Lecture 1: Examples, connectedness, paths and cycles

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2011-10-22 lör
Outline

The course plan

Examples and applications of graphs

Relations

The definition of graphs as relations

Connectedness, paths and cycles
The course plan

Examples and applications of graphs

Relations

The definition of graphs as relations

Connectedness, paths and cycles
Literature, teaching format and examination


**Teaching format:** We have twenty combined lectures and tutorials. Weight will be put on problem solving.

**Examination**  We will have four “problem sessions”, where you can present your own solutions to problems. Problem sheets will be handed out and posted at least a week in advance. Details about dates etc., will be posted on the course page later. Participation in the problem sessions (at least 2/4) will give bonus at the written exam in December; the first exam-question will automatically be graded with full points.
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Topic

The course plan

Examples and applications of graphs

Relations

The definition of graphs as relations

Connectedness, paths and cycles
A simple symmetric relation (undirected graphs)

(a) A symmetric graph on 30 vertices
(b) Another 3-regular graph

Figure: Undirected simple graphs: Main object
Automata and state transition graphs

Directed graphs, where arrows stand for a transition in time between states are also very common: Markov chains, automata, game theory, ...
Dependency graphs, factor graphs, etc.

Many application of graphs starts from graphs where nodes representing variables are connected if they are dependent in some way. Examples are Bayesian networks, graphical models, factor graphs, . . .

Figure: A Bayesian network
In Computer Science the use of graphs has a long tradition. Here is an example of a entity–relation diagram used in database constructions.
Topic

The course plan

Examples and applications of graphs

Relations

The definition of graphs as relations

Connectedness, paths and cycles
Naive set theory

Examples of sets are \{1, 2\}, \{1, 1, 2\} = \{1, 2\}, \{3n \in \mathbb{Z} : n \text{ is prime}\}, \{\{1, 3\}, \{2\}\}, etc.

The statement that \(x\) is an element of \(A\), is written \(x \in A\).

Two sets \(A\) and \(B\) are equal when they have precisely the same elements, that is, \(x \in A \iff x \in B\).

A set encode as a “geometric” object a unary relation on objects — or a proposition about outcomes. Logic is obtained by set operations.
Sets of sets and the power set $2^X$

Given a set $X$, we obtain the power set $2^X$ consisting of all subsets of $X$. The set $\binom{X}{k}$ is the set of all $k$-subsets of $X$.

If $X = \{1, 2, 3\}$ then

$$2^X = \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}.$$ 

and

$$\binom{X}{2} = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\} \quad \text{and} \quad \binom{X}{3} = \{\{1, 2, 3\}\}.$$ 

Note the cardinalities (number of elements)

$$|2^X| = 2^{|X|}, \quad \left| \binom{X}{k} \right| = \binom{|X|}{k}.$$
The Cartesian product of sets

Given two sets \(A\) and \(B\), the Cartesian product, \(A \times B\) is the set of pairs (two-tuples) \((a, b)\) such that \(a \in A\) and \(b \in B\).

For example, given the set of playing card ranks and suits

\[
R = \{A, K, Q, J, 10, 9, \ldots, 3, 2\}
\]

and

\[
S = \{\text{♣, ♥, ♦, ♣}\}
\]

we obtain the 52-element set

\[
R \times S = \{(A, \text{♣}), (K, \text{♣}), \ldots, (2, \text{♣}), (A, \text{♥}), \ldots, (3, \text{♠}), (2, \text{♠})\}.
\]

A poker-hand is thus a 5-subset of \(R \times S\), i.e. an element in \(\binom{R \times S}{5}\).
The Cartesian product of sets

Given two sets \( A \) and \( B \), the Cartesian product, \( A \times B \) is the set of pairs (two-tuples) \((a, b)\) such that \( a \in A \) and \( b \in B \).

For example, given the set of playing card ranks and suits

\[
R = \{A, K, Q, J, 10, 9, \ldots, 3, 2\}
\]

and

\[
S = \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}
\]

we obtain the 52-element set

\[
R \times S = \{(A, \spadesuit), (K, \spadesuit), \ldots, (2, \spadesuit), (A, \heartsuit), \ldots, (3, \clubsuit), (2, \clubsuit)\}.
\]

A poker-hand is thus a 5-subset of \( R \times S \), i.e. an element in \( \binom{R \times S}{5} \).
A (binary) relation $R$ between elements of $A$ and elements of $B$ is (can be interpreted as) a subset $R \subseteq A \times B$, where $aRb$ precisely if $(a, b) \in R$.

If $A = B = X$, we say a relation $Q \subseteq X \times X$ is a relation on $X$. 
A relation \( f \subset A \times B \) is a function from \( A \) to \( B \) if for every \( a \in A \) there is exactly one element in \( B \) related to \( a \). This element is written \( f(a) \).

A function is surjective if every element in \( B \) is related to at least one element in \( A \).

A function is injective if no element in \( B \) is related to more than one element in \( A \).

A bijective function \( f : A \rightarrow B \) is one which is both surjective and injective. In this case we have \( |B| = |f(A)| = |A| \).
A relation \( Q \) on \( X \) is symmetric if, for all \( x, y \in X \), \( xQy \) implies \( yQx \).

It is reflexive if \( xQx \) for all \( x \in X \).

A relation \( Q \) on \( X \) is transitive if, for all \( x, y, z \in X \),

\[
xQy \land yQz \implies xQz.\]

A symmetric, reflexive and transitive relation is an equivalence relation.
Symmetric, transitive and reflexive relations

A relation $Q$ on $X$ is *symmetric* if, for all $x, y \in X$, $xQy$ implies $yQx$.

It is *reflexive* if $xQx$ for all $x \in X$.

A relation $Q$ on $X$ is *transitive* if, for all $x, y, z \in X$,

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\begin{array}{cccccc}
\text{a} & \text{b} & \text{c} & \text{d} & \text{e} & \text{f} \\
\text{c} & \text{b} & \text{d} & \text{e} & \text{f} & \\
\text{d} & \text{e} & \text{f} & & & \\
\text{e} & \text{f} & & & & \\
\text{f} & & & & & \\
\end{array}
\]
Symmetric, transitive and reflexive relations

A relation $Q$ on $X$ is symmetric if, for all $x, y \in X$, $xQy$ implies $yQx$.

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A symmetric, reflexive and transitive relation is an equivalence relation.
Topic

The course plan

Examples and applications of graphs

Relations

The definition of graphs as relations

Connectedness, paths and cycles
Definition of simple directed graphs

**Def:** Let $V$ be a set and $E \subset V \times V$ a binary relation on $V$. A simple *digraph* $G$ is the pair $G = (V, E)$, where $V$ is called the set of *vertices* and $E$ the set of directed *edges* or *arcs*.

We say $i \in V$ is *adjacent to* $j \in V$ if $ij = (i, j) \in E$ and adjacent *from* $k \in V$ if $(k, i) \in E$.

The edge $(i, j) \in E$ is an *out-edge* at $i$ and an *in-edge* at $j$; $j$ is an out-neighbor of $i$ and $i$ is an in-neighbor of $j$.

If $E$ does not include edges of the form $(i, i)$, the graph is *without loops*. 
Definition of simple directed graphs

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If $E$ does not include edges of the form $(i, i)$, the graph is *without loops*.
**Undirected simple graph**

**Def:** An *undirected simple graph* is a pair $G = (V, E)$, where direction $E \subset (V^2)$. We often write elements $\{i, j\}$ of $E$ as $ij$ — direction is disregarded.

An undirected graph corresponds to a *symmetric* adjacency relation; $i$ adjacent to $j$ and vice versa precisely if $\{i, j\} \in E$.

This correspond to an underlying symmetric reflexive or anti-reflexive relation.

We say that a vertex $v$ is *incident* with an edge $e \in E$ if $v \in e$. 
General *multigraphs*; head and tail functions; incidence relation

Let $V$ and $E$ be two finite sets. A general (multi-) digraph $G = (V, E, \text{head} \times \text{tail})$ is given by two functions

$$\text{head} : E \to V \quad \text{and} \quad \text{tail} : E \to V.$$ 

The multiplicity of an edge $(u, v) \in V \times V$ is the number edges $e$ such that $\text{head} e = u$ and $\text{tail} e = v$.

The *incidence relation*, Inc, from $E$ to $V$, is defined as follows: $e$ is incident with $v$ if $v = \text{head} e$ or $v = \text{tail} e$.

An *undirected* multi-graph $G = (V, E)$ is an *equivalence class* with a fixed incidence relation. Multiplicity of edge $\{u, v\}$ is the number of edges such that $\text{Inc}(\{e\}) = \{u, v\}$. Every digraph $\vec{G}$ with the same incidence relation is called an *orientation* of $G$.

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General multigraphs; head and tail functions; incidence relation

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An undirected multi-graph $G = (V, E)$ is an equivalence class with a fixed incidence relation. Multiplicity of edge $\{u, v\}$ is the number of edges such that $\text{Inc}(\{e\}) = \{u, v\}$. Every digraph $\vec{G}$ with the same incidence relation is called an orientation of $G$. 

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An undirected multi-graph $G = (V, E)$ is an equivalence class with a fixed incidence relation. Multiplicity of edge $\{u, v\}$ is the number of edges such that $\text{Inc}(\{e\}) = \{u, v\}$. Every digraph $\vec{G}$ with the same incidence relation is called an orientation of $G$. 

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The Cashier/Guard problem

Consider the following (simple, undirected) graph. (The normal case!)

Place a minimum set of guards so that each cashier has at least one guard in some neighbouring aisle. Find a minimum set $S$ such that every vertex is either in $S$ or adjacent to some vertex in $S$.

What is the minimum size of a set $S \subset V$, such that every edge is incident with at least one vertex in $S$?
Topic

The course plan

Examples and applications of graphs

Relations

The definition of graphs as relations

Connectedness, paths and cycles
Walks and reachability

The problem of Reachability

Can we reach vertex $v$ from vertex $u$ by “walking around” in the graph? If so, in how many ways can we do this?

**Def:** An $uv$-walk $W$ of length $n$ in the (multi-di-) graph $G = (V, E)$ is an alternating sequence of vertices $x_i \in V$, $i = 0, 1, \ldots, n$, and edges $e_j \in E$, $j = 1, \ldots, n$,

$$W : \quad u = x_0, e_1, x_1, e_2, x_2, \ldots, e_{n-1}, x_{n-1}, e_n, x_n = v,$$

such that, for all $i = 1, \ldots, n$, both $x_{i-1}$ and $x_i$ are incident with $e_i$.

If for all $i$, $x_{i-1} = \text{head } e_i$ and $x_i = \text{tail } e_i$ it is a *directed* walk otherwise it is an *oriented* walk.

If $G$ is simple then a walk $W$ is specified by the sequence

$$W := (x_0, x_1, \ldots, x_n) \in V^* := V \cup V^2 \cup \cdots \cup V^n \cup \ldots.$$
Closed walks, trails, circuits, paths and cycles

An $uv$-walk

$$u = x_0, x_1, \ldots, x_n = v,$$

is

- **closed** if $a = b$
- a **trail** if all edges are distinct
- a **circuit** if it is a closed trail
- a **path** if all vertices are distinct
- a **cycle** if it is a closed path
Closed walks, trails, circuits, paths and cycles

An \( uv \)-walk

\[ u = x_0, x_1, \ldots, x_n = v, \]

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A \( a_0a_2 \)-walk
Closed walks, trails, circuits, paths and cycles

An \textit{uv}-walk

\[ u = x_0, x_1, \ldots, x_n = v, \]

is

- \textbf{closed} if \( a = b \)
- \textbf{a trail} if all edges are distinct
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\[ A \text{ } a_0a_2 \text{-trail} \]
Closed walks, trails, circuits, paths and cycles

An *uv*-walk

\[ u = x_0, x_1, \ldots, x_n = v, \]

is

- closed if \( a = b \)
- a *trail* if all edges are distinct
- a *circuit* if it is a closed trail
- a *path* if all vertices are distinct
- a *cycle* if it is a closed path

A *\( a_0a_2 \)-path
An \( uv \)-walk

\[ u = x_0, x_1, \ldots, x_n = v, \]

is

- **closed** if \( a = b \)
- a **trail** if all edges are distinct
- a **circuit** if it is a closed trail
- a **path** if all vertices are distinct
- a **cycle** if it is a closed path

A \( a_0a_0 \)-circuit
Closed walks, trails, circuits, paths and cycles

An uv-walk

\[ u = x_0, x_1, \ldots, x_n = v, \]

is

- closed if \( a = b \)
- a trail if all edges are distinct
- a circuit if it is a closed trail
- a path if all vertices are distinct
- a cycle if it is a closed path

A \( a_0a_0 \)-cycle
Compute the number of closed walks in a lollipop graph

Consider the following graph with a loop.

![Diagram of a lollipop graph with a loop](image)

**Figure:** The lollipop graph

Determine a *recursion formula* for \( a_n \) — the number of closed vv-walks in this graph.
Paths determine the connectivity structure

We can describe a path/cycle in any type of graph as a subset of edges and (possibly) the starting point.

Paths and cycles are also subgraphs. (More next time)

**Theorem (Paths are enough)**

*There exists an (directed/oriented) uv-walk if and only if there exists a (directed/oriented) uv-path.*

**Task:** Prove this!
A vertex $u \in V$ is *reachable* from $v \in V$ if there exists an oriented/directed $uv$-walk — or a path.

For oriented walks/paths — for undirected graphs — this is an equivalence relation on $V$.

The equivalence classes are called *connected components*. The number of components of a graph $G$ is denoted by $c(G)$. ($\kappa(G)$ in the book) A graph is *connected* if $c(G) = 1$.

For digraphs we can say that two vertices $u$ an $v$ are *strongly connected* if there are both a directed $uv$-walk and a directed $vu$-walk. This is again an equivalence relation; partitioning $V$ into *strong components*.

$c(G) = 3$. 

\[ 0 \quad 7 \quad 8 \quad 9 \]

\[ 6 \quad 4 \quad 5 \quad 1 \]

\[ 3 \quad 2 \]
Components, connectedness

A vertex $u \in V$ is reachable from $v \in V$ if there exists an oriented/directed $uv$-walk — or a path.

For oriented walks/paths — for undirected graphs — this is an equivalence relation on $V$.

The equivalence classes are called connected components. The number of components of a graph $G$ is denoted by $c(G)$. ($\kappa(G)$ in the book) A graph is connected if $c(G) = 1$.

For digraphs we can say that two vertices $u$ and $v$ are strongly connected if there are both a directed $uv$-walk and a directed $vu$-walk. This is again an equivalence relation; partitioning $V$ into strong components.
Distance, diameter and girth

Given a graph $G = (V, E)$ the length (or $+\infty$) of a shortest path between $u$ and $v$, $u, v \in V$, is called the distance between $u$ and $v$. It is denoted $\text{dist}_G(u, v)$.

The diameter, $\text{diam}(G)$, of a graph is the maximum distance between two vertices.

The girth, $g(G)$, of a graph is the length of the smallest cycle. (If the graph contains no cycles it is called acyclic.)

1. What is the diameter and girth of the Petersen graph?
2. Prove that for every non-acyclic graph $g(G) \leq 2 \text{diam}(G) + 1$.

Figure: The Petersen graph