Lecture 6: Hamiltonian graphs

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Definition of a Hamilton cycle

Let $G = (V, E)$, where $|V| = n$. A cyclic subgraph $C \subset G$ such that $C \cong C_n$ is called a *Hamilton cycle* in G. It is thus a cycle passing through every vertex. If G has a Hamilton cycle it is said to be Hamiltonian.

1. How many Hamilton cycles does K_4 have?
2. Show that the dodecahedron is Hamiltonian.
3. Show that the following graph is non-Hamiltonian.
4. What about the Petersen graph?
5. Show that a bipartite graph is Hamiltonian only if it is *balanced*.
6. Show that Q_n has a Hamilton cycle. Use induction on n.

Independent sets of Hamiltonian graphs

Let G be a graph with independent set $S \subset V$. An independent set must not take up to many edges for the graph to be Hamiltonian.

1. Show that
2. Show that the following graph is non-hamiltonian

Tournaments and ranking path

After an all-meets-all table-tennis tournament, show that we can rank the players so that every player (except the last one) beat the following player in the ranking.

A *tournament* on n vertices is an orientation T of the complete graph K_n.

1. Show that there is a *directed* Hamilton path in T. (Redei's theorem)

Diracs theorem

Theorem Given $G = (V, E)$, $|V| = n$, if the minimum degree $\delta(G) \geq n/2$ then G is Hamiltonian.

Proof sketch: Take a edge-maximal counterexample G. Then $G + uv$, $uv \in E$, has a H-path $P : u = x_1 x_2 x_3 \dots x_n = v$.

We aim to get the following construction.

Let $\vec{P}$ be the orientation of P from u to v. Then

$$|N_+(N(v, G), \vec{P}) \cap N(v)| > 0.$$

Here $N_+(S, \vec{P})$ means the set S shifted one step along the path $\vec{P}$.

4 Graph colourings

Definition of (proper) colourings and k-colourability.

A *proper colouring* of a (simple) graph is a labeling of vertices where adjacent vertices never share a label. The labels are then often called *colours*. We say that a graph is k-colourable if it can be coloured using (at most) k colours.

The complete K_4 is an example of a planar simple graph which is not 3-colourable.

1. Determine $\chi(G)$ for $G = K_n$, $\overline{K_n}$, C_n?.

Let $[k] := \{1, 2, \ldots, k\}$ (or any set with k elements). An (improper) k-/colouring/ is an element of $[k]^V := \{f : V \to [k]\}$. It is thus a *proper* k-colouring if u adjacent to v implies that $f(u) \neq f(v)$. Equivalently, the inverse image $f^{-1}(\alpha)$ of each colour $\alpha \in [k]$ is an *independent set*.

The smallest number k for which the graph G is k-colourable, is called the *chromatic number* of G, denoted by $\chi(G)$.

The chromatic number

The *chromatic number*, $\chi(G)$, of G is sthe smallest k such that G has a proper $[k]$-colouring.

Thus $\chi(G) \leq k$ if and only if there exists a proper k-colouring. We say that G is *k-colourable* or *k-partite*; it means that we can partition the vertex set into k independent sets. A graph is bipartite is the same stating that $\chi(G) \leq 2$.

1. Give example of a K_3-free graph such that $\chi(G) \geq 3$.

The Four-Colour theorem and Heawoods Five-Colour theorem

There is the celebrated **Theorem (The Four-Color Theorem)** Every simple planar graph is 4-colourable.

The only known proofs require extensive computer runs. The first such proof was obtained by Kenneth Appel and Wolfgang Haken in 1976.

If we require only 5-colourability, then there is **Theorem 6.6. (Heawood's Theorem or The Five-Color Theorem)** Every simple planar graph is 5-colourable.

- **Proof** We may think of $G = (V, E)$ as a plane graph. Use induction on the number $n = |V|$. (Induction Basis: $n = 1$ is 1-colourable since there are no edges.) We assum the theorem is true for $1 \leq |V| \leq n$ and aim to show theorem is true for $|V| = n + 1$.

  Recall the Minimum Degree Bound for Planar Graph (Derived from Euler's theorem), i.e. there is $v \in V(G)$ of degree at most 5. On the other hand, according to the Induction Hypothesis the graph $G - v$ is 5-colourable. If, in this colouring, the set of vertices adjacent to v, $N(v)$, are coloured using at most four colours, then clearly we can 5-colour G.

So we are left with the case where the neighbours $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$ are coloured using different colours. We may assume that the indexing of the vertices proceeds clockwise, and we label the colours with the numbers $1, 2, 3, 4, 5$ (in this order). We show that the colouring of $G - v$ can be changed so that (at most) four colours suffice for colouring $N(v)$. We denote by

$$H_{i,j} := (G - v)[f^{-1}(\{i, j\})]$$

the bichromatic subgraph of $G - v$ induced by the vertices coloured with i and j. We have two cases:

1. v_1 and v_3 are in different components H_1 and H_3 of $H_{1,3}$. We then *interchange* the colours 1 and 3 in the vertices of, say, H_3 leaving the other colours untouched. In the resulting 5-colouring of $G - v$ the vertices v_1 and v_3 both have the colour 1. We can then give the colour 3 to v.
2. v_1 and v_3 are connected in $H_{1,3}$. Then there is a $v_1 - v_3$ path in $H_{1,3}$. Including the vertex v we get from this path a circuit C. Now, since we indexed the vertices $\{v_1, v_2, v_3, v_4, v_5\}$ clockwise, exactly one of the vertices v_2 and v_4 is inside C. We deduce that v_2 and v_4 are in different components of $H_{2,4}$, and we have a case similar to the previous one.

Note that he proof gives a simple algorithm for 5-colouring a planar graph.

Brook's theorem

Another theorem we will talk about is Brook's theorem.

- (Brook's theorem) A connected graph G has chromatic number bounded by the *maximum degree*, i.e.

  $$\chi(G) \leq \Delta(G) = \max_v \deg(v, G),$$

  unless it is a *complete graph* or an odd cycle.
- The proof is quite similar in many ways to the 5-color theorem: One works with bichromatic components, etc.