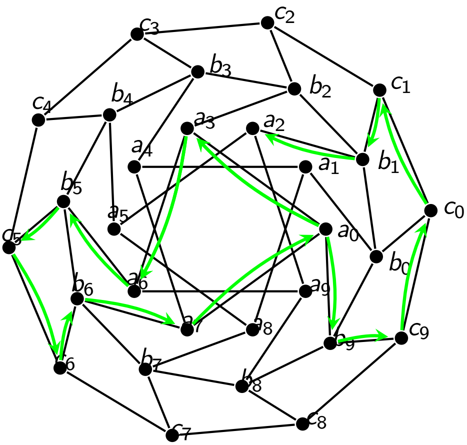


A a_0a_2-trail

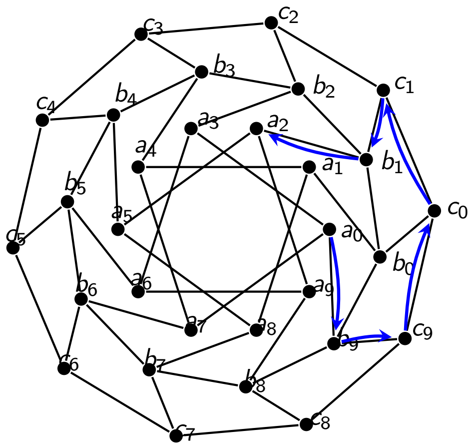
Closed walks, trails, circuits, paths and cycles

An uv-walk

$$u = x_0, x_1, \ldots, x_n = v,$$

is

closed	if $a = b$
a *trail*	if all edges are distinct
a *circuit*	if it is a closed trail
a *path*	if all vertices are distinct
a *cycle*	if it is a closed path



A a_0a_2-path

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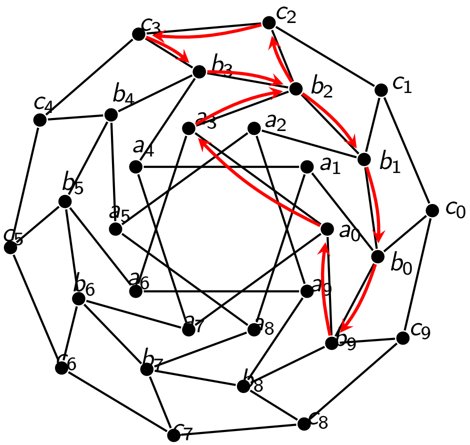

A $a_0 a_0$-circuit

Closed walks, trails, circuits, paths and cycles

An uv-walk

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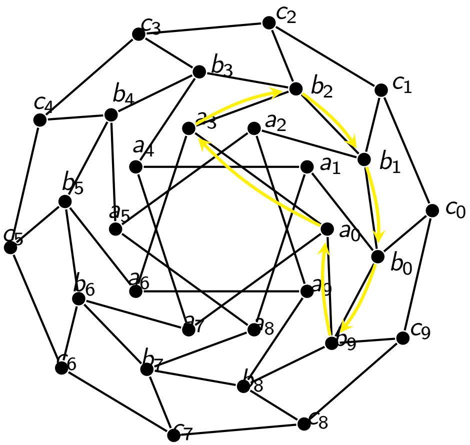


A $a_0 a_0$-cycle

Compute the number of closed walks in a lollipop graph

Consider the following graph with a loop.

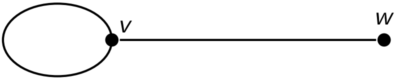


Figure: The lollipop graph

Determine a *recursion formula* for a_n — the number of closed vv-walks in this graph.

Paths determine the connectivity structure

We can describe a path/cycle in any type of graph as a subset of edges and (possibly) the starting point.

Paths and cycles are also *subgraphs*. (More next time)

Theorem (Paths are enough)

There exists an (directed/oriented) uv-walk if and only if there exists a (directed/oriented) uv-path.

Task: Prove this!

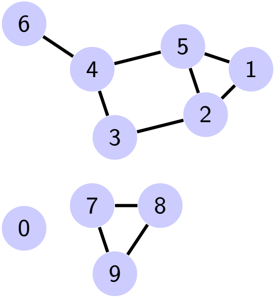
Components, connectedness

A vertex $u \in V$ is *reachable* from $v \in V$ if there exists an oriented/directed uv-walk — or a path.

For oriented walks/paths — for undirected graphs — this is an equivalence relation on V.

The equivalence classes are called *connected components*. The number of components of a graph G is denoted by $c(G)$. ($\kappa(G)$ in the book) A graph is *connected* if $c(G) = 1$.

For digraphs we can say that two vertices u an v are *strongly connected* if there are both a directed uv-walk and a directed vu-walk. This is again an equivalence relation; partitioning V into *strong components*.



$c(G) = 3$.

Components, connectedness

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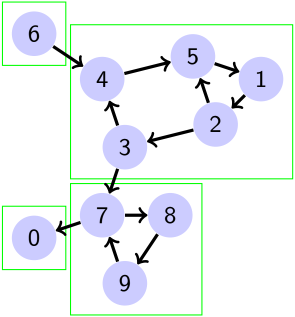

Strong components

Distance, diameter and girth

Given a graph $G = (V, E)$ the length (or $+\infty$) of a *shortest path* between u and v, $u, v \in V$, is called the *distance* between u and v. It is denoted $\mathrm{dist}_G(u, v)$.

The *diameter*, $\mathrm{diam}(G)$, of a graph is the maximum distance between two vertices.

The *girth*, $g(G)$, of a graph is the length of the smallest cycle. (If the graph contains no cycles it is called *acyclic*.)

1. What is the diameter and girth of the Petersen graph?
2. Prove that for every non-acyclic graph $g(G) \leq 2\,\mathrm{diam}(G) + 1$.

Figure: The Petersen graph