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Prov i matematik
Funktionalanalys
Kurs: F3B, F4Sy, 1MA283
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SOLUTIONS:

1. If $x \in X$ and $\sum \langle x, x_j \rangle x_j = 0$, then

$$0 = \left\langle \sum \langle x, x_j \rangle x_j, x \right\rangle = \sum |\langle x, x_j \rangle|^2.$$

Hence $x \perp x_j$ for all j, and thus $x \in Y^{\perp}$. The opposite inclusion is obvious.

2. Since $M \subset c \subset l^{\infty}$ and c is closed in l^{∞}, it suffices to show that for each $x \in c$ there exists a sequence $(x^{(n)}) \subset M$ such that $\|x^{(n)} - x\| \to 0$ as $n \to \infty$. Take $x = (\xi_j) \in c$ and let $\xi = \lim_{j \to \infty} \xi_j$. Define

$$x^{(n)} = (\xi_j^{(n)}) \in M \text{ by the formula } \xi_j^{(n)} = \begin{cases} \xi_j, & j \leq n, \\ \xi_n, & j > n. \end{cases}$$

Take $\epsilon > 0$. There exists $N > 0$ such that $|\xi_j - \xi| <$ for all $j \geq N$. Then for all $n \geq N$

$$\|x^{(n)} - x\| = \sup_{j \geq n} |\xi_n - \xi_j| \leq |\xi_n - \xi| + \sup_{j \geq n} |\xi - \xi_j| \leq 2\epsilon.$$

3. Given x we have to find $t \in \mathbf{R}$ such that $x + tv \perp u$. But this is equivalent to

$$t = -\frac{\langle x, u \rangle}{\langle v, u \rangle}.$$

Hence

$$Tx = x - \frac{\langle x, u \rangle}{\langle v, u \rangle} v.$$

The operator is bounded because

$$\|Tx\| \leq \|x\| + \frac{\|x\| \|u\| \|v\|}{\langle v, u \rangle}.$$

Clearly $\mathcal{R}(T) = u^{\perp}$. ("$\subset$" is obvious; if $y \perp u$, then $y = T(y + 0v)$, which gives the opposite inclusion.)

4. If $u = \lambda v$, then T becomes the orthogonal projection onto $u^{\perp}$.

5. $Tx = \sum \langle Tx, e_j \rangle e_j = \sum \langle x, T^* e_j \rangle e_j$.

6. Let $a_j = (\overline{\alpha}_{jk})_{k \geq 1} \in l^2$. Then $Tx = (\langle x, a_j \rangle)_{j \geq 1}$. Define

$$T_n x = (\langle x, a_1 \rangle, \ldots, \langle x, a_n \rangle, 0, 0, \ldots).$$

Then $T_n : l^2 \longrightarrow l^2$ is compact and

$$\|(T - T_n)x\|^2 = \sum_{j \geq n} |\langle x, a_j \rangle|^2 \leq \|x\|^2 \sum_{j \geq n} \|a_j\|^2.$$

Since

$$\lim_{n \to \infty} \sum_{j \geq n} \sum_{k} |\alpha_{jk}|^2 = 0,$$

the result follows.

7. $S^* = S^{-1} \Longrightarrow Q^* = S^* P^* S^{-1*} = S^{-1} P S^{**} = Q$ and $QQ = S^{-1} P S S^{-1} P S = S^{-1} P S = Q$.

8. If

$$\begin{array}{ccc} (x_n, y_n) & \in & \mathcal{G}(T_1 + T_2) \\ \downarrow & & \\ (x, y), & & \end{array}$$

then

$$\begin{array}{ccccc} y_n & = & T_1 x_n & + & T_2 x_n \\ \downarrow & & & & \downarrow \\ y & & & & T_2 x. \end{array}$$

Hence $T_1 x_n = (y_n - T_2 x_n) \to (y - T_2 x)$. Thus

$$\mathcal{G}(T_1) \ni (x_n, T_1 x_n) \to (x, y - T_2 x) \in \mathcal{G}(T_1).$$

Hence $T_1 x = y - T_2 x$ and so $y = T_1 x + T_2 x$ as required.

9. Suppose that $0 \in \rho(T)$. Then T^{-1} exists, is bounded and its range $\mathcal{R}(T)$ is dense in X. Thus $\dim \mathcal{R}(T) = \infty$. Take a sequence $(y_n) \subset \tilde{B}(0,1) \cap \mathcal{R}(T)$. The sequence $(T^{-1} y_n)$ is bounded, and so (y_n) has a bounded subsequence. Therefore $\tilde{B}(0,1) \cap \mathcal{R}(T)$ is compact. Contradiction.

10. Because $\sum (1 - 1/n)^{k-1}/n = 1$, we have $|f_n(x)| \leq \|x\|$. Moreover

$$f_n\left(\sum_{k=1}^{m} e_k\right) = \sum_{k=1}^{m} \frac{(1 - 1/n)^{k-1}}{n} \longrightarrow 1 \text{ as } m \to \infty,$$

and thus $\|f_n\| = 1$. Also $\|f\| = 1$, because $|f(x)| = |\lim_{j \to \infty} \xi_j| \leq \sup_j |\xi_j| = \|x\|$, where $x = (\xi_j)$, with equality achieved for $x = (1, 1, 1, \ldots)$. Let $x = (\xi_j)$, where $\xi_j \to \xi$. Take $\epsilon > 0$. $\exists N \forall j \geq N : |\xi_j - \xi| < \epsilon$. Therefore

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \sum_{k=1}^{\infty} \frac{(1 - 1/n)^{k-1}}{n} \xi_k - \sum_{k=1}^{\infty} \frac{(1 - 1/n)^{k-1}}{n} \xi \right| \\ &= \left| \sum_{k=1}^{\infty} \frac{(1 - 1/n)^{k-1}}{n} (\xi_k - \xi) \right| \\ &\leq 2\|x\| \sum_{k=1}^{N-1} \frac{(1 - 1/n)^{k-1}}{n} + \epsilon. \end{aligned}$$

But

$$\sum_{k=1}^{N-1} \frac{(1 - 1/n)^{k-1}}{n} \to 0 \text{ as } n \to \infty.$$