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Graph Theory
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Problem sheet 3

1. Show that any tree has at least $\Delta(T)$ leafs (vertices of degree one). Here $\Delta(G)$ denotes the maximum degree of a graph.

2. For a graph $G = (V, E)$, show that the number of subgraphs isomorphic to P_2, i.e. a path of length 2, is given by

$$\sum_{x \in V} d(x)\,\big(d(x) - 1\big)\,.$$

3. Assume that the graph $G = (V, E)$ satisfies

$$\omega(H)\omega(\bar{H}) \geq |V(H)|,$$

for every *induced* subgraph $H \subset G$. Show that the chromatic number of G equals $\chi(G) = \omega(G)$.

4. For a sequence $x = x_1x_2 \ldots x_n$ in $\mathbb{Z}_2^n$ and an integer k with $1 \leq k \leq n$, let $P_k(x)$ be the sequence $x_1x_2 \cdots x_{k-1}x_{k+1} \cdots x_n$ in $\mathbb{Z}_2^{n-1}$ with x_k removed. For a subset $S \subset \mathbb{Z}_2^n$ on n elements show that there is a k such that $|P_k(S)| = |S|$. (Hint: Let H be the induced subgraph $Q_n[S]$ where each edge xy is given colour i if x and y differ in coordinate i. Show that any subgraph F of H having different colours on all edges can have at most $n-1$ edges.)

5. Let G be a simple graph on $n \geq 4$ vertices with (strictly) more than $3(n-1)/2$ edges. Prove that G contains a subgraph consisting of three internally vertex-disjoint paths connecting the same pair of vertices.

6. Prove that one can remove any $k-1$ edges from the complete bipartite graph $K_{k,k}$ so that the resulting graph still has a 1-factor (a perfect matching).

7. Construct a connected balanced bipartite graph on $2n$ vertices, such that each edge is contained in exactly one 1-factor. What if the maximum degree is required to be at least three? (Hint: The last question is difficult, but you can use *Kotzigs theorem:* that any bipartite graph has an *even* number of Hamilton cycles.)

8. If G and H are two graphs on the same vertex set. Show that

$$\chi(G \cup H) \leq \chi(G) \cdot \chi(H).$$

9. A graph with chromatic number $\chi(G) = k$, but, with $\chi(G - v) = k - 1$ for all $v \in V(G)$ is said to be *critical k-chromatic.* Show that any critical k-chromatic graph G is $(k-1)$-edge-connected, i.e. it contains no edge-cut-set of size less than or equal to $(k-2)$. (Hint: You can the result stated in Problem 6, although it is not necessary.)

10. If T is a tree, show that if $\varphi : V(T) \to V(T)$ is an auto-morphism, then φ fixes some vertex or edge. That is, either there is some vertex $x \in V(T)$ such that $\varphi(x) = x$ or some edge $\{x, y\} \in E(T)$ such that $\varphi(\{x, y\}) = \{x, y\}$. Give also an example of a graph where this does not hold.

Terminology

Auto-morphism For a simple graph $G = (V, E)$, a bijective map $\phi : V \to V$ such that $\{x, y\} \in E$ if and only if $\{\phi(x), \phi(y)\} \in E$ is called an *auto-morphism.*

k-factor A k-factor of a graph $G = (V, E)$ is a spanning subgraph F which is k-regular, i.e. $d(x, F) = k$ for all $x \in V$. For balanced bipartite graphs, a one-factor is also called a *perfect matching.*

Degree and Neighbourhood The degree, $d(v)$, of a vertex v is the number of edges with which it is incident. Two vertices are adjacent if they are incident to a common edge. The set of neighbours (neighbourhood), $N(v)$, of a vertex v is the set of vertices which are adjacent to v. For a simple graph, the degree of a vertex is also the cardinality of its neighbour set.

Paths and cycles and more A walk is an alternating sequence of vertices and edges, with each edge being incident to the vertices immediately preceeding and succeeding it in the sequence. A trail is a walk with no repeated edges. A path is a walk with no repeated vertices. A walk is closed if the initial vertex is also the terminal vertex. A cycle is a closed trail with at least one edge and with no repeated vertices except that the initial vertex is the terminal vertex. We refer to paths and cycles, also identified the *subgraphs* spanned by the edges occurring.

Maximal, average and minimal degree The average degree of a graph G is

$$\bar{d}(G) := \frac{1}{|V|} \sum_{v \in V} d(v, G).$$

The minimal degree is $\delta(G) := \min_v d(v, G)$ and the maximal degree is $\Delta(G) := \max_v d(v, G)$. Clearly, $\delta(G) \leq \bar{d}(G) \leq \Delta(G)$.

Spanning tree A tree is a connected graph without cycles. Every connected graph on n vertices has at least one tree as a spanning subgraph — a spanning tree.

Internally disjoint Paths are internally vertex disjoint if the corresponding vertex-sets only intersect at end-points.

Induced subgraphs For a set of vertices X, we use $G[X]$ to denote the induced subgraph of G whose vertex set is X and whose edge set is the subset of $E(G)$ consisting of those edges with both ends in X. For a set S of edges, we use $G[S]$ to denote the edge induced subgraph of G whose edge set is S and whose vertex set is the subset of $V(G)$ consisting of those vertices incident with any edge in S. If Y is a subset of $V(G)$, we write $G - Y$ for the subgraph $G[V(G) - Y]$.

Clique, $\omega(G)$ A (sub-)graph graph is complete, or a clique, if every pair of distinct vertices is adjacent. We write K_m for the (isomorphism class of) complete graph on m vertices. We write $\omega(G)$ for the largest clique of a graph. (The clique-number of G.)

Hamiltonian and Eulerian A Hamiltonian graph is a graph that has a Hamilton-cycle. An Eulerian graph is one that admits an Euler-cycle, i.e. connected and all vertices have even degree.

Colouring and proper colouring A colouring simply means an assignment $\sigma : V \to S$, where S is a finite set of colours. A *proper* colouring is such that adjacent vertices receives different colours. A proper edge-colouring is a mapping $\sigma : E \to S$ such that no two incident edges obtain the same colour.

Orienting a graph An undirected graph can be *oriented*, i.e. made into a digraph $\overrightarrow{G}$, by choosing, for each edge $e = \{u, v\}$ one of (u, v) or (v, u) as the corresponding edge in $\overrightarrow{G}$.

Bipartite graph A bipartite graph is a graph such that V is composed of two non-empty disjoint parts X and Y and all edges connects vertices in X with vertices in Y. A bipartite graph is *balanced* if $|X| = |Y|$.

k-regular A graph is k-regular if every vertex has degree k.

Number of components of a graph G is denoted by $c(G)$.

Oriented cycle In a digraph an *oriented walk* is a sequence $v_1 e_1 v_2 \cdots v_k e_k v_{k+1}$, of vertices v_i and edges e_j, such that either $e_i = v_i v_{i+1}$ or $e_i = v_{i+1} v_i$; i the second case we say the edge is oppositely oriented. Oriented trails, circuits, paths and cycles are defined in an analogue manner.

Directed cycle A normal, i.e. not oriented, cycle in a digraph.

Strongly connected A digraph is strongly connected if for any pair of vertices (i, j) there is a directed path from i to j. (And thus also a directed path from j to i.)

Cut-set, k-edge-connected A cut-set in a connected graph G is a set of edges W such that $G - W$ is not connected. A graph is k-connected if every cut-set has at least k edges.

k-connected, $\kappa(G)$ A connected graph G is *k-connected* (or k-vertex connected) if $|V(G)| \geq k+1$ and $G - S$ is connected for any set $S \subset V(G)$ of at most $k-1$ vertices. In other words, no two vertices are separated by a set of $k - 1$ vertices. The largest k, such that G is k-connected, is called the (vertex-) connectivity of G and is denoted $\kappa(G)$.