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Matematiska institutionen
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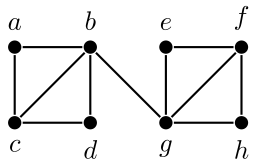
Graph Theory
Frist, KandMa, IT
2010–12–10

Problem sheet 4 — Exam type problems

Solve (or discuss) a subset of, say, four questions for the problem session.

1. Consider the following graph G.



(a) State the minimum, maximum and *average* degree of G.

(b) State the clique-number $\omega(G)$ and the chromatic number $\chi(G)$.

(c) Draw the tree $B(G)$ given by *blocks and cut-vertices*. (Biconnected components and articulation vertices.)

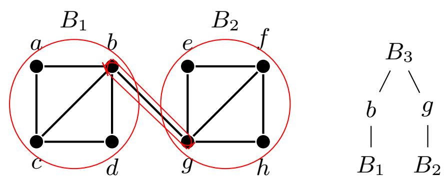
(d) Give the number $\tau(G)$ of spanning trees in G.

(e) Determine the *chromatic polynomial* $P(G, \lambda)$ of G.

Solution: (a) $\delta(G) = 2$, $\Delta(G) = 4$ and $\bar{d}(G) = 2 \cdot 11/8 = 11/4$.

(b) $\chi(G)$: $\chi(G) = \omega(G) = 3$

(c) By inspection we have three blocks: B_1, B_2 and $B_3 =$ the bridge bg and the cut vertices: b and g.



(d) Since every tree contains bg, the number of trees is

$$\tau(G) = \tau(B_1) \cdot \tau(B_2) = (4 + 4) \cdot 1 \cdot (4 + 4) = 64.$$

(e) We can colour b and g in $\lambda(\lambda - 1)$ different ways and each choice gives us a colouring of the two paths acd and efh using $\lambda - 1$ colours. Hence,

$$P(G, \lambda) = \lambda(\lambda - 1) \cdot \left(P(\bullet\!-\!\bullet\!-\!\bullet, \lambda - 1)\right)^2 .$$

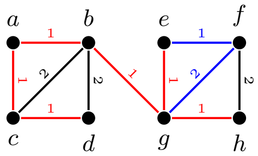
$\square$

2. Consider the graph $G = (V, E)$ depicted in Problem 1. Construct a weighting $w : E \to \{1, 2\}$ such that (a) the weighted graph has a unique minimum spanning tree. (b) has more than one minimum spanning tree.

Solution: In (a) we use the weight 1 on any spanning tree and the weight 2 on the complement. In (b) we use the weight 1 on all edges and then any spanning tree is minimal. $\square$

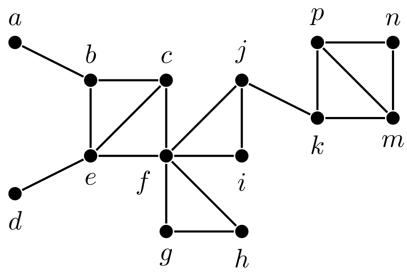
3. Consider the graph $G = (V, E)$ depicted in Problem 1. Construct a weighting $w : E \to \{1, 2\}$ such that the weighted graph has exactly two distance-trees (shortest-path trees) rooted at a.

Solution: The following assignment of lengths/weights give us exactly two distance-trees: We can choose between the edges $\{ef, gf\}$.



□

4. Consider the following graph G.



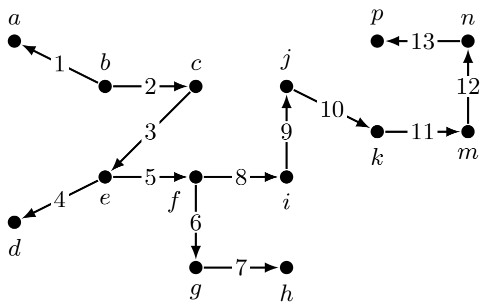
(a) Find the depth first tree of G, rooted at b, provided the vertices are ordered alphabetically.

(b) Determine the *blocks* and *cut-vertices* of G.

(c) Draw or explain how to construct a graph with the same number of spanning trees.

(d) Determine the number of spanning trees in G.

Solution: (a) Using the depth first search algorithm, we obtain the depicted tree where numbers indicate the sequence when edges are added to the tree

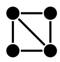


(b) By inspection — or by analysing the back-edges in (a): The blocks are the sub graphs induced by

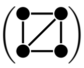
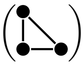
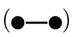
$$\{a, b\}, \{b, c, e, f\}, \{e, d\}, \{f, g, h\}, \{f, i, j\}, \{j, k\}, \{k, m, n, p\}.$$

Cut vertices are b, e, f, j, k.

(c) The number of spanning trees $\tau(G)$ is given by the product of $\tau(B_i)$ over blocks B_i. Hence we obtain the same number of trees if we replace a block by any 2-connected graph having the same number of spanning trees. For instance, we can replace

 with 

(d) The number of trees $\tau(G)$ is given by the product of $\tau(B_i)$ over blocks B_i.

τ () $= 8$, τ () $= 3$, τ () $= 1$.

□

5. (a) Give an example of (loop-free) graph with chromatic number $\chi(G) = 3$, but G contains no triangle K_3 as a subgraph. (b) Describe two *types* of graphs, where $\chi(G) = \Delta + 1$.

Solution: (a) Any odd cycle C_{2k+1} will do. (b) Complete graphs and odd cycles. (By Brook's theorem, these are the only types.) □

6. Let $n \geq 1$.

(a) How many edges has the hypercube Q_n?

(b) For what values of n, does Q_n have an Euler circuit?

(c) What is the length of the longest path in Q_n?

Solution: (a) $|V(Q_n)| = 2^n$ and Q_n is n-regular. Hence, by the handshake lemma,

$$2|E(Q_n)| = 2^n \cdot n \implies |E(Q_n)| = n2^{n-1}.$$

(b) An Euler circuit exists if and only if the degree n at each vertex is even.
(c) We know, (Fact!) that Q_n is Hamiltonian — has a Hamilton cycle H. Hence, the longest path must be a Hamilton path of length $|V(Q_n)| - 1 = 2^n - 1$, since we can remove any edge of H. (The fact is established by induction on n.) □

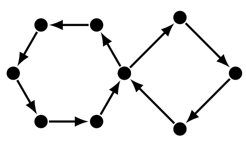
7. (a) For which graphs does it hold that every induced subgraph is connected? (b) Construct a graph G with minimum degree $\delta(G) \geq 3$ such that no induced subgraph is isomorphic to the 3-star $K_{1,3}$. (It is also called a claw-free graph.)

Solution: (a) Complete graphs — any non-edge induces a disconnected graph: .
(b) We can take K_n, $n \geq 4$ — Any set of 4 vertices then induces a K_4-graph. □

8. (a) Give an example of a strongly connected simple digraph without a directed Hamiltonian path.

(b) Prove that a connected directed graph G is *strongly connected* if the out-degree $d_+(v, G)$ equals the in-degree $d_-(v, G)$ at each vertex v.

Solution: (a) We can take two directed cycles joined by a cut-vertex as in the figure below.

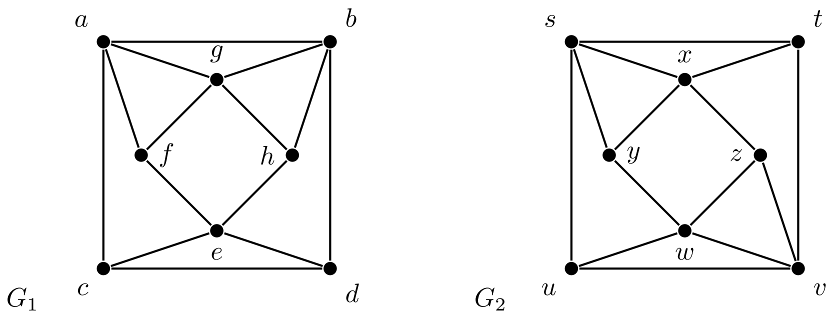


It is clearly non-hamiltonian and is strongly connected by (b) below.

(b) The directed version of Euler's theorem, states that such graphs have an directed Eulerian circuit C. For any pair of vertices u and v, we can construct a directed uv-walk by following the circuit C (from u) until we reach v.

□

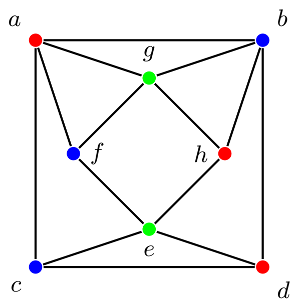
9. Consider the following two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.



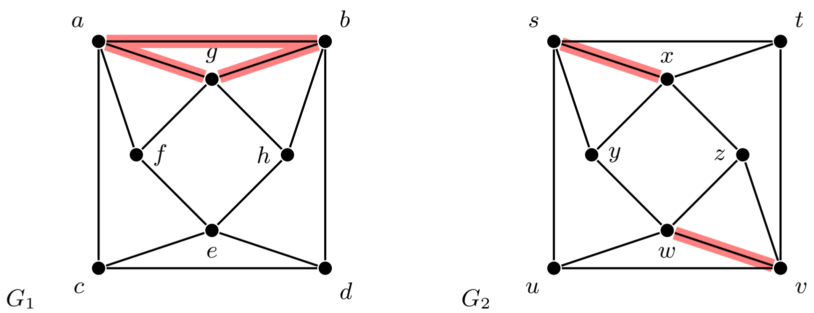
(a) Define a proper 3-colouring $\sigma : V(G_1) \to \{1, 2, 3\}$.

(b) Determine whether G_1 and G_2 are isomorphic graphs.

Solution: (a) Here is a colouring with colours red, green and blue



(b) They are not isomorphic, as can be seen as follows: If we take the sub graphs H_1 and H_2 induced by vertices of degree four in G_1 and G_2, respectively, then we obtain non-isomorphic graphs.



□

10. (a) What is the minimum n such that the complement $\overline{C_n}$ of the n-cycle C_n is Hamiltonian. (b) What is the maximum $\alpha(G)$ possible for a Hamiltonian graph G on n vertices. (c) What is the maximum minimum degree $\delta(G)$ among non-Hamiltonian graphs G on n vertices.

Solution: (a) The answer is $n = 5$. For $n = 5$ it is clear, since the complement of C_5 is isomorphic to C_5. For $n \geq 6$ it follows from Dirac's theorem — that any graph with minimum degree $\delta(G) \geq n/2$, has a Hamilton cycle. The graph $\overline{C_n}$ is $n - 1 - 2 = n - 3$-regular and, since $n - 3 \geq n/2$ for $n \geq 6$, it follows.
(b) Clearly, $\alpha(G) = \alpha(C_n) = n/2$, since G is obtained from $H \cong C_n$ by adding edges — we can not increase the independence number α by adding edges.
(c) By Dirac's theorem it is $n/2$. □

11. Let $T = (V, E)$ is a *complete* binary tree. Let $l(T)$ be the number of leaves, $i(T)$ the number of internal vertices, and $n = l(T) + i(T)$ the number of vertices.

(a) Prove that $l(T) = i(T) + 1$.

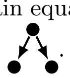
(b) Give the minimum height of a complete binary tree if the number of leaves is 33.

Solution: Definitions: An internal vertex is a vertex with at least one child and a binary tree is complete if every internal vertex has exactly two children. The root is therefore internal unless the tree is a tree on one vertex.

(a) Let u be an internal vertex in T with the level $\ell(u)$ maximum. Then u has two leaves l and l' as children. Removing l and l' results in a tree T' with u a leaf. By induction, we can assume that $l(T') = i(T') + 1$ holds for T'. (It holds for the the tree on 1 vertex.)

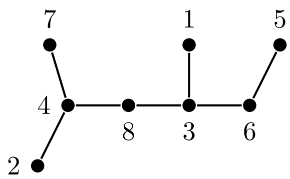
(b) As is easily seen by induction, the number of leaves are at most $2^{h(T)}$, with equality if and only if the tree is perfect — all leaves x are on level $\ell(x) = h(T)$. Hence,

$$h(T) \geq \lceil \log_2 l(T) \rceil = \lceil \log_2 33 \rceil = 5 + 1 = 6.$$

We obtain equality by starting from a perfect tree with 32 leaves and then replace one leaf ● with the sub tree . This gives 33 leaves.

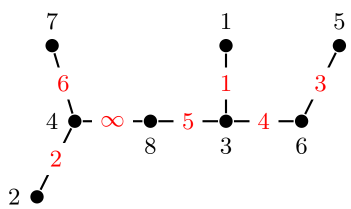
□

12. (a) Find the six-digit Prüfer code for the following tree



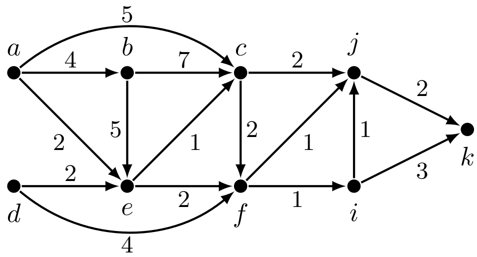
(b) Which trees have a code with only one integer occurring?

Solution: (a) The code obtained is 3, 4, 6, 3, 8, 4. Below we number the edges in the order they are removed from the tree in the construction of the code.



(b) The stars $K_{1,r}$! □

13. Consider the following weighted network $G = (V, E, w)$.

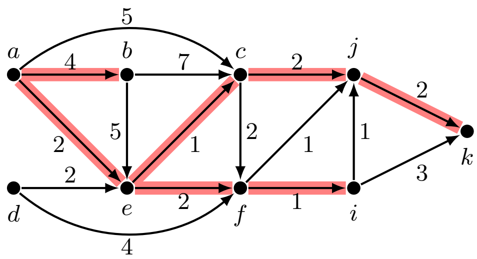


(a) Interpret the weights as *lengths* and find a shortest path-tree (directed case) rooted at a. You should also label each node with its (directed) distance from a. (If there is no directed path from a the label is ∞.)

(b) If you interpret the weights as *costs* and want to find a *minimum-cost* flow with source $a = +2$ and sinks $f = -1$ and $k = -1$, show that the distance tree obtained in (a) give a *basic* optimal solution by demonstrating a *basic optimal dual solution.*

(c) Interpret the weights as capacities for a flow network. Two producers situated at a and d can produce 2 units each of a commodity. The consumer at k requires 4 units. Use the max-flow min-cut theorem and demonstrate a maximum flow and a corresponding minimum cut to answer the question.

Solution: (a) We obtain, using Dijkstra's algorithm, the following tree



(We can replace the edge cj with the edge fj if we choose to.)
The labels are given by the (directed) distance from a in this tree (which also is the distance in G.) We get,

$$L(a) = 0,\ L(b) = 4,\ L(e) = 2,\ L(c) = 3,\ L(f) = 4,\ L(j) = L(i) = 5, L(k) = 7,\ \text{and}\ L(d) = \infty$$

We can check optimality by Bellman's condition: There is no edge (i, j) such that $L(j) > L(i) + \ell(ij)$.

(b) The tree gives us a basic solution x, given by

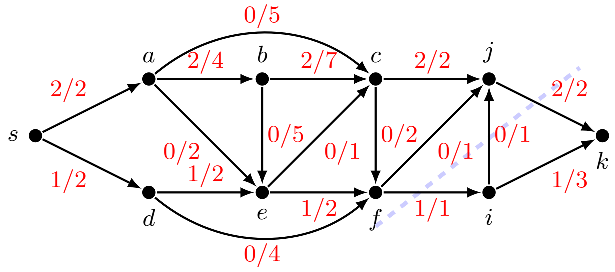
$$x(ac) = 2,\ x(ec) = x(cj) = x(jk) = 1,\ x(ef) = 1,$$

and $x(ij) \equiv 0$ for all other edges $ij \in E(G)$.
Bellman's condition gives that $L(j) - L(i) \geq \ell(ij)$ for all edges ij in G. This means that $y(i) = -L(i)$ is a feasible dual solution, i.e.

$$y(i) - y(j) \leq \ell(ij)$$

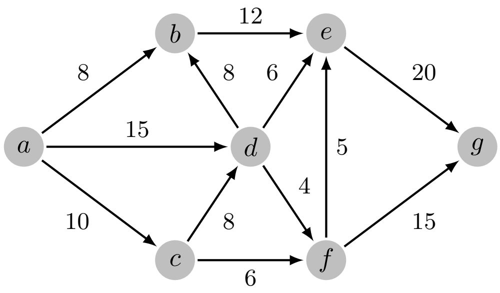
for all edges $ij \in E(G)$. Moreover, equality holds if ij belongs to the tree. It follows that x and y is a pair of primal and dual (basic) feasible solutions satisfying complementary slackness — thus they are optimal.

(c) We add a source s connected to a and d with capcities 2. In this way we make it a two-point flow problem. The optimal flow is obtained by adding augmenting path-flows according to the Ford–Fulkerson method. We obtain the following optimal flow of value 3 with the minimum cut given by $S = \{s, a, d, b, e, c, f, j\}$ of capacity 3. There is thus no possibility to obtain a flow that satisfy demands.



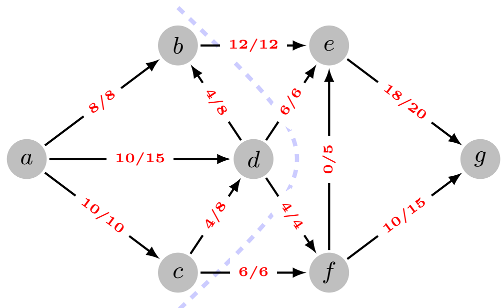
□

14. Consider the following weighted network $G = (V, E, c)$.



Interpret the weights as capacities for a flow network and find a maximum flow from node a to node g together with a minimum cut.

Solution: The following pair of flow and cut are clearly optimal.



□

15. For the following two weighted alphabets, construct a corresponding Huffman code in the form of a Huffman tree. Describe the construction of the tree. (a) For the alphabet $\mathcal{A} = \{A, B, C, D, E, F\}$ with weights $\{8, 9, 12, 34, 2, 39\}$. (b) For the alphabet $\mathcal{A} = \{a, b, c, d, e, f\}$ with weights $\{3, 7, 1, 12, 31, 15\}$.

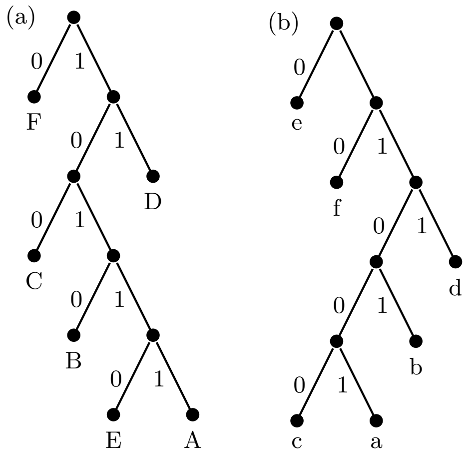
Solution: (a) Writing the weights in order and reducing according to Huffman's algorithm gives

$2, 8, 9, 12, 34, 39 \to 9, (2, 8), 12, 34, 39 \to 12, (9, (2, 8)), 34, 39 \to (12, (9, (2, 8))), 34, 39 \to 39, ((12, (9, (2, 8))), 34)$

Giving the tree $(F, ((C, (B, (E, A))), D))$.
(b) Writing the weights in order and reducing according to Huffman's algorithm gives

$1, 3, 7, 12, 15, 31 \to (1, 3), 7, 12, 15, 31 \to ((1, 3), 7), 12, 15, 31 \to 15, (((1, 3), 7), 12), 31 \to 31, (15, (((1, 3), 7), 12)),$

which corresponds to the tree $(e, (f, (((c, a), b), d)))$.



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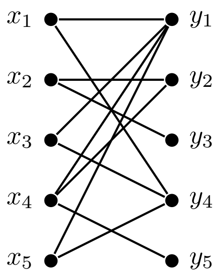
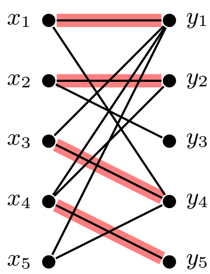
16. (a) Find a maximum matching in the graph below and use Hall's theorem to prove optimality.



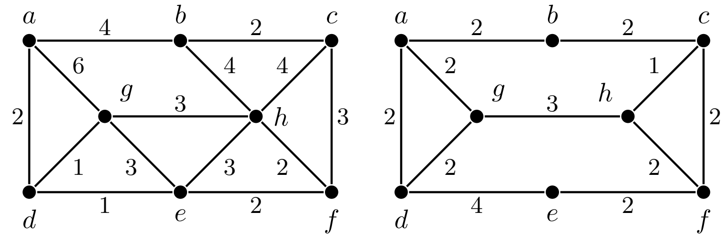
(b) Describe the equivalent maximum-flow problem and the corresponding minimum cut.

Solution: We see that with $S = \{x_1, x_3, x_5\}$ we have $N(S) = \{y_1, y_4\}$, so S has deficiency 1. A maximum matching can thus have at most 4 edges as the matching depicted below has. It is thus optimal.

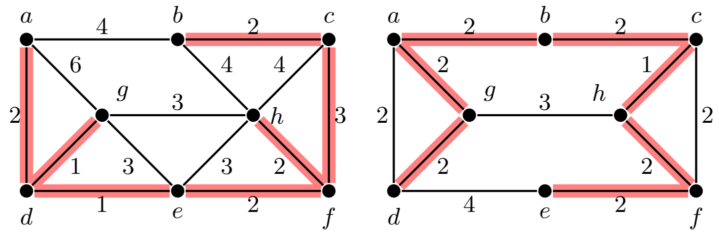


□

17. (a) Find one minimum spanning trees in the following weighted graphs, using a greedy algorithm; Prim's algorithm or Kruskal's algorithm. State which algorithm you use and the sequence in which the edges are added to tree. (b) Determine the number of minimum spanning trees for each graph.



Solution: We use Prim's algorithm for the first graph and Kruskal's for the second.



The order of edges added is, using Prim's in the first graph is $de, dg, da, ef, fh, fc, cb$. In the second graph (Kruskal's) it is $hc, ef, dg, ab, bc, fh, ga$.
(b) We see that the first tree is unique, so there is only one MST for the first graph. In the second graph, we can choose two edges from the cycle $\{ag, gd, da\}$ at random (3 choices) and then choose one of the edges fc or fh, to construct a tree of the same weight. We have thus 6 different MSTs. □

18. Construct a graph G according to the following recipe. Let S be a set of 5 elements. The vertices of G consists of all subsets of S of size 2 and two such subsets are adjacent in G if they are disjoint.

(a) Draw the graph G. (You do not need to name the vertices of G.)

(b) What is the chromatic number of G?

(c) Does G have a perfect matching (1-factor)?

(d) How many edges must be added to G in order for G to have a Hamiltonian cycle?

Solution: (a) It is the Petersen graph.

(b) We can colour with 3 colours.

(c) Yes, it has a perfect matching.

(d) One edge; we already have a Hamilton path between non-adjacent pairs on the "outer" or "inner" cycle.

□

19. (a) State Eulers formula for a planar map $G = (V, E, F)$.

(b) If G is a 5-regular simple graph and $|V| = 10$, prove that G is non-planar.

Solution: (a) Let $G = (V, E, F)$ be a planar map. Then

$$|V| - |E| + |F| = 2.$$

(b) To prove this, one can argue as follows. Assume G is planar with a map $(G = (V, E; F)$. By regularity, we have $|E| = 5 \cdot 10/2 = 25$. We obtain for Euler's theorem

$$2 = 10 - 25 + |F| \implies |F| = 17.$$

Since G is simple, we have that $2|E| \geq 3|F|$; each face is incident with at least 3 edges and each edge is incident with two faces. ("Dual" handshake lemma) But this is a contradiction, since $3 \cdot 17 > 2 \cdot 25$.

□

20. For a graph $G = (V, E)$.

(a) Define what a *greedy* colouring of G is.

(b) Prove that $\chi(G) \leq \Delta + 1$, where Δ is the maximum degree.

(c) If $\chi(G - v) = \chi(G) - 1$ for all vertices $v \in V$, show that G is connected.

Solution: (a) Given a graph $G = (V, E)$ with an ordering $V = \{v_{\pi_1}, v_{\pi_2}, \ldots, v_{\pi_n}\}$ of the vertices. The greedy (proper) colouring of G is the mapping $f : V \to \Gamma = \{1, 2, 3, \ldots\}$, recursively defined for $k = 1, 2, 3, \ldots, n$ by

$$f(v_k) = \min \Gamma \setminus \left\{ f(v_j) : j < k,\ v_j \in N(v_k) \right\}.$$

In other words, $f(v_k)$ is defined to be the minimal colour that has not been used on any neighbour v_j to v_k that precedes v_k in the ordering.

(b) Since each vertex v_k can have at most Δ neighbours preceding v_k, it follows that $f(v_k) \leq \Delta + 1$. Hence, we use at most $\Delta + 1$ colours in any greedy colouring.

(c) If G has two components G_1 and G_2 then

$$\chi(G) = \max\{\chi(G_1), \chi(G_2)\},$$

and we can assume $\chi(G) = \chi(G_2)$. But, if $v \in V(G_1)$ then

$$\chi(G - v) = \max\{\chi(G_1 - v), \chi(G_2)\} = \chi(G_2) = \chi(G),$$

since $\chi(H - v) \leq \chi(H)$. Contradiction.

□

21. Prove that if the complete graph K_n, $n \geq 4$, can be decomposed into edge-disjoint cycles of length 4, then $n = 1 \mod 8$.

Solution: We shall prove that (a) $n \equiv 1 \mod 2$ and (b) $\binom{n}{2} \equiv 0 \mod 4$. It follows from (a) that $n(n-1)$ is divisible by 8 but since (a) gives that n is not divisible by 2, it follows that $n - 1$ must be divisible by $8 = 2^3$.
To show (a), it is enough to note that the 4-cycles that contain a fixed vertex i partitions the $n - 1$ edges incident to i and hence $n - 1$ is divisible by 2.
To show (b), it is enough to note that $\binom{n}{2}$ must be divisible by 4 since the decomposition partitions the $\binom{n}{2}$ edges of K_n into sets of size four. □

22. (a) State Euler's Theorem relating the number of faces, edges and vertices for a planar graph.

(b) State and prove a bound for the minimum degree of a simple planar graph.

Solution: (a) Let $G = (V, E, F)$ be a planar map. Then

$$|V| - |E| + |F| = 2.$$

(b) The best possible bound is that

$$\delta(G) \leq 5.$$

(An icosahedron graph shows that this is best possible.)

To prove this, one can argue as follows. First, we have that $2|E| \geq 3|F|$ since each face is incident with at least 3 edges and each edge is incident with two faces. ("Dual" handshake lemma) Further, $2|E| \geq \delta|V|$ by the handshake lemma. Hence, we obtain

$$2 = |V| - |E| + |F| \leq |V| - \delta|V|/2 + \delta|V|/3$$

$$\iff \frac{2}{|V|} \leq 1 - \frac{\delta}{6}.$$

Since l.h.s. in the last inequality is > 0 this shows that $\delta < 6$, i.e. $\delta \leq 5$ since δ is an integer.

□

23. (a) Show that a graph is bipartite if and only if every *induced* cycle, i.e., every induced subgraph isomorphic to a cycle, has even length. (b) Show that a planar bridge-less graph G is bipartite if and only if a dual graph G^* is an eulerian graph.

Solution: (a) We shall show that every cycle is even, since this is equivalent to a graph being bipartite. Let $C \subset G$ be an odd cycle of minimum length, then C is not induced and thus there must be a chord $uv \in E(G)$, where $u, v \in V(C)$. But the chord gives two cycles C_1 and C_2 containing uv and segments of C. Clearly, $\ell(C) = \ell(C_1) + \ell(C_2) - 2$ and thus one of C_1 and C_2 are an odd cycle of length less than C. Contradiction.

(b) Let (V, E, F) be a map corresponding to the dual graph G^*. If G is bipartite then each face $f \in F$ has an even boundary cycle and hence the degree of $f \in V(C^*)$ is even.

For the other direction one may argue similarly s in (a) above: The assumption implies that each face is an even cycle. Take an odd cycle C bounding a minimal region D in the plane. Then C can not be a face by assumption and there must be a path P in the interior of D connecting $u, v \in V(C)$. But $C + P$ spans two cycles C_1 and C_2 of which one must be odd and bounding a smaller region. Contradiction. □

24. Prove that for $n \geq 2$ the hypercube Q_n has a Hamilton cycle.

Solution: Use induction on n; the case $n = 2$ is clear. The graph Q_n contains two induced copies Q^0_{n-1} and Q^1_{n-1} isomorphic to Q_{n-1}; the isomorphisms $f^i : V(Q_{n-1}) \to V(Q^i_{n-1})$, $i = 0, 1$ are given by

$$f^i(x_1x_2 \ldots x_{n-1}) = x_1x_2 \ldots x_{n-1}\, i, \quad i = 0, 1.$$

Note that $f^0(v)$ is adjacent to $f^1(v)$ for every $v \in V(Q_n)$. By induction we have a Hamilton cycle $H = v_1v_2 \ldots v_Nv_1 \subset Q_{n-1}$ from which we obtain the Hamilton cycle

$$f^0(v_1)f^0(v_2)\cdots f^0(v_N)f^1(v_N)f^1(v_{N_1})\ldots f^1(v_2)f^1(v_1)f^0(v_1).$$

□

25. Given a connected graph $G = (V, E)$ and $a \in V$. Prove that G is bipartite if and only if for all edges $xy \in E$

$$\mathrm{dist}_G(x, a) \neq \mathrm{dist}_G(y, a).$$

Solution: Assume G is bipartite; we shall show that equality of distances may not hold.
Let $a \neq x, y$ and let T be an (undirected) distance tree rooted at a. That is, all paths T_{av} in T with a and v as end-points is also shortest paths between a and v in G, so that $\mathrm{dist}_G(a, v) = \ell(T_{av})$.
If the edge $xy \in T$ then, clearly,

$$|\,\mathrm{dist}_G(x, a) - \mathrm{dist}_G(y, a)| = |\ell(T_{ax}) - \ell(T_{ay})| = 1,$$

since one of the paths T_{ax} and T_{ay} must use the edge xy, i.e. $T_{ax} = T_{ay} + xy$ or $T_{ay} = T_{ax} + xy$.
If $xy \notin T$ then $T + xy$ has the unique cycle C_{xy} and let v be the vertex common to C_{xy}, T_{ax} and T_{ay}. Then $\ell(T_{ax}) = \ell(T_{av} + \ell(T_{vx})$ and $\ell(T_{ay}) = \ell(T_{av}) + \ell(T_{vy})$ and

$$\ell(C_{xy}) = \ell(T_{vx}) + \ell(T_{vy}) + 1.$$

If $\mathrm{dist}_G(x, a) = \mathrm{dist}_G(y, a)$ then $\ell(T_{vx}) = \ell(T_{vy})$ and this implies that $\ell(C_{xy})$ is odd. Contradiction.
For the other direction we shall prove that if G is not bipartite there is an edge xy and $a \in V$ such that $\mathrm{dist}_G(x, a) = \mathrm{dist}_G(y, a)$.
Notice that G is bipartite if and only if all *isometric* cycles are even; that is, cycles C such that, for all $u, v \in V(C)$, $\mathrm{dist}_G(u, v) = \mathrm{dist}_C(u, v)$. If C is an odd cycle of minimum length then C must be an isometric cycle, since the existence of a shorter path P_{uv} in G, for a pair of vertices $u, v \in V(C)$, would give a shorter odd cycle. If C is an odd isometric cycle with an edge $xy \in C$, then if we take $v \in V(C)$ as the antipodal node, we clearly have $\mathrm{dist}(x, v) = \mathrm{dist}(y, v)$, where the distance is both C and G distance. □

26. Use Hall's theorem to prove that every k-regular bipartite graph has a perfect matching (1-factor).

Solution: A k-regular bipartite graph with bipartition X and Y is necessarily balanced since $|E(G)| = k|X| = k|Y|$ which implies $|X| = |Y|$. Hence, it is enough to show that we have an X-complete matching. Hall's theorem — the "non-quantitative" version states that G has an X-complete matching if and only if $|N(S)| \geq |S|$. This condition is deduced by counting the edges $E(S, N(S))$ in the graph induced by S and $N(S)$: We have

$$|E(S, N(S))| = k|S| \leq k|N(S)| \implies |S| \leq |N(S)|.$$

(Not all edges incident with vertices in $N(S)$ have to be in $E(S, N(S))$, but all edges incident with vertices in S must.) □

27. Show that there is some tree with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ *if and only if* $d_1 \geq 1$ and

$$\sum_{i=1}^{n} d_i = 2(n-1).$$

Solution: The "if" part follows since T must be connected ($\implies d_1 \geq 1$) and every tree has $n-1$ edges, which, by the handshake lemma, implies the identity for the sum of degrees.
To see that there is some tree T with the sequence d satisfying the conditions: Let d' be a sequence

$$d' = (d_2, \ldots, d_k - 1, \ldots, d_n), 2 \leq k \leq n$$

where $d_k \geq 2$; such k must exist since otherwise the sum would be equal to n. By induction we have a tree T' on $n-1$ vertices $\{v_2, \ldots, v_k, \ldots, v_n\}$ with degrees given by d' and we obtain T as the tree $T' + v_1 v_k$. □

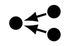
28. Find a recurrence relation for a_n, $n \geq 0$, where a_n denotes the number of *independent vertex sets* in a path P_n of length n.

Solution: Let $V(P_n) = \{0, 1, 2, 3, \ldots, n\}$ and $\mathcal{I}_n$ denote the independent sets in P_n. (We include the empty set $\emptyset$ in $\mathcal{I}_n$.) A general independent set $I\mathcal{I}_n$ has the form

$$I = \begin{cases} I',\ I' \in \mathcal{I}_{n-1} & \text{n} \notin I \\ \{n\} \cup I'',\ I'' \in \mathcal{I}_{n-2} & \text{n} \in I. \end{cases}$$

We obtain the recursion $a_n := |\mathcal{I}_n| = a_{n-1} + a_{n-2}$, where it is easy to check that $a_0 = 2$ and $a_1 = 3$ by inspection. □

29. (a) Prove that every *tournament* has a directed Hamilton path. (b) Construct a tournament on $n = 5$ vertices having the maximum number of directed 3-cycles.

Solution: (Hints)
(a) Let $P = v_1 v_2 \dots v_k$ be a longest directed path and $v \notin V(P)$. Then there is some j such that $v_j v \in E$ and $v v_{j+1} \in E$. (The argument is in Grimaldi)
(b) Notice that every non-directed 3-cycle corresponds uniquely to a pair of in-edges at a vertex 
It follows that the number of directed 3-cycles in a tournament on n vertices is given by

$$\binom{n}{3} - \sum_{v=1}^{n} \binom{d_-(v)}{2},$$

where we have the constraint $\sum_{v=1}^{n} d_-(v) = \binom{n}{2}$.
To maximise this, we should make the in degrees $d_-(v)$ "as equal" as possible; which for $n = 5$ means an Eulerian orientation. □

30. (a) For each $n \geq 1$, give an example of a *color-critical* graph $G = (V, E)$ with $|V| = n$, i.e. a graph G such that $\chi(G - v) < \chi(G)$ for all $v \in V$.
(b) If G is color-critical, show that $d(v, G) \geq \chi(G) - 1$ for all $v \in V$.

Solution: (a) The only choice is K_n.
(b) Assume $\chi(G - v) = k = \chi(G) - 1$ and that $d(v) \leq k - 1$. Then a k-colouring of $G - v$ can be extend to G since there is some colour not used on $N(v)$. Contradiction.

□

31. Assume that G is a simple graph containing a cycle $C \subset G$. Assume that two vertices $x, y \in V(C)$ on C are connected in G by a path P of length k. Show that G contains a cycle of length at least $\sqrt{2k}$.

Solution: Let $l = \ell(C)$, The path P is divided into $P = P_1 P_2 \dots P_r$, $1 \leq r \leq l$ of minimal edge-disjoint sub paths P_i of P having endpoints in $V(C)$. We assume $r \geq 2$; if $r = 1$ then we obtain a cycle of length at least $l/2 + k \geq \sqrt{2k}$ since $l \geq 3$ and $k \geq 1$.
From any pair P_i, P_j, $1 \leq i < j \leq r$, we can construct a cycle C_{ij} that contains these paths and segments of C and the length of the constructed cycle is thus

$$\ell(C_{ij}) \geq \ell(P_i) + \ell(P_j).$$

By averaging we can assume that $\ell(P_i) + \ell(P_j) \geq 2k/r \geq 2k/l$ and thus the length of a longest cycle in G is greater or equal to $\max\{l, 2k/l\}$ but this is greater than the geometric mean $(l \cdot k(2k/l))^{1/2} = \sqrt{2k}$. □

32. Let $T_1, T_2, \dots, T_k$ be subtrees of a fixed tree T which pairwise have vertices in common, i.e. $V(T_i) \cap V(T_j) \neq \emptyset$ for all $1 \leq i, j \leq k$. Show that

$$V(T_1) \cap V(T_2) \cap \cdots \cap V(T_k) \neq \emptyset.$$

Solution: Use induction on k. (For $k = 1$ there is nothing to prove.)
Let $S = V(T_2) \cap \cdots \cap V(T_k)$, which by induction is non-empty. We shall show that $V(T_1) \cap S \neq \emptyset$ and assume for contradiction that $V(T_1) \cap S = \emptyset$. Note that T_1 can only contain vertices from one component F_1 in the forest $F = G - S$ — otherwise T_1 would be a disconnected graph. Furthermore there must be a tree, say T_i, such that $V(T_i) \cap V(F_1) = \emptyset$ — otherwise $V(T_j)$, $j = 2, \dots, k$, all should contain the unique vertex $w \in V(F_1)$ connected to S and that would mean $w \in S$. But this shows that $V(T_1) \cap V(T_i) = \emptyset$. Contradiction. □

33. (a) Prove that in every directed graph $\overrightarrow{G}$, the set of vertices of in-degree zero is an independent set in the underlying undirected graph G.

(b) For a directed graph $\overrightarrow{G}$, let $L(\overrightarrow{G})$ denote the maximum length of a directed path in $\overrightarrow{G}$. For a given undirected graph G, show that the chromatic number

$$\chi(G) = 1 + \min_{\overrightarrow{G}} L(\overrightarrow{G})$$

where the minimum is taken over all acyclic orientations of G. (Acyclic means that it contains no directed cycles.)

Solution: (a) Obvious: If u and v only have out-edges there can be no edge uv since that would mean v has an in-edge.

(b) Assume $\overrightarrow{G}$ is the acyclic orientation that minimises $L = L(\overrightarrow{G})$ and S the set of vertices with in-degree zero in $\overrightarrow{G}$.

To show $\chi(G) \leq 1 + L$, we use induction on L. When $L = 0$ this is trivial since then G is an empty graph.

We show that $L(\overrightarrow{G} - S) \leq L - 1$: since $\overrightarrow{H} = \overrightarrow{G} - S$ must be the acyclic orientation of $H = G - S$ minimising $L(\overrightarrow{H})$ — otherwise if we have a longest directed path $P \subset \overrightarrow{H}$ of length $\geq L + 1$ then the start vertex v has an in-edge xv in $\overrightarrow{G}$, but since $\overrightarrow{H}$ is acyclic the path $P' = xv + P$ is a path of length $\ell(P) + 1$ in $\overrightarrow{G}$.

Induction then gives that $\chi(H) \leq 1 + L(\overrightarrow{H}) \leq L$ and since any k-colouring of H can be extended to $(k+1)$-colouring of G by colouring all vertices in S with a new colour, we obtain that $\chi(G) \leq L + 1$.

To show $\chi(G) \geq 1 + L$ note that from a colouring $f : V \to \{1, \ldots, k\}$ we can orient the graph acyclically by orienting each edge $\{x, y\} \in G$ to $(x, y) \in \overrightarrow{G}$ for $f(x) < f(y)$, i.e. "from lower to higher colour." The length of longest directed path is clearly less than $k - 1$ since the colour number increases strictly along the path.

□

Terminology

Auto-morphism For a simple graph $G = (V, E)$, a bijective map $\phi : V \to V$ such that $\{x, y\} \in E$ if and only if $\{\phi(x), \phi(y)\} \in E$ is called an *auto-morphism.*

k-factor A k-factor of a graph $G = (V, E)$ is a spanning subgraph F which is k-regular, i.e. $d(x, F) = k$ for all $x \in V$. A one-factor is also called a *perfect matching.*

Decomposition A set of edge-disjoint subgraphs which together cover all edges in the given graph.

Degree and Neighbourhood The degree, $d(v)$, of a vertex v is the number of edges with which it is incident. Two vertices are adjacent if they are incident to a common edge. The set of neighbours (neighbourhood), $N(v)$, of a vertex v is the set of vertices which are adjacent to v. For a simple graph, the degree of a vertex is also the cardinality of its neighbour set.

Paths and cycles and more A walk is an alternating sequence of vertices and edges, with each edge being incident to the vertices immediately preceeding and succeeding it in the sequence. A trail is a walk with no repeated edges. A path is a walk with no repeated vertices. A walk is closed if the initial vertex is also the terminal vertex. A cycle is a closed trail with at least one edge and with no repeated vertices except that the initial vertex is the terminal vertex. We refer to paths and cycles, also identified the *subgraphs* spanned by the edges occurring.

Independent set of vertices A set $S \subset V(G)$ is independent if no two vertices in S are adjacent, i.e. the graph induced by S is an empty graph.

Maximal, average and minimal degree The average degree of a graph G is

$$\bar{d}(G) := \frac{1}{|V|} \sum_{v \in V} d(v, G).$$

The minimal degree is $\delta(G) := \min_v d(v, G)$ and the maximal degree is $\Delta(G) := \max_v d(v, G)$. Clearly, $\delta(G) \leq \bar{d}(G) \leq \Delta(G)$.

Spanning tree A tree is a connected graph without cycles. Every connected graph on n vertices has at least one tree as a spanning subgraph — a spanning tree.

Internally disjoint Paths are internally vertex disjoint if the corresponding vertex-sets only intersect at end-points.

Induced subgraphs For a set of vertices X, we use $G[X]$ to denote the induced subgraph of G whose vertex set is X and whose edge set is the subset of $E(G)$ consisting of those edges with both ends in X. For a set S of edges, we use $G[S]$ to denote the edge induced subgraph of G whose edge set is S and whose vertex set is the subset of $V(G)$ consisting of those vertices incident with any edge in S. If Y is a subset of $V(G)$, we write $G - Y$ for the subgraph $G[V(G) - Y]$.

Clique, $\omega(G)$ A (sub-)graph graph is complete, or a clique, if every pair of distinct vertices is adjacent. We write K_m for the (isomorphism class of) complete graph on m vertices. We write $\omega(G)$ for the largest clique of a graph. (The clique-number of G.)

Hamiltonian and Eulerian A Hamiltonian graph is a graph that has a Hamilton-cycle. An Eulerian graph is one that admits an Euler-cycle, i.e. connected and all vertices have even degree.

Colouring and proper colouring A colouring simply means an assignment $\sigma : V \to S$, where S is a finite set of colours. A *proper* colouring is such that adjacent vertices receives different colours. A proper edge-colouring is a mapping $\sigma : E \to S$ such that no two incident edges obtain the same colour.

Orienting a graph An undirected graph can be *oriented*, i.e. made into a digraph $\overrightarrow{G}$, by choosing, for each edge $e = \{u, v\}$ one of (u, v) or (v, u) as the corresponding edge in $\overrightarrow{G}$.

Bipartite graph A bipartite graph is a graph such that V is composed of two non-empty disjoint parts X and Y and all edges connects vertices in X with vertices in Y. A bipartite graph is *balanced* if $|X| = |Y|$.

k-regular A graph is k-regular if every vertex has degree k.

Number of components of a graph G is denoted by $c(G)$.

Oriented cycle In a digraph an *oriented walk* is a sequence $v_1 e_1 v_2 \cdots v_k e_k v_{k+1}$, of vertices v_i and edges e_j, such that either $e_i = v_i v_{i+1}$ or $e_i = v_{i+1} v_i$; i the second case we say the edge is oppositely oriented. Oriented trails, circuits, paths and cycles are defined in an analogue manner.

Directed cycle A normal, i.e. not oriented, cycle in a digraph.

Strongly connected A digraph is strongly connected if for any pair of vertices (i, j) there is a directed path from i to j. (And thus also a directed path from j to i.)

Cut-set, k-edge-connected,bridge A cut-set in a connected graph G is a set of edges W such that $G - W$ contains at least two components. A graph is k-edge connected if every cut-set has at least k edges. A bridge is a cut-set of one edge.

k-connected, $\kappa(G)$ A connected graph G is *k-connected* (or k-vertex connected) if $|V(G)| \geq k+1$ and $G-S$ is connected for any set $S \subset V(G)$ of at most $k-1$ vertices. In other words, no two vertices are separated by a set of $k-1$ vertices. The largest k, such that G is k-connected, is called the (vertex-) connectivity of G and is denoted $\kappa(G)$.

Cut-vertex, block A cut-vertex is a vertex v such that $G - v$ have more components than G. A *block* a is a maximal (induced) subgraph without any cut-vertex, a maximal 2-connected graph or a bridge.

Tournament A tournament is a complete oriented graph, i.e. a simple directed graph such that the underlying undirected graph is isomorphic to K_n for some n.