Lecture 7: Hamiltonian graphs and colourings

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Contents

Warning: These notes are not complete and many figures are missing.

Hamilton cycles

- Definition of a Hamilton cycle Let $G = (V, E)$, where $|V| = n$. A cyclic subgraph $C \subset G$ such that $C \cong C_n$ is called a *Hamilton cycle* in G. It is thus a cycle passing through every vertex. If G has a Hamilton cycle it is said to be Hamiltonian.

  1. How many Hamilton cycles does K_4 have?
  2. Show that the dodecahedron is Hamiltonian.
  3. Show that the following graph is non-Hamiltonian.
  4. What about the Petersen graph?
  5. Show that a bipartite graph is Hamiltonian only if it is *balanced*.
  6. Show that Q_n has a Hamilton cycle. Use induction on n.

- Independent sets of Hamiltonian graphs Let G be a graph with independent set $S \subset V$. An independent set must not take up to many edges for the graph to be Hamiltonian.

  1. Show that $|E| - \sum_{v \in S} \deg(v) < |V| - 2|S|$ implies that G is non-hamitonian.
  2. Show that the Herschel graph is non-hamiltonian.

- Tournaments and ranking path After an all-meets-all table-tennis tournament, show that we can rank the players so that every player (except the last one) beat the player that immediately follows in the ranking.

  A *tournament* on n vertices is an orientation T of the complete graph K_n.

1. Show that there is a *directed* Hamilton path in T. (Redei's theorem)

- Diracs theorem

  Theorem 1. *Given $G = (V, E)$, $|V| = n$, if the minimum degree $\delta(G) \geq n/2$ then G is Hamiltonian.*

  Proof. Sketch: Take a edge-maximal counterexample G. Then $G + uv$, $uv \in E$, has a H-path $P : u = x_1 x_2 x_3 \ldots x_n = v$.

  We aim to prove the existence of a cycle of the following form. (Figure)

  Let $\vec{P}$ be the orientation of P from u to v. Then

  $$|N_+(N(v, G), \vec{P}) \cap N(v)| > 0.$$

  Here $N_+(S, \vec{P})$ means the set S shifted one step along the path $\vec{P}$. □

Graph colourings

- Definition of (proper) colourings and k-colourability. How can schedule meetings for n committees into k time-slots (parallell sessions are OK) so that no delegate is double booked?

  Think of each committee as the corresponding set of delegates and construct the intersection graph on n vertices. We shall label so that no adjacent vertices get the same label.

  A *proper colouring* of a (simple) graph is a labeling of vertices where adjacent vertices never share a label. The labels are called *colours*. We say that a graph is k-colourable if it can be coloured using (at most) k colour and say then that the chromatic number $\chi(G) \leq k$.

  If a graph is colourable for any k then it obviously can not have loops. Equally obviously, parallel edges can be reduced to one, so we may assume our graphs here to be simple.

  The complete K_4 is an example of a planar simple graph which is not 3-colourable.

  1. Determine $\chi(G)$ for $G = K_n$, $\overline{K_n}$, C_n?.

- Equivalent formulations Let $[k] := \{1, 2, \ldots, k\}$ (or any set with k elements). An (improper) k-/colouring/ is an element of $[k]^V := \{f : V \to [k]\}$. It is thus a *proper* k-colouring if u adjacent to v implies that $f(u) \neq f(v)$. Equivalently, the inverse image $f^{-1}(\alpha)$ of each colour $\alpha \in [k]$ is an *independent set*.

  The smallest number k for which the graph G is k-colourable, is called the *chromatic number* of G, denoted by $\chi(G)$.

- The chromatic number The *chromatic number*, $\chi(G)$, of G is the smallest k such that G has a proper $[k]$-colouring.

  We say that G is *k-colourable* or *k-partite*; it means that we can partition the vertex set into k independent sets. A graph is bipartite is the same stating that $\chi(G) \leq 2$.

  1. Give example of a K_3-free graph such that $\chi(G) \geq 3$.

- The Four-Colour theorem and Heawoods Five-Colour theorem There is the celebrated **Theorem (The Four-Color Theorem)** Every simple planar graph is 4-colourable.

  The only known proofs require extensive computer runs. The first such proof was obtained by Appel and Haken in 1976.

  If we require only 5-colourability, then there is.

  Theorem 2 ((Heawood's Theorem or The Five-Color Theorem)). *Every simple planar graph is* 5*-colourable.*

  We may think of $G = (V, E)$ as a plane graph. Use induction on the number $n = |V|$. (Induction Basis: $n = 1$ is 1-colourable since there are no edges.) Assume the statement is true for $1 \leq |V| \leq n - 1$ and aim to show it for $|V| = n$.

  Recall $\delta(G) \leq 5$ for a planar graph. By induction, the graph $G - v$ is 5-colourable. If, in this colouring, the set of vertices adjacent to v, $N(v)$, are coloured using less than five colours, then clearly we can 5-colour G.

  So we are left with the case where the neighbours $N(v) = \{v_1, v_2, v_3, v_4, v_5\}$ are coloured using different colours. We may assume that the indexing of the vertices proceeds clockwise, and we label the colours with the numbers $1, 2, 3, 4, 5$ (in this order). We show that the colouring of $G - v$ can be changed so that (at most) four colours suffice for colouring $N(v)$. We denote by

  $$H_{i,j} := (G - v)[f^{-1}(\{i, j\})]$$

  the bichromatic subgraph of $G - v$ induced by the vertices coloured with i and j. We have two cases:

  1. v_1 and v_3 are in different components H_1 and H_3 of $H_{1,3}$. We then *interchange* the colours 1 and 3 in the vertices of, say, H_3 leaving the other colours untouched. In the resulting 5-colouring of $G - v$ the vertices v_1 and v_3 both have the colour 1. We can then give the colour 3 to v.
  2. v_1 and v_3 are connected in $H_{1,3}$. Then there is a $v_1 - v_3$ path in $H_{1,3}$. Including the vertex v we get from this path a circuit C. Now, since we indexed the vertices $\{v_1, v_2, v_3, v_4, v_5\}$ clockwise, exactly one of the vertices v_2 and v_4 is inside C. We deduce that v_2 and v_4 are in different components of $H_{2,4}$, and we have a case similar to the previous one.

Note that he proof gives a simple algorithm for 5-colouring a planar graph.

- Greedy colouring Perhaps the simplest way to obtain a proper colouring is to use a *greedy colouring*: For a given (perhaps random) vertex-order $V = \{v_1, v_2, \ldots, v_n\}$, define recursively $f : V \to \mathbb{Z}_+$ by $f(v_1) = 1$ and

$$f(v_i) := \min f\,(N(v_i) \cap \{v_1, v_2, \ldots, v_n\}.$$

This is a proper colouring! Note that $f(v_i) \leq d(v_i)+1$ and thus the greedy colouring uses at least $\Delta(G) + 1$ colours.

1. Show that for some ordering $\{v_1, v_2, \ldots, v_n\}$ of the vertices the greedy colouring uses $\chi(G)$ colours.

- Bounding the chromatic number Recall $\omega(G)$ is the largest *clique* (complete subgraph) in G. We have the basic duality

$$\omega(G) \leq \chi(G).$$

This is a type of broken LP-duality. If you define rational relaxations with fractional colourings, then the corresponding quantities are equal. (Note: Perfect graphs.)

What about upper bounds? We saw that a greedy colouring gave $\chi(G) \leq \Delta(G) + 1$. This is sharp for cycles and complete graphs.

Theorem 3 (Brook's theorem)**.** *A connected graph G has chromatic number bounded by the* maximum degree*, i.e.*

$$\chi(G) \leq \Delta(G) = \max_v \deg(v, G),$$

unless it is a complete graph *or an odd cycle.*

The proof is quite similar in many ways to the 5-color theorem: One works with bichromatic components in hte neighbourhood

- Enumeration of colourings: The chromatic polynomial $P(G, \lambda)$.

1. In how many ways can we colour K_n ($\bar{K}_n$) with k colours?
2. In how many ways can we colour P_n (C_n) with k colours?
3. If we have a cut-vertex v, such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$, express the number of ways we can k-colour G in terms of the number of ways we can colour G_1 and G_2.

Definition 4. The *chromatic polynomial* of a graph G, $P(G, \lambda)$, is defined as

$$P(G, \lambda) = \{\text{The number of ways to colour } G \text{ with } \lambda \text{ colours.}\}$$

- The basic recursion for $P(G, \lambda)$

Theorem 5. *If we discard loops when contracting then*

$$P(G, \lambda) = P(G - e, \lambda) - P(G/e, \lambda).$$

This shows that

1. $P(G, \lambda)$ is a polynomial.
2. The sum of coefficients are 0 unless $P(G, \lambda) = \lambda^n$.
3. Leading term is λ^n.

Trees and forests

- Characterisation of trees (Figure of trees)

  Theorem 6. *The following are equivalent for a simple undirected graph T on n vertices*

  1. *T is a tree;*
  2. *T is connected and contains no cycles;*
  3. *T is connected and has $n - 1$ edges;*
  4. *T is connected and every edge is a bridge;*
  5. *any two vertices is connected by exactly one path;*
  6. *T is acyclic but the addition of any new edge creates exactly one new cycle.*

  1. What are the corresponding statements for forests?
  2. Prove, say, (iii) $\implies$ (iv).

Spanning trees and spanning forests

Given a graph G, a spanning subgraph $T \subset G$ which is a tree is called a *spanning tree*. A *spanning forest* is a subgraph $F \subset G$ which is the vertex disjoin union of spanning trees in each component of G.

1. The graph G is connected $\iff$ G has a spanning tree T
2. For a spanning tree $T \subset G$, every edge in T corresponds to a unique bond B_e of G. (Recall a bond is a minimal cut-set.) Every bond contain some edge of T.
3. Every edge e of G not in T corresponds to a unique cycle $C_e \subset G$ and every cycle contain an edge from $G - E(T)$.
4. If a set $W \subset E(G)$ is such that $W \cap E(T) \neq \emptyset$ for *every* spanning tree T, then W is a (not necessarily minimal) cut-set.

5. Let $\tau(G)$ denote the number of spanning trees in G. Show that

$$\tau(G) = \tau(G - e) + \tau(G/e).$$

The cycles $\{C_e : e \notin T\}$ and the bonds $\{B_e : e \in T\}$ constitute basises for the cycle space $Z(G)$ and the cut-space $Z^{\perp}(G)$, respectively.

Leafs and Prüfer codes

A *leaf* is a vertex v in a tree T of degree one. The handshake lemma readily gives that

Every tree has at least two leaves.

Deleting a leaf v from T gives a tree $T - v$.

A Prüfer code is a way to code a tree T "bottom up". Assume $V(T) = \{1, \ldots, n\}$ (or order the vertices using a labeling). Consider the following tree:

(Figure) The tree on 7 vertices with code $(6, 5, 6, 5, 1)$.

To construct the Prüfer code we iterate the following procedure.

- Take the first leaf in order, say, i and write down its neighbour j.
- Delete the leaf i for T.

The code is thus a sequence $\mathbf{C} = (c_1, \ldots, c_{n-2}) \in V(T)^{n-2}$ of length $n - 2$ of vertices.

We can reconstruct the tree from its code $\mathbf{C}$ by first listing $\mathbf{L} = (1, 2, 3, \ldots, n)$ all vertices in the assumed order. Then iterate the following

- Let i be the first vertex in the list $\mathbf{L}$ not in the code. Add the edge ic where c is the first label in the code.
- Delete (Cross out) the first symbol c from $\mathbf{c}$ and delete i from $\mathbf{L}$.

As a final step, we add the edge between the two remaining vertices in $\mathbf{L}$.

Note that we have n^{n-2} Prüfer codes for trees $T \subset K_n$. Since we have established a bijection between the codes and trees, we conclude **Cayleys theorem:** There are n^{n-2} (labeled) trees.