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Graph Theory
Frist, KandMa, IT
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# Problem sheet 1

**1.** Show that
$$c(G) + |E(G)| \geq |V(G)|.$$

**2.** (a) Is there a graph with degree sequence 3, 3, 3, 3, 5, 6, 6, 6, 6, 6, 6?
(b) Is there a bipartite graph with degree sequence 3, 3, 3, 3, 3, 5, 6, 6, 6, 6, 6, 6, 6, 6?
(c) Is there a simple graph with degrees 1, 1, 3, 3, 3, 3, 5, 6, 8, 9?

**3.** Let $G_1$ and $G_2$ be two graphs, with $V(G_1) = V(G_2)$.

1. Show that
$$c(G_1) + c(G_2) \leq c(G_1 \cup G_2) + c(G_1 \cap G_2).$$
2. Show that this holds without assuming $V(G_1) = V(G_2)$.

**4.** Let $G = (V, E)$ be a connected digraph. Let $w : E \to \mathbb{R}$ be a weighting, where $w(xy)$ is considered as the work needed to go from the tail $x$ to the head $y$ and that $-w(xy)$ is the work to go in the opposite direction; from head $y$ to tail $x$. Show that $w$ can be associated to a *potential* $p : V \to \mathbb{R}$, such that $w(xy) = p(y) - p(x)$ if and only if going round any *oriented cycle* the total work needed is 0.

**5.** Show that a $k$-regular connected bipartite graph is 2-connected.

**6.** Consider a *strongly connected* digraph $G$. Show that there is a two-colouring $\varphi : V(G) \to \{-1, +1\}$ such that for every vertex $i$ there is some out-edge $ij$ where $j$ has opposite colour to $i$ if and only if $G$ contains at least one directed even cycle.

**7.** Show that a digraph $G$ is strongly connected if and only if for every non-empty subset $X \subsetneq V(G)$ there is some edge going out from $X$.

**8.** Let $G = (V, E)$ be a digraph with $s, t \in V$ two distinct vertices. Suppose that the edges are coloured red, green and black in an arbitrary manner. Show that exactly one of the following two assertions hold.

(i) There is a black and red oriented $st$-path, such that no black edge is oppositely oriented.
(ii) There is a set $S$ such that $s \in S$ and $t \in V \setminus S$ and such that no red edge connects $S$ with $V \setminus S$ in any direction and no black edge goes from $S$ to $V \setminus S$.

**9.** If $G$ is a graph with even degrees, then it can be oriented in such way that the out-degree equals the in-degree at each vertex.

## Terminology

**Orienting a graph** An undirected graph can be *oriented*, i.e. made into a digraph $\overrightarrow{G}$, by choosing, for each edge $e = \{u, v\}$ one of $(u, v)$ or $(v, u)$ as the corresponding edge in $\overrightarrow{G}$.

**Bipartite graph** A bipartite graph is a graph such that $V$ is composed of two non-empty disjoint parts $X$ and $Y$ and all edges connects vertices in $X$ with vertices in $Y$.

**$k$-regular** A graph is $k$-regular if every vertex has degree $k$.

**Number of components** of a graph $G$ is denoted by $c(G)$.

**Oriented cycle** In a digraph an *oriented walk* is a sequence $v_1e_1v_2 \cdots v_ke_kv_{k+1}$, of vertices $v_i$ and edges $e_j$, such that either $e_i = v_iv_{i+1}$ or $e_i = v_{i+1}v_i$; i the second case we say the edge is oppositely oriented. Oriented trails, circuits, paths and cycles are defined in an analogue manner.

**Directed cycle** A normal, i.e. not oriented, cycle in a digraph.

**Strongly connected** A digraph is strongly connected if for any pair of vertices $(i, j)$ there is a directed path from $i$ to $j$. (And thus also a directed path from $j$ to $i$.)

**$k$-connected, $\kappa(G)$** A connected graph $G$ is *$k$-connected* if $|V(G)| \geq k+1$ and $G-S$ is connected for any set $S \subset V(G)$ of at most $k-1$ vertices. The largest $k$ is called the (vertex-)connectivity of $G$ and is denoted $\kappa(G)$.