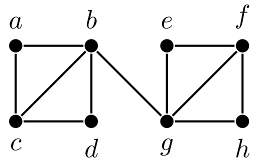
UPPSALA UNIVERSITET
Matematiska institutionen
Anders Johansson

Graph Theory
Frist, KandMa, IT
2010–12–10

Problem sheet 4 — Exam type problems

Solve (or discuss) a subset of, say, four questions for the problem session.

1. Consider the following graph G.



(a) State the minimum, maximum and *average* degree of G.

(b) State the clique-number $\omega(G)$ and the chromatic number $\chi(G)$.

(c) Draw the tree $B(G)$ given by *blocks and cut-vertices.* (Biconnected components and articulation vertices.)

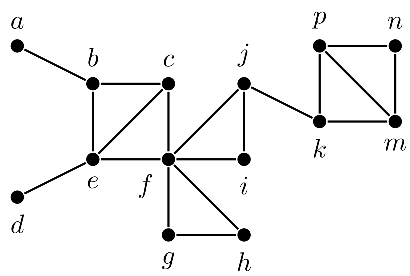
(d) Give the number $\tau(G)$ of spanning trees in G.

(e) Determine the *chromatic polynomial* $P(G, \lambda)$ of G.

2. Consider the graph $G = (V, E)$ depicted in Problem 1. Construct a weighting $w : E \to \{1, 2\}$ such that (a) the weighted graph has a unique minimum spanning tree. (b) has more than one minimum spanning tree.

3. Consider the graph $G = (V, E)$ depicted in Problem 1. Construct a weighting $w : E \to \{1, 2\}$ such that the weighted graph has exactly two distance-trees (shortest-path trees) rooted at a.

4. Consider the following graph G.



(a) Find the depth first tree of G, rooted at b, provided the vertices are ordered alphabetically.

(b) Determine the *blocks* and *cut-vertices* of G.

(c) Draw or explain how to construct a graph with the same number of spanning trees.

(d) Determine the number of spanning trees in G.

5. (a) Give an example of (loop-free) graph with chromatic number $\chi(G) = 3$, but G contains no triangle K_3 as a subgraph. (b) Describe two *types* of graphs, where $\chi(G) = \Delta + 1$.

6. Let $n \geq 1$.

(a) How many edges has the hypercube Q_n?

(b) For what values of n, does Q_n have an Euler circuit?

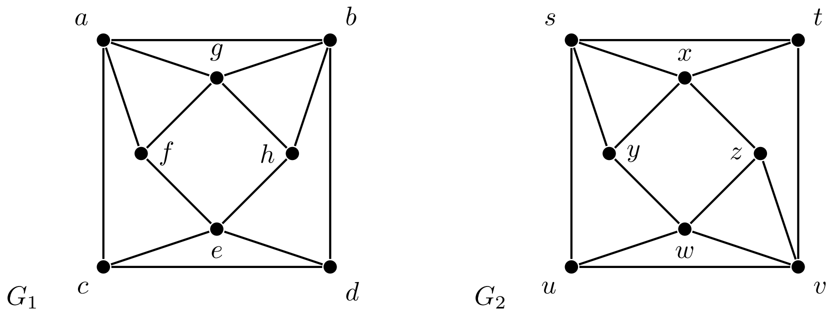
(c) What is the length of the longest path in Q_n?

7. (a) For which graphs does it hold that every induced subgraph is connected? (b) Construct a graph G with minimum degree $\delta(G) \geq 3$ such that no induced subgraph is isomorphic to the 3-star $K_{1,3}$. (It is also called a claw-free graph.)

8. (a) Give an example of a strongly connected simple digraph without a directed Hamiltonian path.

(b) Prove that a connected directed graph G is *strongly connected* if the out-degree $d_+(v, G)$ equals the in-degree $d_-(v, G)$ at each vertex v.

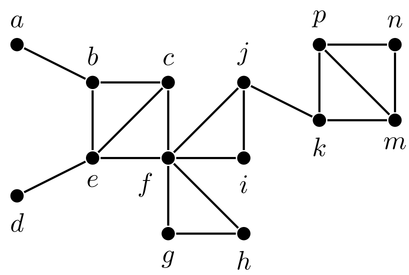
9. Consider the following two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.



(a) Define a proper 3-colouring $\sigma : V(G_1) \to \{1, 2, 3\}$.

(b) Determine whether G_1 and G_2 are isomorphic graphs.

10. Consider the following graph G.



(a) Find the depth first tree of G, rooted at b, provided the vertices are ordered alphabetically.

(b) Determine the *blocks* and *cut-vertices* of G.

(c) Draw or explain how to construct a graph with the same number of spanning trees.

(d) Determine the number of spanning trees in G.

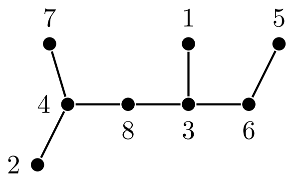
11. (a) What is the minimum n such that the complement $\overline{C_n}$ of the n-cycle C_n is Hamiltonian. (b) What is the maximum $\alpha(G)$ possible for a Hamiltonian graph G on n vertices. (c) What is the maximum minimum degree $\delta(G)$ among non-Hamiltonian graphs G on n vertices.

12. Let $T = (V, E)$ is a *complete* binary tree. Let $l(T)$ be the number of leaves, $i(T)$ the number of internal vertices, and $n = l(T) + i(T)$ the number of vertices.

(a) Prove that $l(T) = i(T) + 1$.

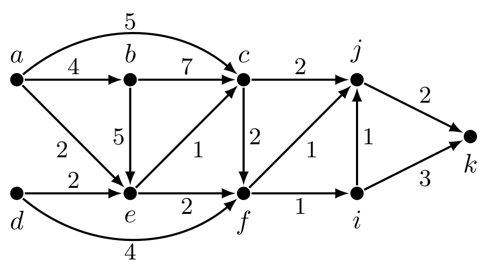
(b) Give the minimum height of a complete binary tree if the number of leaves is 33.

13. (a) Find the six-digit Prüfer code for the following tree



(b) Which trees have a code with only one integer occurring?

14. Consider the following weighted network $G = (V, E, w)$.

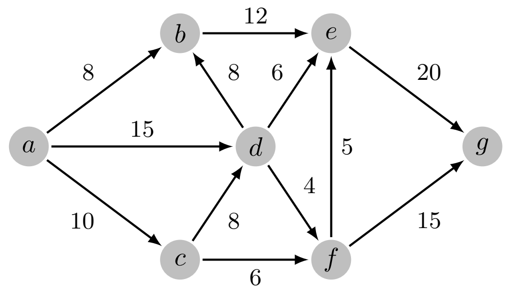


(a) Interpret the weights as *lengths* and find a shortest path-tree (directed case) rooted at a. You should also label each node with its (directed) distance from a. (If there is no directed path from a the label is ∞.)

(b) If you interpret the weights as *costs* and want to find a *minimum-cost* flow with source $a = +2$ and sinks $f = -1$ and $k = -1$, show that the distance tree obtained in (a) give a *basic* optimal solution by demonstrating a *basic optimal dual solution.*

(c) Interpret the weights as capacities for a flow network. Two producers situated at a and d can produce 2 units each of a commodity. The consumer at k requires 4 units. Use the max-flow min-cut theorem and demonstrate a maximum flow and a corresponding minimum cut to answer the question.

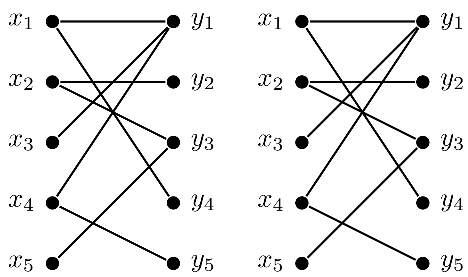
15. Consider the following weighted network $G = (V, E, c)$.



Interpret the weights as capacities for a flow network and find a maximum flow from node a to node g together with a minimum cut.

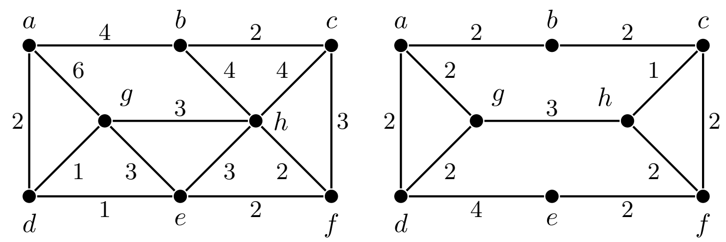
16. For the following two weighted alphabets, construct a corresponding Huffman code in the form of a Huffman tree. Describe the construction of the tree. (a) For the alphabet $\mathcal{A} = \{A, B, C, D, E, F\}$ with weights $\{8, 9, 12, 34, 2, 39\}$. (b) For the alphabet $\mathcal{A} = \{a, b, c, d, e, f\}$ with weights $\{3, 7, 1, 12, 31, 15\}$.

17. (a) Find a maximum matching in the graphs below and use Hall's theorem to prove optimality.



(b) Describe for first graph the equivalent maximum-flow problem and the corresponding minimum cut.

18. (a) Find one minimum spanning trees in the following weighted graphs, using a greedy algorithm; Prim's algorithm or Kruskal's algorithm. State which algorithm you use and the sequence in which the edges are added to tree. (b) Determine the number of minimum spanning trees for each graph.



19. Construct a graph G according to the following recipe. Let S be a set of 5 elements. The vertices of G consists of all subsets of S of size 2 and two such subsets are adjacent in G if they are disjoint.

(a) Draw the graph G. (You do not need to name the vertices of G.)

(b) What is the chromatic number of G?

(c) Does G have a perfect matching (1-factor)?

(d) How many edges must be added to G in order for G to have a Hamiltonian cycle?

20. (a) State Eulers formula for a planar map $G = (V, E, F)$.

(b) If G is a 5-regular graph and $|V| = 10$, prove that G is non-planar.

21. For a graph $G = (V, E)$.

(a) Define what a *greedy* colouring of G is.

(b) Prove that $\chi(G) \leq \Delta + 1$, where Δ is the maximum degree.

(c) If $\chi(G - v) = \chi(G) - 1$ for all vertices $v \in V$, show that G is connected.

22. Prove that if the complete graph K_n, $n \geq 4$, can be decomposed into edge-disjoint cycles of length 4, then $n = 1 \mod 8$.

23. (a) State Euler's Theorem relating the number of faces, edges and vertices for a planar graph.

(b) State and prove a bound for the minimum degree of a simple planar graph.

24. (a) Show that a graph is bipartite if and only if every *induced* cycle, i.e., every induced subgraph isomorphic to a cycle, has even length. (b) Show that a planar bridge-less graph G is bipartite if and only if the dual graph G^* is an eulerian graph.

25. Prove that for $n \geq 2$ the hypercube Q_n has a Hamilton cycle.

26. Given a connected graph $G = (V, E)$ and $a \in V$. Prove that G is bipartite if and only if for all edges $xy \in E$

$$\mathrm{dist}_G(x, a) \neq \mathrm{dist}_G(y, a).$$

27. Use Hall's theorem to prove that every k-regular bipartite graph has a perfect matching (1-factor).

28. Show that there is some tree with degree sequence $d_1 \le d_2 \le \cdots \le d_n$ *if and only if* $d_1 \ge 1$ and

$$\sum_{i=1}^{n} d_i = 2(n-1).$$

29. Find a recurrence relation for a_n, $n \ge 0$, where a_n denotes the number of *independent vertex sets* in a path P_n of length n.

30. (a) Prove that every *tournament* has a directed Hamilton path. (b) Construct a tournament on $n = 6$ vertices having the maximum number of directed 3-cycles.

31. (a) For each $n \ge 1$, give an example of a *color-critical* graph $G = (V, E)$ with $|V| = n$, i.e. a graph G such that $\chi(G - v) < \chi(G)$ for all $v \in V$.

(b) If G is color-critical, show that $d(v, G) \ge \chi(G) - 1$ for all $v \in V$.

32. Assume that G is a simple graph containing a cycle $C \subset G$. Assume that two vertices $x, y \in V(C)$ on C are connected in G by a path P of length k. Show that G contains a cycle of length at least $\sqrt{2k}$.

33. Let $T_1, T_2, \ldots, T_k$ be subtrees of a fixed tree T which pairwise have vertices in common, i.e. $V(T_i) \cap V(T_j) \ne \emptyset$ for all $1 \le i, j \le k$. Show that

$$V(T_1) \cap V(T_2) \cap \cdots \cap V(T_k) \ne \emptyset.$$

34. (a) Prove that in every directed graph $\overrightarrow{G}$, the set of vertices of in-degree zero is an independent set in the underlying undirected graph G.

(b) For a directed graph $\overrightarrow{G}$, let $L(\overrightarrow{G})$ denote the maximum length of a directed path in $\overrightarrow{G}$. For a given undirected graph, show that the chromatic number

$$\chi(G) = 1 + \min_{\overrightarrow{G}} L(\overrightarrow{G})$$

where the minimum is taken over all acyclic orientations of G. (Acyclic means that it contains no directed cycles.)

Terminology

Auto-morphism For a simple graph $G = (V, E)$, a bijective map $\phi : V \to V$ such that $\{x, y\} \in E$ if and only if $\{\phi(x), \phi(y)\} \in E$ is called an *auto-morphism.*

k-factor A k-factor of a graph $G = (V, E)$ is a spanning subgraph F which is k-regular, i.e. $d(x, F) = k$ for all $x \in V$. A one-factor is also called a *perfect matching.*

Decomposition A set of edge-disjoint subgraphs which together cover all edges in the given graph.

Degree and Neighbourhood The degree, $d(v)$, of a vertex v is the number of edges with which it is incident. Two vertices are adjacent if they are incident to a common edge. The set of neighbours (neighbourhood), $N(v)$, of a vertex v is the set of vertices which are adjacent to v. For a simple graph, the degree of a vertex is also the cardinality of its neighbour set.

Paths and cycles and more A walk is an alternating sequence of vertices and edges, with each edge being incident to the vertices immediately preceeding and succeeding it in the sequence. A trail is a walk with no repeated edges. A path is a walk with no repeated vertices. A walk is closed if the initial vertex is also the terminal vertex. A cycle is a closed trail with at least one edge and with no repeated vertices except that the initial vertex is the terminal vertex. We refer to paths and cycles, also identified the *subgraphs* spanned by the edges occurring.

Independent set of vertices A set $S \subset V(G)$ is independent if no two vertices in S are adjacent, i.e. the graph induced by S is an empty graph.

Maximal, average and minimal degree The average degree of a graph G is

$$\bar{d}(G) := \frac{1}{|V|} \sum_{v \in V} d(v, G).$$

The minimal degree is $\delta(G) := \min_v d(v, G)$ and the maximal degree is $\Delta(G) := \max_v d(v, G)$. Clearly, $\delta(G) \leq \bar{d}(G) \leq \Delta(G)$.

Spanning tree A tree is a connected graph without cycles. Every connected graph on n vertices has at least one tree as a spanning subgraph — a spanning tree.

Internally disjoint Paths are internally vertex disjoint if the corresponding vertex-sets only intersect at end-points.

Induced subgraphs For a set of vertices X, we use $G[X]$ to denote the induced subgraph of G whose vertex set is X and whose edge set is the subset of $E(G)$ consisting of those edges with both ends in X. For a set S of edges, we use $G[S]$ to denote the edge induced subgraph of G whose edge set is S and whose vertex set is the subset of $V(G)$ consisting of those vertices incident with any edge in S. If Y is a subset of $V(G)$, we write $G - Y$ for the subgraph $G[V(G) - Y]$.

Clique, $\omega(G)$ A (sub-)graph graph is complete, or a clique, if every pair of distinct vertices is adjacent. We write K_m for the (isomorphism class of) complete graph on m vertices. We write $\omega(G)$ for the largest clique of a graph. (The clique-number of G.)

Hamiltonian and Eulerian A Hamiltonian graph is a graph that has a Hamilton-cycle. An Eulerian graph is one that admits an Euler-cycle, i.e. connected and all vertices have even degree.

Colouring and proper colouring A colouring simply means an assignment $\sigma : V \to S$, where S is a finite set of colours. A *proper* colouring is such that adjacent vertices receives different colours. A proper edge-colouring is a mapping $\sigma : E \to S$ such that no two incident edges obtain the same colour.

Orienting a graph An undirected graph can be *oriented*, i.e. made into a digraph $\overrightarrow{G}$, by choosing, for each edge $e = \{u, v\}$ one of (u, v) or (v, u) as the corresponding edge in $\overrightarrow{G}$.

Bipartite graph A bipartite graph is a graph such that V is composed of two non-empty disjoint parts X and Y and all edges connects vertices in X with vertices in Y. A bipartite graph is *balanced* if $|X| = |Y|$.

k-regular A graph is k-regular if every vertex has degree k.

Number of components of a graph G is denoted by $c(G)$.

Oriented cycle In a digraph an *oriented walk* is a sequence $v_1e_1v_2 \cdots v_ke_kv_{k+1}$, of vertices v_i and edges e_j, such that either $e_i = v_iv_{i+1}$ or $e_i = v_{i+1}v_i$; i the second case we say the edge is oppositely oriented. Oriented trails, circuits, paths and cycles are defined in an analogue manner.

Directed cycle A normal, i.e. not oriented, cycle in a digraph.

Strongly connected A digraph is strongly connected if for any pair of vertices (i, j) there is a directed path from i to j. (And thus also a directed path from j to i.)

Cut-set, k-edge-connected,bridge A cut-set in a connected graph G is a set of edges W such that $G - W$ contains at least two components. A graph is k-edge connected if every cut-set has at least k edges. A bridge is a cut-set of one edge.

k-connected, $\kappa(G)$ A connected graph G is *k-connected* (or k-vertex connected) if $|V(G)| \geq k+1$ and $G-S$ is connected for any set $S \subset V(G)$ of at most $k-1$ vertices. In other words, no two vertices are separated by a set of $k-1$ vertices. The largest k, such that G is k-connected, is called the (vertex-) connectivity of G and is denoted $\kappa(G)$.

Cut-vertex, block A cut-vertex is a vertex v such that $G - v$ have more components than G. A *block* a is a maximal (induced) subgraph without any cut-vertex, a maximal 2-connected graph or a bridge.

Tournament A tournament is a complete oriented graph, i.e. a simple directed graph such that the underlying undirected graph is isomorphic to K_n for some n.