1. Consider the following graph $G$.

   ![Graph Diagram](image)

   (a) State the minimum, maximum and average degree of $G$.
   (b) State the clique-number $\omega(G)$ and the chromatic number $\chi(G)$.
   (c) Give the number of spanning trees in $G$.

2. Consider the graph $G = (V,E)$ depicted in Problem 1. Construct a weighting $w : E \rightarrow \{1, 2\}$ such that (a) the weighted graph has a unique minimum spanning tree. (b) has at least two minimum spanning trees.

3. (a) State Eulers formula for a planar map $G = (V,E,F)$.
   (b) If $G$ is a 5-regular graph and $|V| = 10$, prove that $G$ is non-planar.

4. (a) Draw the Petersen graph.
   (b) Use Kuratowski’s theorem to prove that the Petersen graph is non-planar.

5. Let $n \geq 1$.
   (a) How many edges has the hypercube $Q_n$?
   (b) What is the length of the longest path in $Q_n$?
   (c) For what values of $n$, does $Q_n$ have an Euler circuit?

6. Prove that if the complete graph $K_n$, $n \geq 4$, can be decomposed into edge-disjoint cycles of length 4, then $n = 1 \mod 8$.

7. (a) Give an example of a strongly connected simple digraph without a directed Hamiltonian path.
(b) Prove that a connected directed graph $G$ is strongly connected if the out-degree $d_+(v,G)$ equals the in-degree $d_-(v,G)$ at each vertex $v$.

8. (a) Draw the Petersen graph.
(b) Determine the chromatic number of the Petersen graph $P = (V,E)$.
(c) Can you construct a proper 3-colouring $\sigma : V(P) \rightarrow \{1,2,3\}$ of $P$ such that the third colour class consists of only one vertex, i.e. $|\sigma^{-1}\{3\}| = 1$.

9. For a graph $G = (V,E)$.
(a) Prove that $\chi(G) \leq \Delta + 1$, where $\Delta$ is the maximum degree.
(b) If $\chi(G - v) = \chi(G) - 1$ for all vertices $v \in V$, show that $G$ is connected.

10. Find a recurrence relation for $a_n$, $n \geq 0$, where $a_n$ denotes the number of independent vertex sets in a path $P_n$ of length $n$.

11. Prove König’s theorem: The edge-chromatic number $\chi'(G)$ of a bipartite graph is given by $\Delta(G)$, the maximum degree.

12. Prove that every tournament has a directed Hamilton path.

13. Consider the following graph $G$.

(a) Find the depth first tree of $G$, rooted at $b$, provided the vertices are ordered alphabetically.
(b) Determine the blocks and cut-vertices of $G$.
(c) Draw or explain how to construct a graph with the same number of spanning trees.
(d) Determine the number of spanning trees in $G$.

14. Let $T = (V,E)$ is a complete binary tree. Let $\ell$ be the number of leaves, $i$ the number of internal vertices, and $n$ the number of vertices.
(a) Prove that $\ell = i + 1$.

(b) Give the minimum height has a binary decision tree if the *total number of outcomes* (leafs) is 33.

15. For the alphabet $\mathcal{A} = \{a, b, c, d, e, f\}$ with weights $\{3, 7, 1, 12, 31, 15\}$, construct a corresponding Huffman code $\varphi : \mathcal{A} \to \mathbb{Z}_2^*$ and a Huffman tree. State also the optimal weight of the code.

16. Consider the following network $G = (V, E, w)$.

   ![Network Diagram]

   (a) Interpret the weights as lengths and find a shortest path-tree (directed case) rooted at $a$. You should also label each node with its (directed) distance from $a$. (If there is no directed path from $a$ the label is $\infty$.)

   (b) Interpret the weights as capacities for a flow network. Two producers situated at $a$ and $d$ can produce 2 units each of a commodity. The consumer at $k$ requires 4 units. Use the max-flow min-cut theorem and demonstrate a maximum flow and a corresponding minimum cut to answer the question.

17. Find a minimum spanning tree in the following weighted graph, using a greedy algorithm.

   ![Minimum Spanning Tree Diagram]

18. Use Hall’s theorem to prove that every $k$-regular bipartite graph has a perfect matching (1-factor).

19. Find a maximum matching in the graph below and use Hall’s theorem to show that it is indeed maximum.
20. Given a connected graph $G = (V, E)$ and $a \in V$. Prove that $G$ is bipartite if and only if for all edges $xy \in E$ \[ \text{dist}_G(x, a) \neq \text{dist}_G(y, a). \]

21. Show that a graph is bipartite if and only if every induced cycle, i.e., every induced subgraph isomorphic to a cycle, has even length.

22. (a) Prove that in every directed graph $\overrightarrow{G}$, the set of vertices of in-degree zero is an independent set in the underlying undirected graph $G$.

(b) For a directed graph $\overrightarrow{G}$, let $L(\overrightarrow{G})$ denote the maximum length of a directed path in $\overrightarrow{G}$. For a given undirected graph, show that

\[ \chi(G) = 1 + \min_{\overrightarrow{G}} L(\overrightarrow{G}) \]

where the minimum is taken over all acyclic orientations of $G$.

Terminology

Auto-morphism For a simple graph $G = (V, E)$, a bijective map $\phi : V \to V$ such that $\{x, y\} \in E$ if and only if $\{\phi(x), \phi(y)\} \in E$ is called an auto-morphism.

$k$-factor A $k$-factor of a graph $G = (V, E)$ is a spanning subgraph $F$ which is $k$-regular, i.e. $d(x, F) = k$ for all $x \in V$. For balanced bipartite graphs, a one-factor is also called a perfect matching.

Decomposition A set of edge-disjoint subgraphs which together cover all edges in the given graph.

Degree and Neighbourhood The degree, $d(v)$, of a vertex $v$ is the number of edges with which it is incident. Two vertices are adjacent if they are incident to a common edge. The set of neighbours (neighbourhood), $N(v)$, of a vertex $v$ is the set of vertices which are adjacent to $v$. For a simple graph, the degree of a vertex is also the cardinality of its neighbour set.

Paths and cycles and more A walk is an alternating sequence of vertices and edges, with each edge being incident to the vertices immediately preceding and succeeding it in the sequence. A trail is a walk with no repeated edges. A path is a walk with no repeated vertices. A walk is closed if the initial vertex is also the terminal vertex. A cycle is a closed trail with at least one edge and with no repeated vertices except that the initial vertex is the terminal vertex. We refer to paths and cycles, also identified the subgraphs spanned by the edges occurring.
Independent set of vertices A set $S \subset V(G)$ is independent if no two vertices in $S$ are adjacent, i.e. the graph induced by $S$ is an empty graph.

Maximal, average and minimal degree The average degree of a graph $G$ is

$$
\bar{d}(G) := \frac{1}{|V|} \sum_{v \in V} d(v,G).
$$

The minimal degree is $\delta(G) := \min_{v} d(v,G)$ and the maximal degree is $\Delta(G) := \max_{v} d(v,G)$. Clearly, $\delta(G) \leq \bar{d}(G) \leq \Delta(G)$.

Spanning tree A tree is a connected graph without cycles. Every connected graph on $n$ vertices has at least one tree as a spanning subgraph — a spanning tree.

Internally disjoint Paths are internally vertex disjoint if the corresponding vertex-sets only intersect at end-points.

Induced subgraphs For a set of vertices $X$, we use $G[X]$ to denote the induced subgraph of $G$ whose vertex set is $X$ and whose edge set is the subset of $E(G)$ consisting of those edges with both ends in $X$. For a set $S$ of edges, we use $G[S]$ to denote the edge induced subgraph of $G$ whose edge set is $S$ and whose vertex set is the subset of $V(G)$ consisting of those vertices incident with any edge in $S$. If $Y$ is a subset of $V(G)$, we write $G - Y$ for the subgraph $G[V(G) - Y]$.

Clique, $\omega(G)$ A (sub-)graph graph is complete, or a clique, if every pair of distinct vertices is adjacent. We write $K_m$ for the (isomorphism class of) complete graph on $m$ vertices. We write $\omega(G)$ for the largest clique of a graph. (The clique-number of $G$.)

Hamiltonian and Eulerian A Hamiltonian graph is a graph that has a Hamilton-cycle. An Eulerian graph is one that admits an Euler-cycle, i.e. connected and all vertices have even degree.

Colouring and proper colouring A colouring simply means an assignment $\sigma : V \rightarrow S$, where $S$ is a finite set of colours. A proper colouring is such that adjacent vertices receives different colours. A proper edge-colouring is a mapping $\sigma : E \rightarrow S$ such that no two incident edges obtain the same colour.

Orienting a graph An undirected graph can be oriented, i.e. made into a digraph $\vec{G}$, by choosing, for each edge $e = \{u,v\}$ one of $(u,v)$ or $(v,u)$ as the corresponding edge in $\vec{G}$.

Bipartite graph A bipartite graph is a graph such that $V$ is composed of two non-empty disjoint parts $X$ and $Y$ and all edges connects vertices in $X$ with vertices in $Y$. A bipartite graph is balanced if $|X| = |Y|$.

$k$-regular A graph is $k$-regular if every vertex has degree $k$.

Number of components of a graph $G$ is denoted by $c(G)$.

Oriented cycle In a digraph an oriented walk is a sequence $v_1e_1v_2 \cdots e_kv_{k+1}$, of vertices $v_i$ and edges $e_j$ such that either $e_i = v_iv_{i+1}$ or $e_i = v_{i+1}v_i$; i the second case we say the edge is oppositely oriented. Oriented trails, circuits, paths and cycles are defined in an analogue manner.

Directed cycle A normal, i.e. not oriented, cycle in a digraph.

Strongly connected A digraph is strongly connected if for any pair of vertices $(i,j)$ there is a directed path from $i$ to $j$. (And thus also a directed path from $j$ to $i$.)
**Cut-set, k-edge-connected, bridge** A cut-set in a connected graph $G$ is a set of edges $W$ such that $G - W$ contains at least two components. A graph is $k$-edge connected if every cut-set has at least $k$ edges. A bridge is a cut-set of one edge.

**$k$-connected, $\kappa(G)$** A connected graph $G$ is $k$-connected (or $k$-vertex connected) if $|V(G)| \geq k+1$ and $G - S$ is connected for any set $S \subset V(G)$ of at most $k-1$ vertices. In other words, no two vertices are separated by a set of $k-1$ vertices. The largest $k$, such that $G$ is $k$-connected, is called the (vertex-) connectivity of $G$ and is denoted $\kappa(G)$.

**Cut-vertex, block** A cut-vertex is a vertex $v$ such that $G - v$ have more components than $G$. A block $a$ is a maximal (induced) subgraph without any cut-vertex, a maximal 2-connected graph or a bridge.

**Tournament** A tournament is a complete oriented graph, i.e. a simple directed graph such that the underlying undirected graph is isomorphic to $K_n$ for some $n$. 