Lecture 5: Dual graphs and algebraic duality

Anders Johansson

2011-10-22 lör

Outline

Eulers formula

Thm: For every plane map (V, E, F) we have

$$|V| - |E| + |F| = 2 + c(G) - 1,$$

where $G = (V, E)$ and $c(G)$ denotes the number of components in G. In particular, the right hand side is 2 if G is connected.

(Proof by edge-deletion.)

1. Prove that each planar graph has a vertex of degree at most 5. (The minimum degree $\delta(G) \leq 5$ for planar graphs.)
2. A *platonic solid* is a simple connected d- *regular* graph where each face contains the same number, r, of edges and which is not a cycle. What possibilities are there?

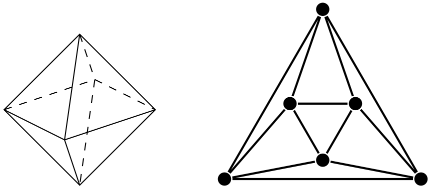


Figure: The octahedron

The dual graph

Given a plane map $G = (V, E, F)$, where $F = \{f_1, \dots, \}$ is the set of faces, define the "abstract" graph $G^* = (F, E)$ where $e \in E$ is incident with $f \in F$ if and only if $e \subset F$.

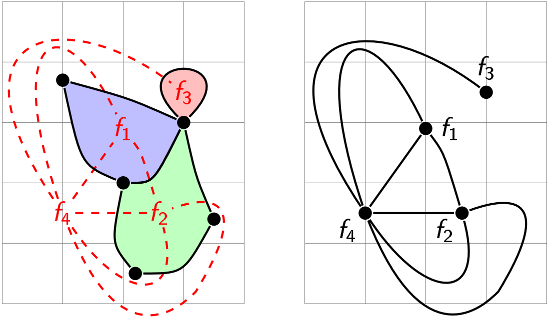
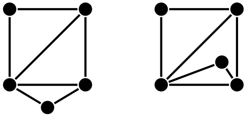


Figure: A dual map

Some problems regarding the definition of dual graphs

1. Draw the dual graphs to the following two maps:

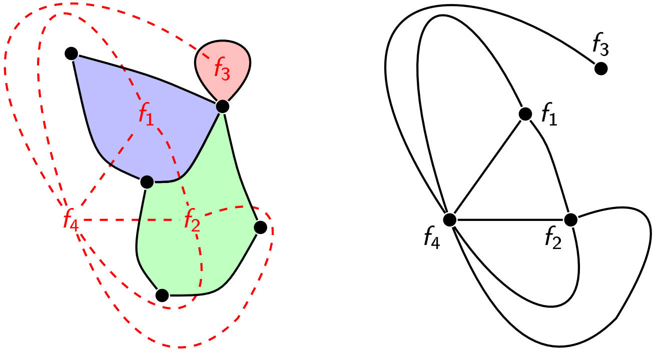


2. What is the dual of a *forest* (an acyclic graph)?
3. Show that he dual graph is planar and there is a natural *dual map* (F, E, V).
4. What is the dual to the octahedron? Will any platonic solid have a platonic solid as a dual graph?

Duality between cuts and cycles in dual planar maps

A *cut-set* in a graph $G = (V, E)$ is a set of edges $S \subset E$ such that $S = E(V_1, V_2)$ for some partition $V = V_1 \dot\cup V_2$. A cut-set (or just cut) is *minimal* if either V_1 or V_2 are connected.

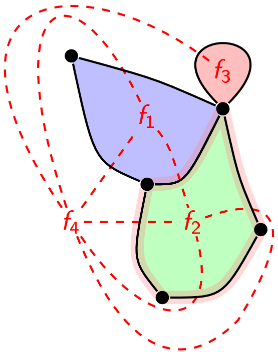
There is a one-to-one correspondence between minimal cuts in G and cycles in G^* and vice versa; between cycles in G and minimal cut-sets in G^*.

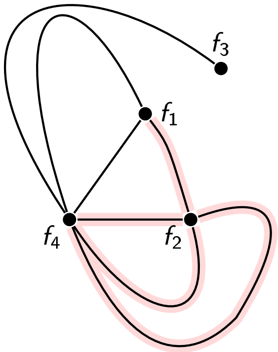


Duality between cuts and cycles in dual planar maps

A *cut-set* in a graph $G = (V, E)$ is a set of edges $S \subset E$ such that $S = E(V_1, V_2)$ for some partition $V = V_1 \dot\cup V_2$. A cut-set (or just cut) is *minimal* if either V_1 or V_2 are connected.

There is a one-to-one correspondence between minimal cuts in G and cycles in G^* and vice versa; between cycles in G and minimal cut-sets in G^*.

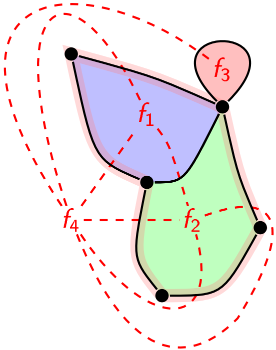


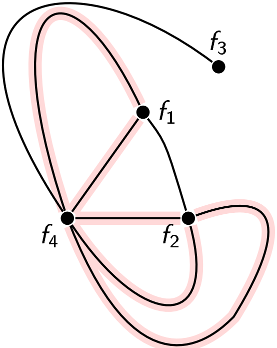


Duality between cuts and cycles in dual planar maps

A *cut-set* in a graph $G = (V, E)$ is a set of edges $S \subset E$ such that $S = E(V_1, V_2)$ for some partition $V = V_1 \dot{\cup} V_2$. A cut-set (or just cut) is *minimal* if either V_1 or V_2 are connected.

There is a one-to-one correspondence between minimal cuts in G and cycles in G^* and vice versa; between cycles in G and minimal cut-sets in G^*.



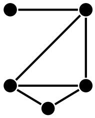


The incidence matrix of a graph

Let $G = (V, E)$ be a connected (di-)graph (without loops) and, if G is undirected, let $\vec{G}$ be a *reference orientation* of G.

Define the *incidence matrix* $B \in \mathbb{R}^{V \times E}$ (or $B(G)$) to G, (or $\vec{G}$) as follows: Let $e \in E$ index the columns and $v \in V(G)$ index the rows and let $B(v, e)$ be $+1$ if $v = \text{tail}\, e$, -1 if $v = \text{head}\, e$ and 0 otherwise. (We index using "function notation"!)

1. What is the incidence matrix of the following graph?



2. What does it say about the vector $\mathbf{x} \in \mathbb{R}^E$ that $B\mathbf{x}(v) = 0$? That $B^\mathsf{T}\mathbf{y} = 0$, $\mathbf{y} \in \mathbb{R}^V$, B^T is the transpose?
3. Compute the *Laplacian* $L = B^\mathsf{T} B$, for some graphs. How would you describe the structure of L? What does it mean when $L\mathbf{y}(v) = 0$
4. Describe a set of linearly *independent* columns in B. Describe a *minimal* set of linearly *dependent* columns in B.

The cycle space and the cut space

The *cycle space* $Z(G)$ is the set of vectors $\mathbf{z} \in \mathbb{R}^E$, such that $B\mathbf{z} = 0$. The *cut space* $Z^\perp(G) \subset \mathbb{R}^E$ is the orthogonal complement to $Z(G)$, with respect to the natural inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\mathsf{T}\mathbf{v} = u(e_1)v(e_1) + \cdots + u(e_m)v(e_m),$$

where we enumerate $E = \{e_1, e_2, \ldots, e_m\}$.

1. What are the dimensions of $Z(G)$ and $Z^\perp(G)$?
2. The *support* of a vector $x \in \mathbb{R}^E$ in $Z(G)$ is the set of $e \in E$, such that $x(e) \neq 0$. Show that the vectors of minimal support in $Z(G)$ are the cycles.

Algebraic duality

Let G be a connected graph. An algebraic dual of G is a graph $G^\star$ so that G and $G^\star$ have the same set of edges and

$$Z(G) = Z^{\perp}(G^\star).$$

This means that any cycle of G is a cut of $G^\star$, and any cut of G is a cycle of $G^\star$.

Every planar graph has an algebraic dual and Whitney showed that any connected graph G is planar if and only if it has an algebraic dual.

Mac Lane showed that a graph is planar if and only if there is a *basis* of cycles for the cycle space, such that every edge is contained in at most two such basis-cycles.

Kissing graphs and complex analysis

Wagner's theorem (1936): A planar graph has a plane embedding using only straight lines as edges.

Koebe's theorem (1936): A graph is planar if and only if it is a intersection graph of circles in the plane, where each pair of circles are *tangent*.

Koebe–Andreev–Thurston theorem: If G is a finite triangulated planar graph, then the circle packing whose tangency graph is (isomorphic to) G is unique, up to Möbius transformations and reflections in lines.