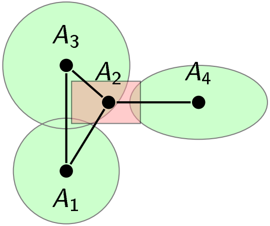
Lecture 3: Degrees and parity

Anders Johansson

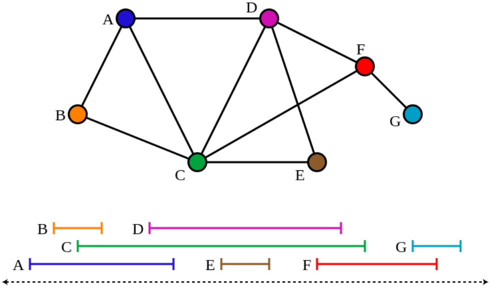
2011-10-22 lör

Outline

Intersection graphs



(a) An intersection graph

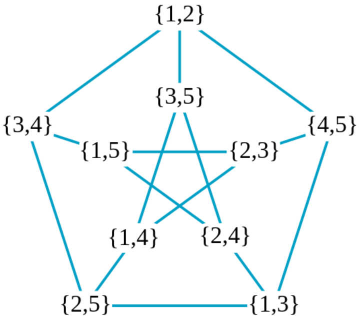


(b) An interval graph

Def: Let $V = \{A_1, A_2, \ldots, A_n\}$ be a set system — a finite set of subsets $A_i \subset X$ of a set X. We say that $G = (V, E)$ is the *intersection graph* of V, if $\{A_i, A_j\} \in E$ precisely if $A_i \cap A_j \neq \emptyset$.

There are many specialisations depending on which type of sets one considers. In particular, *interval graphs*.

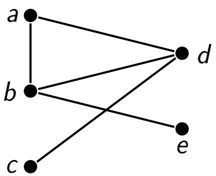
The Petersen graph as a complement to the intersection graphs



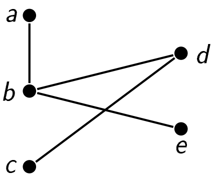
1. Prove that every graph is an intersection graph for some set system.

Edge- and vertex deletion

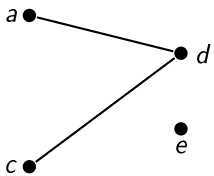
Given a graph $G = (V, E)$, we define the graph obtained by **deletion of edge** $e \in E$ as the graph $G - e := (V, E \setminus \{e\})$. The graph obtained by **deletion of vertex** $v \in V$ gives the graph $G - v := (V \setminus \{v\}, E \setminus E(v, G))$.



(c) The graph G



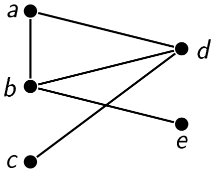
(d) $G - ad$



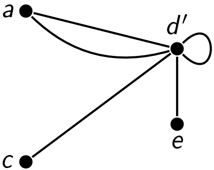
(e) $G - b$

Edge contractions

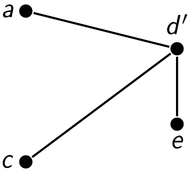
Given a graph $G = (V, E)$, we define the graph obtained by **contraction of edge** $e = uv \in E$ as a graph $G/e := (V \setminus \{u, v\} \cup \{v'\}, E)$, where $e \in E$ now is incident with the "new" vertex v' if and only if e is incident with u *or* v. Sometimes one remove multiple edges and loops to keep the graphs simple.



(a) The graph G



(b) The graph G/bd



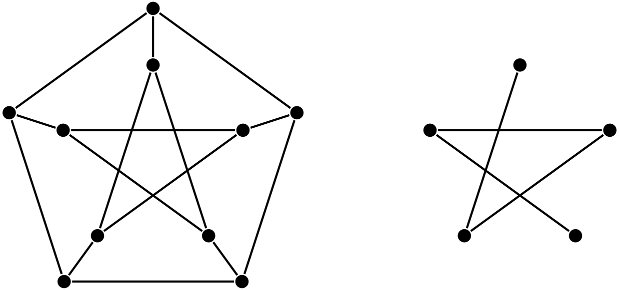
(c) The simplified graph G/bd

Subgraphs and spanning subgraphs

Let G and H be two graphs. We say that H is a *subgraph* of G, write it $H \subset G$, if

$$V(H) \subset V(G) \text{ and } E(H) \subset E(G).$$

If $V(H) = V(G)$ then H is a *spanning subgraph*, and H is just a subset of edges. We can write $H \subset_{sp} G$ and obtain H from G by *deleting* edges.



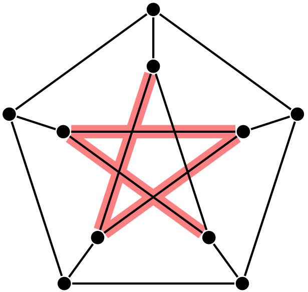
(a) The Petersen graph G (b) The subgraph $H \subset G$

Subgraphs and spanning subgraphs

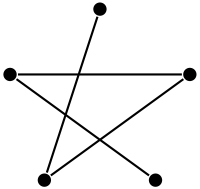
Let G and H be two graphs. We say that H is a *subgraph* of G, write it $H \subset G$, if

$$V(H) \subset V(G) \text{ and } E(H) \subset E(G).$$

If $V(H) = V(G)$ then H is a *spanning subgraph*, and H is just a subset of edges. We can write $H \subset_{sp} G$ and obtain H from G by *deleting* edges.



(a) The Petersen graph G



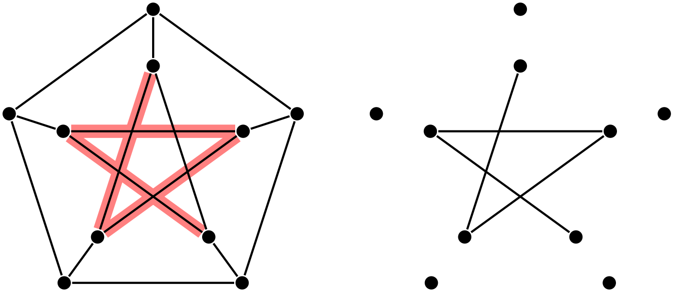
(b) The subgraph $H \subset G$

Subgraphs and spanning subgraphs

Let G and H be two graphs. We say that H is a *subgraph* of G, write it $H \subset G$, if

$$V(H) \subset V(G) \text{ and } E(H) \subset E(G).$$

If $V(H) = V(G)$ then H is a *spanning subgraph*, and H is just a subset of edges. We can write $H \subset_{sp} G$ and obtain H from G by *deleting* edges.

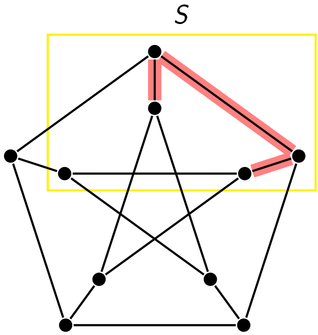


(a) The Petersen graph G

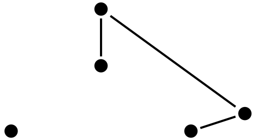
(b) The spanning subgraph $H \subset_{sp} G$

Subgraphs induced by a set of vertices

We say that H is an *induced subgraph* of G if $V(H) = S \subset V(G)$ and $E(H)$ consists of all edges with both endpoints in $V(H)$. We write $H = G[S]$ and $H \triangleleft G$ and it means that H can be obtained from G by *deleting vertices*.



(c) G and S



(d) The induced subgraph H

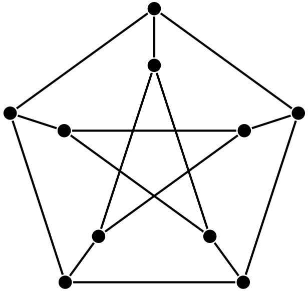
Questions about subgraphs and induced subgraphs

1. How many spanning subgraphs does a graph $G = (V, E)$ have? How many induced subgraphs? The number of subgraphs is harder to determine . . .
2. If every induced subgraph of a graph is connected. What is the graph?
3. Show that the shortest cycle in any graph is an induced cycle, if it exists.

Minors

A final "subgraph" relation is the minor relation. A graph H is *minor* in G if we can obtain H by deleting edges and vertices and *contracting* edges.

Figure: The Petersen graph has a K_5-minor

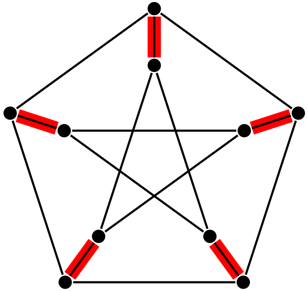


1. Show that the Petersen graph has a $K_{3,3}$-minor.

Minors

A final "subgraph" relation is the minor relation. A graph H is *minor* in G if we can obtain H by deleting edges and vertices and *contracting* edges.

Figure: The Petersen graph has a K_5-minor



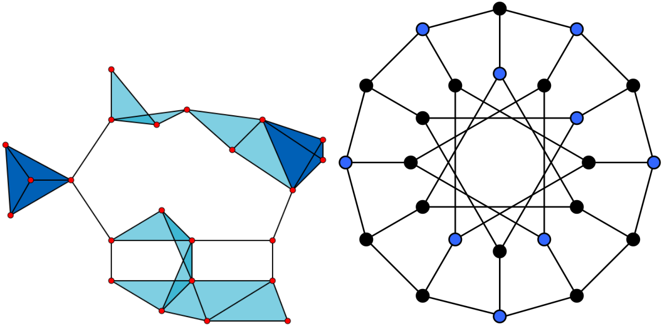
1. Show that the Petersen graph has a $K_{3,3}$-minor.

Independence number and clique number

The *clique number*, $\omega(G)$, is the largest k such that G has a subgraph isomorphic to a complete graph K_k on k-vertices. (A k-clique.)

The *independence number*, $\alpha(G)$, is the largest k such that G has a set of k vertices no two of which are adjacent. Note that $\alpha(G) = \omega(\overline{G})$.

Both $\alpha(G) \leq k$ and $\omega(G) \leq k$ can be stated as forbidden induced subgraphs.



(a) A graph with $\omega(G) = 4$

(b) An independent set of size 9

Monotone properties

Each notion of subgraphs, subgraphs, spanning subgraphs, induced subgraphs and the minor relation, give rise to a *partial order* ($\prec$) on the set $\mathcal{G}$ of graphs where $\prec$ can be $\subset, \subset_{sp}$ or $\triangleleft$.

We say that graph parameter $f : \mathcal{G} \to \mathbb{R}$ is *increasing* (decreasing) in $\prec$ if $G \prec H$ implies $f(G) \leq f(H)$ $(f(G) \geq f(H))$. For instance, the number of components, $c(G)$, is decreasing under the spanning subgraph partial order.

Usually: a property is *monotone increasing* if the property is not destroyed under addition of edges. This means that it is increasing visavi the spanning subgraph property.

A property is *hereditary* if it holds under deletion of vertices. It is thus monotone decreasing under the *induced subgraph* relation $\triangleleft$.

1. Give examples of properties that are monotone/not monotone under these relations.

Degrees and parity

Def: For a general digraph $D = (V, A)$ and a vertex $v \in V$, define the *out-degree*, $\deg_+(v, \vec{G})$ ($\deg_-(v, \vec{G})$) as the number of out-edges (in-edges) to v, i.e. the number of edges $e \in A$ such that $v = \operatorname{tail} e$ ($v = \operatorname{head} v$).

Def: For a graph G and $v \in V(G)$, let the *degree*

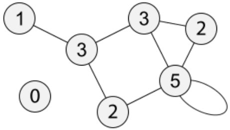
$$\deg(v, G) := \deg_+(v, \vec{G}) + \deg_-(v, \vec{G}),$$

where $\vec{G}$ is any admissable orientation of G.

A graph is *k-regular* if every vertex has degree k.

We can also say that $\deg(v, G)$ is the number of regular edges incident with v plus 2 times the number of loops incident with v.
For simple graphs without loops, we have

$$\deg_G(v) = |N(v, G)| = |E(v, G)|,$$



where $N(v, G)$ is the set of *neighbours* to v and $E(v, G)$ the set of edges incident to v.

Handshake lemma

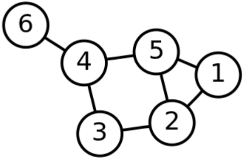
If we sum all the degrees, we count each edge twice.

Thm: (Handshake Lemma) The sum of degrees equals twice the number of edges, i.e.

$$\sum_{v \in V(G)} \deg(v, G) = 2|E(G)|.$$

Cor: Reducing   mod 2 gives that *the number of vertices with odd degree is even*.

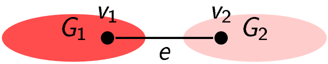
Figure: A graph with four odd vertices.



1. What is $|V(G)|$ if $|E(G)| = 9$ and all edges have degree 3. (*3-regular* graph)

Some problems

1. What is the number of edges in the hypercube Q_n?
2. Can the sequence $1, 1, 1, 2, 3, 4, 5, 7$ be a degree sequences in a simple loop-free connected graph? What about loop-free connected multigraphs?
3. Let e be a *bridge* in a connected graph G as in the figure. Show that each of the graphs G_1, G_2 has an odd number of odd vertices.



4. Let D be an orientation of the undirected complete graph K_n. (Such a digraph is called a *tournament*.) Prove that

$$\sum_{v \in V(D)} \left(\deg_+(v, G)\right)^2 = \sum_{v \in V(D)} \left(\deg_-(v, G)\right)^2 .$$

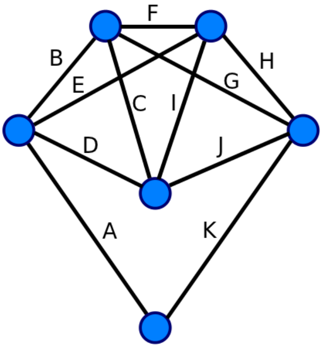
Euler circuits and Euler trails

Problem: Can we draw a graph G on paper without lifting the pen? Can we do it so that we return to the starting point?

Def: An *Euler* circuit (trail) is an circuit using all edges.

Thm: An Euler circuit exists if and only if the graph is connected and all vertices have even degree.

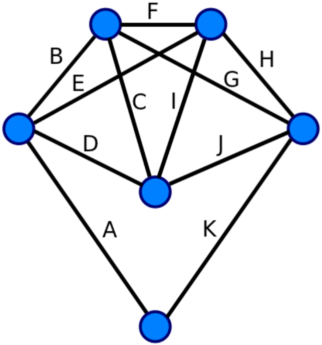
1. Find an Euler circuit for the graph below.
2. Prove the theorem.



Euler circuits as cycle covers and the CDC conjecture

An Euler circuit in a connected graph is equivalent to the existence of a *cycle cover*: A set of cycles $C_1, C_2, \cdots \subset G$, such that every edge $e \in E(G)$ is contained in exactly one of these cycles.

A famous and still unsolved conjecture in Graph Theory states that every *bridge-less* graph has a cycle *double cover*, i.e. a set of, not necessarily distinct, cycles $C_1, C_2, \ldots, C_r \subset G$ such that every edge $e \in E(G)$ is contained in exactly two cycles.



Bipartite graphs

A graph G is *bipartite* — with bipartition V_1, V — if $V = V_1 \dot{\cup} V_2$ and all edges $ij \in E$ has one end in V_1 and V_2.

1. Show that the hypercube Q_n is bipartite.
2. Show that a graph is bipartite if and only if every cycle has even length.

Planar graph

Check this game The game Planarity

Definition of plane graphs and planar graphs

We define a more general concept.

Def: Let S be a topological space. An S-*drawing* is a pair (V, E) if V is a finite set of points in S and where each $e \in E$ is a *arc* in S connecting points in V. We interpret this as a graph G, where the edge (arc) $e \in E$ is incident with its endpoints.

If the edges of an S-drawing are *disjoint* except at the endpoints, then G is S-*embedded*. A graph is S-*embeddable* if G is isomorphic to some S-embedded graph.

When S is the plane $\mathbb{R}^2$ (or S is the sphere S^2), we say that G is *plane* if it is S-embedded and *planar* if it is S-embeddable.

Questions

1. Are the following graphs planar?
2. Is K_5 ($K_{3,3}$) planar? Toroidal?

Thm: The Jordan Curve theorem.

Euler's formula

A plane graph $G = (V, E)$, give rise to a plane *map* (V, E, F) — the union of the edges subdivides the plane into a set F of open regions called *faces* such that each face is bounded by a finite set of edges.

Kuratowski's theorem and homeomorphic graphs

A *subdivision* of a graph is obtained by replacing edges in the graph by paths, equivalently, by iterativel replacing edges ●——● by 2-paths ●—●—●

Two graphs which can be transformed into each other by sequence insertions and "deletions" of vertices of degree two are called *homemorphic*.

Kuratowskis Theorem: A graph is planar *if and only if* it does not contain a subgraph which is a *subdivision* of a K_5 or a $K_{3,3}$.

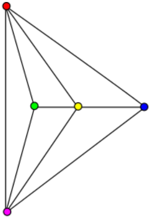
Wagners Theorem: A graph is planar *if and only if* it does not contain K_5 or a $K_{3,3}$ as *minors*.

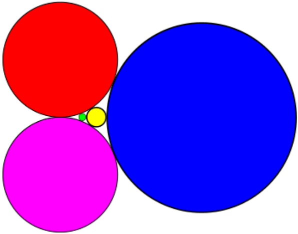
Kissing graphs

Wagner's theorem (1936): A planar graph has a plane embedding using only straight lines as edges.

Koebe's theorem (1936): A graph is planar if and only if it is a intersection graph of circles in the plane, where each pair of circles are *tangent*.

Koebe–Andreev–Thurston theorem: If G is a finite triangulated planar graph, then the circle packing whose tangency graph is (isomorphic to) G is unique, up to Möbius transformations and reflections in lines.





The dual graph