# Lecture 4: Bipartite graphs and planarity

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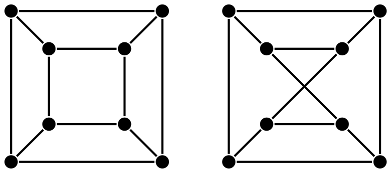
2011-10-22 lör

# Outline

# Bipartite graphs

A graph $G$ is *bipartite* — with bipartition $V_1, V_2$ — if $V = V_1 \dot\cup V_2$ and all edges $ij \in E$ has one end in $V_1$ and $V_2$.

1. Which of the following graphs are bipartite?



2. Show that the hypercube $Q_n$ is bipartite.
3. Show that a graph is bipartite if and only if every cycle has even length.

# Definition of plane graphs and planar graphs

Let $S$ be the plane $\mathbb{R}^2$. How would you define a graph "drawn in the plane without edges crossing"? Here is one definition.

An *arc* is a curve $\gamma \subset S$ such that there is continuous map $\phi : [0,1] \to \gamma$ which is injective, except possibly that $\phi(0) = \phi(1)$, in which case $\gamma$ is a *loop*. The *endpoints* of the arc are $\phi(0)$ and $\phi(1)$.

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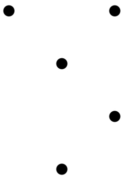

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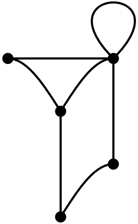
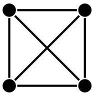
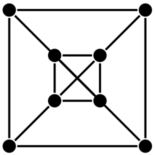


# Questions

To solve this you may need
(**The Jordan Curve theorem:**) A simple closed curve *separates* the plane into two simply connected regions. (The inner and outer).

1. Are the following graphs planar?

2. Prove that $K_5$ and $K_{3,3}$ are non-planar?
3. Prove that planarity is preserved under edge- and vertex deletions *and* edge-contractions. Thus planarity is preserved under taking *minors*. Deduce that any graph with $K_5$ (or $K_{3,3}$) as a minor.

# Kuratowski's theorem (Wagner's theorem) and homeomorphic graphs

**Wagners Theorem:** A graph is planar *if and only if* it does not contain $K_5$ or a $K_{3,3}$ as a *minor*.

A *subdivision* of a graph is obtained by replacing edges in the graph by paths, equivalently, by iteratively replacing edges ●——● by 2-paths ●—●—●

Two graphs which can be transformed into each other by sequence insertions and "deletions" of vertices of degree two are called *homeomorphic*.

**Kuratowskis Theorem:** A graph is planar *if and only if* it does not contain a subgraph which is a *subdivision* of a $K_5$ or a $K_{3,3}$.

# Plane maps and Euler's formula

A plane graph $G = (V, E)$, give rise to a plane *map* $(V, E, F)$ — the union of the arcs the plane into a set of connected open regions. For each such open region call its closure a *face* and let $F$ be the set of faces. Note that each face $f$ is bounded by a unique set of edges $e_1, \ldots, e_r\}$, such that $e_i \subset f$. For all other edges, it holds that $e_j$ either is disjoint from $f$ or else $f \cap e_j \in V$.

1. If the graph is connected each face is *simply connected* "on the Riemann Sphere", i.e. we allow the hole at infinity.
2. If the graph is bridge-less, then every edge is contained in exactly two faces and every face is bounded by a cycle.

# Eulers formula

**Thm:** For every plane map $(V, E, F)$ we have

$$|V| - |E| + |F| = 2 + c(G) - 1,$$

where $G = (V, E)$ and $c(G)$ denotes the number of components in $G$. In particular, the right hand side is 2 if $G$ is connected.

(Proof by edge-deletion.)

1. Prove that each planar graph has a vertex of degree at most 5. (The minimum degree $\delta(G) \leq 5$ for planar graphs.)
2. A *platonic solid* is a simple connected $d$- *regular* graph where each face contains the same number, $r$, of edges and which is not a cycle. What possibilities are there?

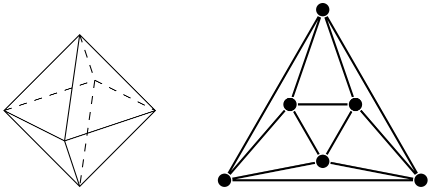


Figure: The octahedron

# The dual graph

Given a plane map $G = (V, E, F)$, where $F = \{f_1, \ldots, \}$ is the set of faces, define the "abstract" graph $G^* = (F, E)$ where $e \in E$ is incident with $f \in F$ if and only if $e \subset F$.

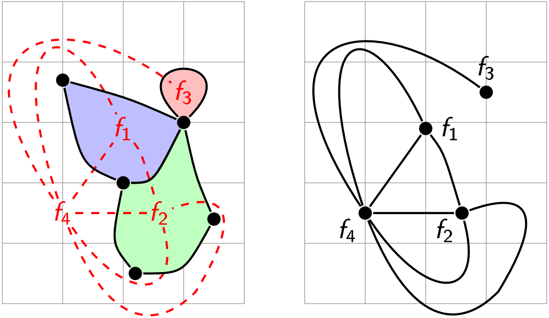
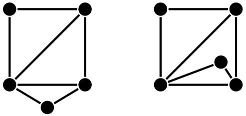



Figure: A dual map

# Some problems regarding the definition of dual graphs

1. Draw the dual graphs to the following two maps:

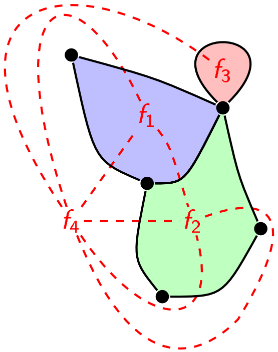


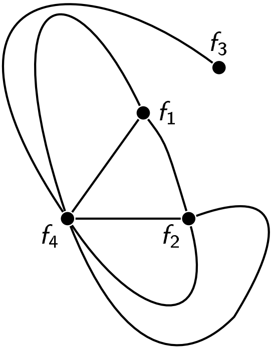
2. What is the dual of a *forest* (an acyclic graph)?
3. Show that he dual graph is planar and there is a natural *dual map* $(F, E, V)$.
4. What is the dual to the octahedron? Will any platonic solid have a platonic solid as a dual graph?

# Duality between cuts and cycles in dual planar maps

A *cut-set* in a graph $G = (V, E)$ is a set of edges $S \subset E$ such that $S = E(V_1, V_2)$ for some partition $V = V_1 \dot\cup V_2$. A cut-set (or just cut) is *minimal* if either $V_1$ or $V_2$ are connected.

There is a one-to-one correspondence between minimal cuts in $G$ and cycles in $G^*$ and vice versa; between cycles in $G$ and minimal cut-sets in $G^*$.



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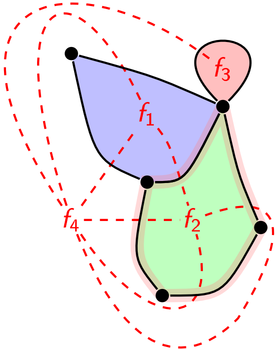


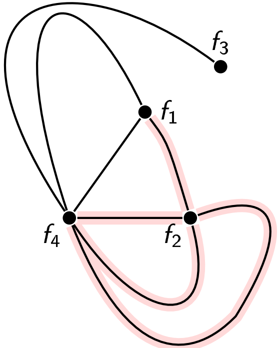

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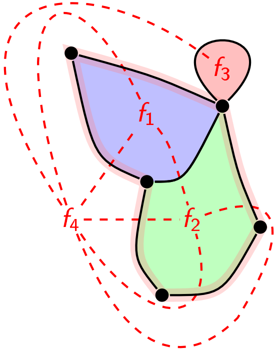
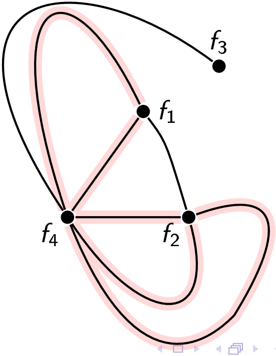


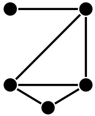



# The incidence matrix of a graph

Let $G = (V, E)$ be a connected (di-)graph (without loops) and, if $G$ is undirected, let $\vec{G}$ be a *reference orientation* of $G$.

Define the *incidence matrix* $B \in \mathbb{R}^{V \times E}$ (or $B(G)$) to $G$, (or $\vec{G}$) as follows: Let $e \in E$ index the columns and $v \in V(G)$ index the rows and let $B(v, e)$ be $+1$ if $v = \text{tail}\, e$, $-1$ if $v = \text{head}\, e$ and 0 otherwise. (We index using "function notation"!)

1. What is the incidence matrix of the following graph?



2. What does it say about the vector $\mathbf{x} \in \mathbb{R}^E$ that $B\mathbf{x}(v) = 0$? That $B^\mathsf{T}\mathbf{y} = 0$, $\mathbf{y} \in \mathbb{R}^V$, $B^\mathsf{T}$ is the transpose?
3. Compute the *Laplacian* $L = B^\mathsf{T} B$, for some graphs. How would you describe the structure of $L$? What does it mean when $L\mathbf{y}(v) = 0$
4. Describe a set of linearly *independent* columns in $B$. Describe a *minimal* set of linearly *dependent* columns in $B$.

# The cycle space and the cut space

The *cycle space* $Z(G)$ is the set of vectors $\mathbf{z} \in \mathbb{R}^E$, such that $B\mathbf{z} = 0$.
The *cut space* $Z^\perp(G) \subset \mathbb{R}^E$ is the orthogonal complement to $Z(G)$, with respect to the natural inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\mathsf{T}\mathbf{v} = u(e_1)v(e_1) + \cdots + u(e_m)v(e_m),$$

where we enumerate $E = \{e_1, e_2, \ldots, e_m\}$.

1. What are the dimensions of $Z(G)$ and $Z^\perp(G)$?
2. The *support* of a vector $x \in \mathbb{R}^E$ in $Z(G)$ is the set of $e \in E$, such that $x(e) \neq 0$. Show that the vectors of minimal support in $Z(G)$ are the cycles.

# Algebraic duality

Let G be a connected graph. An algebraic dual of $G$ is a graph $G^\star$ so that $G$ and $G^\star$ have the same set of edges and

$$Z(G) = Z^\perp(G^\star).$$

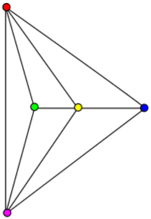
This means that any cycle of $G$ is a cut of $G^\star$, and any cut of $G$ is a cycle of $G^\star$.

Every planar graph has an algebraic dual and Whitney showed that any connected graph G is planar if and only if it has an algebraic dual.

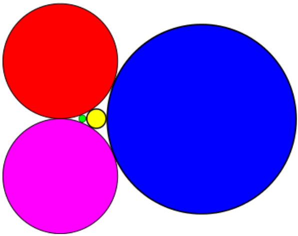
Mac Lane showed that a graph is planar if and only if there is a *basis* of cycles for the cycle space, such that every edge is contained in at most two such basis-cycles.

# Kissing graphs and complex analysis

Wagner's theorem (1936): A planar graph has a plane embedding using only straight lines as edges.



Koebe's theorem (1936): A graph is planar if and only if it is a intersection graph of circles in the plane, where each pair of circles are *tangent*.



Koebe–Andreev–Thurston theorem: If G is a finite triangulated planar graph, then the circle packing whose tangency graph is (isomorphic to) G is unique, up to Möbius transformations and reflections in lines.