Lecture 6: Hamiltonian graphs

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Definition of a Hamilton cycle
Let $G = (V, E)$, where $|V| = n$. A cyclic subgraph $C \subseteq G$ such that $C \cong C_n$ is called a Hamilton cycle in $G$. It is thus a cycle passing through every vertex. If $G$ has a Hamilton cycle it is said to be Hamiltonian.

1. How many Hamilton cycles does $K_4$ have?
2. Show that the dodecahedron is Hamiltonian.
3. Show that the following graph is non-Hamiltonian.
4. What about the Petersen graph?
5. Show that a bipartite graph is Hamiltonian only if it is balanced.
6. Show that $Q_n$ has a Hamilton cycle. Use induction on $n$. 
Independent sets of Hamiltonian graphs

Let $G$ be a graph with independent set $S \subset V$. An independent set must not take up to many edges for the graph to be Hamiltonian.

1. Show that
2. Show that the following graph is non-hamiltonian

Tournaments and ranking path

After an all-meets-all table-tennis tournament, show that we can rank the players so that every player (except the last one) beat the following player in the ranking.

A tournament on $n$ vertices is an orientation $T$ of the complete graph $K_n$.

1. Show that there is a directed Hamilton path in $T$. (Reedi’s theorem)

Diracs theorem

**Theorem** Given $G = (V, E)$, $|V| = n$, if the minimum degree $\delta(G) \geq n/2$ then $G$ is Hamiltonian.

Proof sketch: Take a edge-maximal counterexample $G$. Then $G + uv$, $uv \in E$, has a H-path $P : u = x_1 x_2 x_3 \ldots x_n = v$.

We aim to get the following construction.

Let $\vec{P}$ be the orientation of $P$ from $u$ to $v$. Then

$$|N_+ (N(v,G), \vec{P}) \cap N(v)| > 0.$$ 

Here $N_+ (S, \vec{P})$ means the set $S$ shifted one step along the path $\vec{P}$.

4 Graph colourings

**Definition of (proper) colourings and $k$-colourability.**

A *proper colouring* of a (simple) graph is a labeling of vertices where adjacent vertices never share a label. The labels are then often called *colours*. We say that a graph is *$k$-colourable* if it can be coloured using (at most) $k$ colours.
The complete $K_4$ is an example of a planar simple graph which is not 3-colourable.

1. Determine $\chi(G)$ for $G = K_n, \overline{K_n}, C_n$.

Let $[k] := \{1, 2, \ldots, k\}$ (or any set with $k$ elements). An (improper) $k$-colouring is an element of $[k]^V := \{f : V \rightarrow [k]\}$. It is thus a proper $k$-colouring if $u$ adjacent to $v$ implies that $f(u) \neq f(v)$. Equivalently, the inverse image $f^{-1}(\alpha)$ of each colour $\alpha \in [k]$ is an independent set.

The smallest number $k$ for which the graph $G$ is $k$-colourable, is called the chromatic number of $G$, denoted by $\chi(G)$.

**The chromatic number**

The chromatic number, $\chi(G)$, of $G$ is the smallest $k$ such that $G$ has a proper $[k]$-colouring.

Thus $\chi(G) \leq k$ if and only if there exists a proper $k$-colouring. We say that $G$ is $k$-colourable or $k$-partite; it means that we can partition the vertex set into $k$ independent sets. A graph is bipartite is the same stating that $\chi(G) \leq 2$.

1. Give example of a $K_3$-free graph such that $\chi(G) \geq 3$.

**The Four-Colour theorem and Heawood’s Five-Colour theorem**

There is the celebrated Theorem (The Four-Color Theorem) Every simple planar graph is 4-colourable.

The only known proofs require extensive computer runs. The first such proof was obtained by Kenneth Appel and Wolfgang Haken in 1976.

If we require only 5-colourability, then there is Theorem 6.6. (Heawood’s Theorem or The Five-Color Theorem) Every simple planar graph is 5-colourable.

- Proof We may think of $G = (V, E)$ as a plane graph. Use induction on the number $n = |V|$. (Induction Basis: $n = 1$ is 1-colourable since there are no edges.) We assume the theorem is true for $1 \leq |V| \leq n$ and aim to show theorem is true for $|V| = n + 1$.

Recall the Minimum Degree Bound for Planar Graph (Derived from Euler’s theorem), i.e. there is $v \in V(G)$ of degree at most 5. On the
other hand, according to the Induction Hypothesis the graph $G - v$ is 5-colourable. If, in this colouring, the set of vertices adjacent to $v$, $N(v)$, are coloured using at most four colours, then clearly we can 5-colour $G$.

So we are left with the case where the neighbours $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$ are coloured using different colours. We may assume that the indexing of the vertices proceeds clockwise, and we label the colours with the numbers 1, 2, 3, 4, 5 (in this order). We show that the colouring of $G - v$ can be changed so that (at most) four colours suffice for colouring $N(v)$. We denote by

$$H_{i,j} := (G - v)[f^{-1}(\{i, j\})]$$

the bichromatic subgraph of $G - v$ induced by the vertices coloured with $i$ and $j$. We have two cases:

1. $v_1$ and $v_3$ are in different components $H_1$ and $H_3$ of $H_{1,3}$. We then interchange the colours 1 and 3 in the vertices of, say, $H_3$ leaving the other colours untouched. In the resulting 5-colouring of $G - v$ the vertices $v_1$ and $v_3$ both have the colour 1. We can then give the colour 3 to $v$.

2. $v_1$ and $v_3$ are connected in $H_{1,3}$. Then there is a $v_1 - v_3$ path in $H_{1,3}$. Including the vertex $v$ we get from this path a circuit $C$. Now, since we indexed the vertices $\{v_1, v_2, v_3, v_4, v_5\}$ clockwise, exactly one of the vertices $v_2$ and $v_4$ is inside $C$. We deduce that $v_2$ and $v_4$ are in different components of $H_{2,4}$, and we have a case similar to the previous one.

Note that the proof gives a simple algorithm for 5-colouring a planar graph.

**Brook’s theorem**

Another theorem we will talk about is Brook’s theorem.

- (Brook’s theorem) A connected graph $G$ has chromatic number bounded by the maximum degree, i.e.

$$\chi(G) \leq \Delta(G) = \max_v \deg(v, G),$$

unless it is a complete graph or an odd cycle.

- The proof is quite similar in many ways to the 5-color theorem: One works with bichromatic components, etc.