Lecture 2: Isomorphism and subgraphs

Anders Johansson

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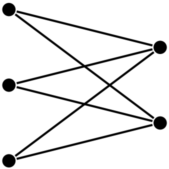
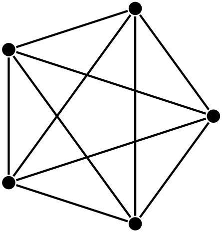
Outline

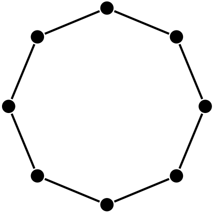
Important classes of graphs



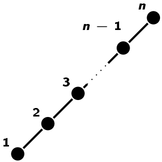
(a) A complete bipartite $K_{3,2}$



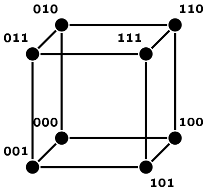
(b) A complete graph K_5



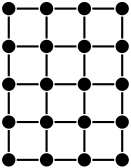
(c) A cycle C_8



(d) A path P_n



(e) A hypercube Q_3



(f) A 4×5 grid

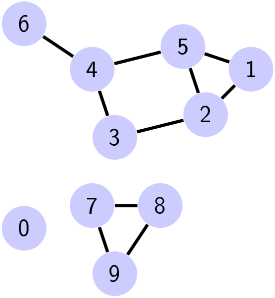
Components, connectedness

A vertex $u \in V$ is *reachable* from $v \in V$ if there exists an oriented/directed uv-path.

For oriented paths — or for undirected graphs — this is an equivalence relation on V.

The equivalence classes are called *connected components*. The number of components of a graph G is denoted by $c(G)$. ($\kappa(G)$ in the book) A graph is *connected* if $c(G) = 1$.

For digraphs we can say that two vertices u an v are *strongly connected* if there are both a directed uv-walk and a directed vu-walk. This is again an equivalence relation; partitioning V into *strong components*.



$c(G) = 3.$

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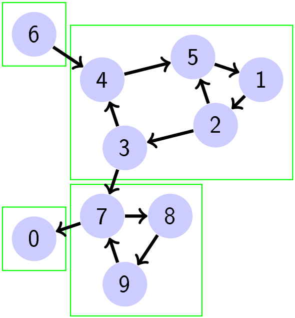


Strong components

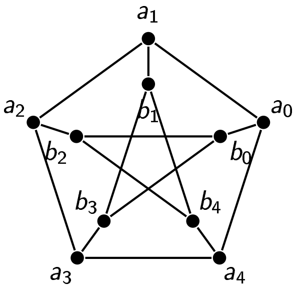
Distance, diameter and girth

Given a graph $G = (V, E)$ the length (or $+\infty$) of a *shortest path* between u and v, $u, v \in V$, is called the *distance* between u and v. It is denoted $\mathrm{dist}_G(u, v)$.

The *diameter*, $\mathrm{diam}(G)$, of a graph is the maximum distance between two vertices.

The *girth*, $g(G)$, of a graph is the length of the smallest cycle. (If the graph contains no cycles it is called *acyclic*.)

1. What is the diameter and girth of the Petersen graph?



More questions

1. Prove that dist_G is a *metric* on $V(G)$: Prove the triangle inequality

$$\text{dist}_G(u, v) \leq \text{dist}_G(u, w) + \text{dist}_G(w, v).$$

2. Let G be a connected loop-free undirected graph and let e be an edge. Prove that that the graph

$$G - e := (V, E \setminus \{e\})$$

is connected if and only if e is a part of a cycle in G.

3. Show that if a graph $G = (V, E)$ is not connected then its *complement*

$$\overline{G} := (V, \binom{V}{2} \setminus E)$$

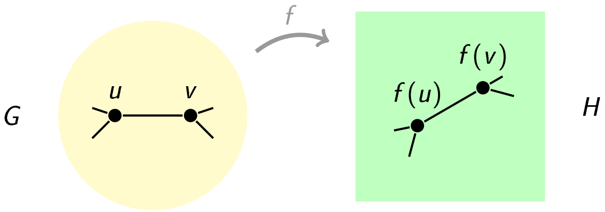
is connected.

4. Prove that for every non-acyclic graph $g(G) \leq 2\,\text{diam}(G) + 1$.

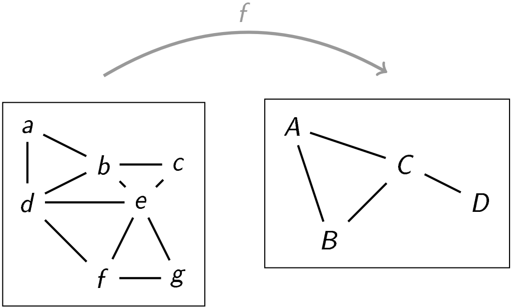
Graph homomorphism

For graphs G and H (possibly with loops), a *graph homomorphism* is a map $f : V(G) \to V(H)$ which take edges to edges, i.e.

$$(u, v) \in E(G) \implies (f(u), f(v)) \in E(H).$$



Example and questions about homomorphisms



x	f(x)
a	A
d	B
b	C
e	A
c	B
f	C
g	B

1. Could we have D in the image $f(V(G))$ of f above?
2. Describe the set $\mathrm{Hom}(P_n, G)$ of homomorphisms between the n-path P_n and G. $\mathrm{Hom}(C_n, G)$? Injective?
3. How should we define homomorphisms for general multigraphs?

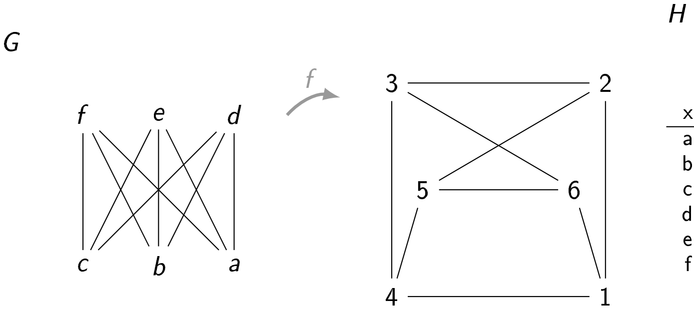
Graph isomorphism

A graph *isomorphism* between graphs G and H is a *bijective* map $f : V(G) \to V(H)$ such that

$$\{u, v\} \in E(G) \iff \{f(u), f(v)\} \in E(H).$$

We write $G \cong H$ and consider in many circumstances two such graphs as the same.

If $G = H$, we talk of an *automorphism*. Makes up a permutation *group*.

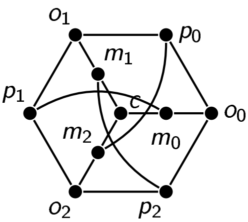


x	f(x)
a	1
b	3
c	5
d	2
e	4
f	6

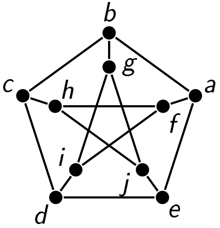
Figure: A graph isomorphism

Questions about isomorphisms

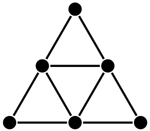
1. Show that (c) and (d) are not isomorphic and show that (a) and (b) are.
2. Show that the number of automorphisms in the Petersen graph (b) is at least 30.



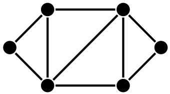
(a)



(b)



(c)



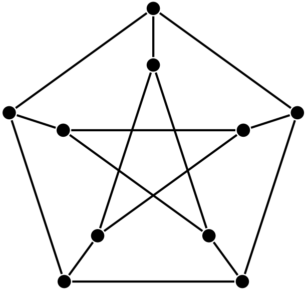
(d)

Subgraphs and spanning subgraphs

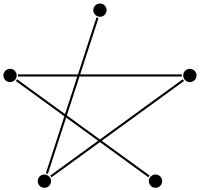
Let G and H be two graphs. We say that H is a *subgraph* of G, write it $H \subset G$, if

$$V(H) \subset V(G) \text{ and } E(H) \subset E(G).$$

If $V(H) = V(G)$ then H is a *spanning subgraph*, and H is just a subset of edges. We can write $H \subset_{sp} G$ and obtain H from G by *deleting* edges.



(a) The Petersen graph G



(b) The subgraph $H \subset G$

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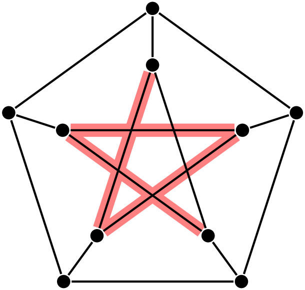

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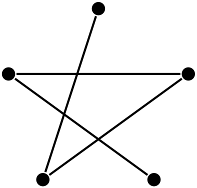

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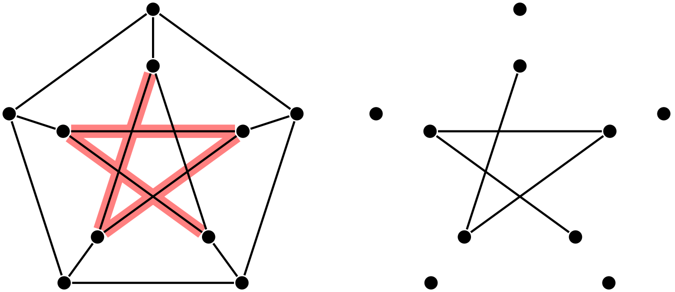
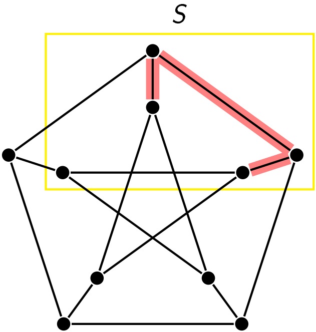


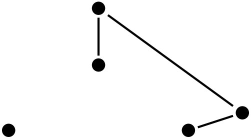
(a) The Petersen graph G (b) The spanning subgraph $H \subset_{sp} G$

Subgraphs induced by a set of vertices

We say that H is an *induced subgraph* of G if $V(H) = S \subset V(G)$ and $E(H)$ consists of all edges with both endpoints in $V(H)$. We write $H = G[S]$ and $H \lhd G$. It corresponds to deletion or addition of vertices.



(c) G and S



(d) The induced subgraph H

Monotone properties

Each notion of subgraphs, subgraphs, spanning subgraphs and induced subraphs, give rise to a *partial order* ($\prec$) on the set $\mathcal{G}$ of graphs where $\prec$ can be $\subset, \subset_{sp}$ or $\lhd$.

We say that graph parameter $f : \mathcal{G} \to \mathbb{R}$ is *increasing* (decreasing) if $G \prec H$ implies $f(G) \leq f(H)$ ($f(G) \geq f(H)$). For instance, the number of components, $c(G)$, is decreasing under the spanning subgraph partial order.

Usually: a property is *monotone increasing* if the property is not destroyed under addition of edges. This means that it is increasing visavi the spanning subgraph property.

A property is *hereditary* if it holds under deletion of vertices. It is thus monotone decreasing under the *induced subgraph* relation $\lhd$.

Questions about subgraphs and induced subgraphs

1. How many spanning subgraphs does a graph $G = (V, E)$ have? How many induced subgraphs? The number of subgraphs is harder to determine . . .
2. If every induced subgraph of a graph is connected. What is the graph?
3. Show that the shortest cycle in any graph is an induced cycle, if it exists.
4. Fill in the diagram

Property/parameter	$\subset$	$\subset_{sp}$	$\triangleleft$
Chromatic number $\chi(G)$	$\nearrow$	$\nearrow$	$\nearrow$
Diameter diam(G)	-	$\searrow$	-
Number of components $c(G)$	?	$\searrow$	?
Size of automorphism group	?	?	?
Girth $g(G)$	?	?	?
Size of max K_n-subgraph	?	?	?
Longest cycle	?	?	?