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Prov i matematik
Funktionalanalys
Kurs: F3B, F4Sy, NVP
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Tillåtna hjälpmedel: Manuella skrivdon, Kreyszigs bok *Introductory Functional Analysis with Applications* och Strömbergssons häfte *Spectral theorem for compact, self-adjoint operators.*

1. Let X and Y be normed spaces and fix some elements $f_1, f_2 \in X'$ and $y_1, y_2 \in Y$. For each $x \in X$ we define

$$T(x) = f_1(x) \cdot y_1 + f_2(x) \cdot y_2.$$

Prove that T is a bounded linear operator from X to Y.

(6p)

2. Let H_1 and H_2 be Hilbert spaces and let $T : H_1 \to H_2$ be a bounded linear operator. Prove that

$$[T(H_1)]^{\perp} = \mathcal{N}(T^*).$$

(6p)

3. (a). Prove that the set $M = \{y_1, y_2, y_3, ...\}$ is *not* total in ℓ^2 if

$$\begin{aligned} y_1 &= (1, 1, 0, 0, 0, ...) \\ y_2 &= (1, 1, 1, 0, 0, 0, ...) \\ y_3 &= (1, 1, 1, 1, 0, 0, 0, ...) \\ y_4 &= (1, 1, 1, 1, 1, 0, 0, 0, ...) \\ &\cdots \end{aligned}$$

(b). Prove that the set $M = \{x_1, x_2, x_3, ...\}$ *is* total in ℓ^2 if

$$\begin{aligned} x_1 &= (1, -1, 0, 0, 0, ...) \\ x_2 &= (1, 1, -1, 0, 0, 0, ...) \\ x_3 &= (1, 1, 1, -1, 0, 0, 0, ...) \\ x_4 &= (1, 1, 1, 1, -1, 0, 0, 0, ...) \\ &\cdots \end{aligned}$$

(*Hint:* One may eg. use Theorem 3.6-2.)

(4p)

4. Define $T : \ell^1 \to \ell^\infty$ by

$$T\big((\xi_1, \xi_2, \xi_3, ...)\big) = \Big(\textstyle\sum_{j=1}^{\infty} \xi_j \ , \ \sum_{j=2}^{\infty} \xi_j \ , \ \sum_{j=3}^{\infty} \xi_j \ , \ \sum_{j=4}^{\infty} \xi_j \ , \ ...\Big)$$

Prove that T is a bounded linear operator $T : \ell^1 \to \ell^\infty$ and compute the norm $||T||$.

(6p)

5. Let $\alpha_{n,m}$ be complex numbers with $|\alpha_{n,m}| \leq 1$ for all $n, m \geq 1$, and assume that the limit

$$\alpha_m = \lim_{n \to \infty} \alpha_{n,m}$$

exists for all $m \geq 1$. For each $n \geq 1$ we let $T_n : \ell^1 \to \ell^1$ be the bounded linear operator given by

$$T_n((\xi_1, \xi_2, \xi_3, ...)) = (\alpha_{n,1}\xi_1, \alpha_{n,2}\xi_2, \alpha_{n,3}\xi_3, ...).$$

Prove that the sequence (T_n) is strongly operator convergent. Also give an example to show that (T_n) is not necessarily uniformly operator convergent.

(6p)

6. Let X be the normed space given by

$$X = \{(\xi_n) \mid \xi_n \in \mathbb{C}, \text{ and } \exists N \in \mathbb{Z}^+ : \forall n \geq N : \xi_n = 0\},$$

$$||(\xi_n)|| := \sqrt{\sum_{n=1}^{\infty} |\xi_n|^2}.$$

Prove that X is meager in itself.

(6p)

7. Define $T : \ell^\infty \to \ell^\infty$ by

$$T((\xi_1, \xi_2, \xi_3, ...)) = (0, \xi_1, \xi_2, \xi_3, ...).$$

Prove that $\frac{1}{2} \in \sigma_r(T)$, i.e. prove that $\lambda = \frac{1}{2}$ belongs to the residual spectrum of T.

(6p)

GOOD LUCK!

Solutions

1. T is linear since for all $x_1, x_2 \in X$ and all $\alpha_1, \alpha_2 \in K$ we have

$$\begin{aligned} &T(\alpha_1 x_1 + \alpha_2 x_2) = f_1(\alpha_1 x_1 + \alpha_2 x_2) \cdot y_1 + f_2(\alpha_1 x_1 + \alpha_2 x_2) \cdot y_2 \\ &= (\alpha_1 f_1(x_1) + \alpha_2 f_1(x_2)) \cdot y_1 + (\alpha_1 f_2(x_1) + \alpha_2 f_2(x_2)) \cdot y_2 \\ &= \alpha_1 \cdot (f_1(x_1) \cdot y_1 + f_2(x_1) \cdot y_2) + \alpha_2 \cdot (f_1(x_2) \cdot y_1 + f_2(x_2) \cdot y_2) \\ &= \alpha_1 T(x_1) + \alpha_2 T(x_2). \end{aligned}$$

T is bounded since for all $x \in X$ we have

$$\begin{aligned} ||T(x)|| &= ||f_1(x) \cdot y_1 + f_2(x) \cdot y_2|| \leq ||f_1(x) \cdot y_1|| + ||f_2(x) \cdot y_2|| \\ &= |f_1(x)| \cdot ||y_1|| + |f_2(x)| \cdot ||y_2|| \\ &\leq ||f_1|| \cdot ||x|| \cdot ||y_1|| + ||f_2|| \cdot ||x|| \cdot ||y_2|| \\ &= \big(||f_1|| \cdot ||y_1|| + ||f_2|| \cdot ||y_2||\big) \cdot ||x||. \end{aligned}$$

2.

$$\begin{aligned} [T(H_1)]^{\perp} &=^1 \{y \in H_2 \mid \forall z \in T(H_1) : \langle y, z \rangle = 0\} \\ &=^2 \{y \in H_2 \mid \forall x \in H_1 : \langle y, Tx \rangle = 0\} \\ &=^3 \{y \in H_2 \mid \forall x \in H_1 : \langle T^* y, x \rangle = 0\} \\ &=^4 \{y \in H_2 \mid T^* y = 0\} \\ &=^5 \mathcal{N}(T^*). \end{aligned}$$

1. By definition of orthogonal complement.
2. By definition of $T(H_1)$.
3. By definition of T^*.
4. By Lemma 3.8-2 and the trivial fact that $\langle 0, x \rangle = 0$ for all $x \in H_1$.
5. By definition of $\mathcal{N}(T^*)$

3. Note that $y_n \perp (1, -1, 0, 0, 0, ...)$ for all $n \geq 1$. Hence by Theorem 3.6-2(a), $M = \{y_1, y_2, ...\}$ is not total in ℓ^2.

(b). Let $x = (\xi_n) \in \ell^2$ be an arbitrary vector which is orthogonal to $M = \{x_1, x_2, ...\}$. Then

$$\begin{aligned}
&(\xi_n) \perp x_1 \Longrightarrow \xi_1 - \xi_2 = 0;\\
&(\xi_n) \perp x_2 \Longrightarrow \xi_1 + \xi_2 - \xi_3 = 0;\\
&(\xi_n) \perp x_3 \Longrightarrow \xi_1 + \xi_2 + \xi_3 - \xi_4 = 0,\\
&\dots\\
&(\xi_n) \perp x_j \Longrightarrow \left(\sum_{n=1}^{j} \xi_n\right) - \xi_{j+1} = 0,
\end{aligned}$$

etc. It follows that

$$\begin{aligned}
&\xi_2 = \xi_1;\\
&\xi_3 = \xi_1 + \xi_2 = 2\xi_1;\\
&\xi_4 = \xi_1 + \xi_2 + \xi_3 = 4\xi_1\\
&\xi_5 = \xi_1 + \xi_2 + \xi_3 + \xi_4 = 8\xi_1\\
&\dots
\end{aligned}$$

We get by induction: $\xi_n = 2^{n-2}\xi_1$ for $n \geq 2$. Hence if $\xi_1 \neq 0$ then

$$\sum_{n=1}^{\infty} |\xi_n|^2 = |\xi_1|(1 + \sum_{n=2}^{\infty} 2^{2(n-2)}) = \infty.$$

This is impossible since $(\xi_n) \in \ell^2$. Hence $\xi_1 = 0$, and thus $\xi_n = 2^{n-2}\xi_1 = 0$ for all $n \geq 2$. Hence there does not exist any nonzero vector $x \in \ell^2$ which is orthogonal to every element in M. By Theorem 3.6-2(b), this proves that M is total in ℓ^2.

4. Note that if $(\xi_j) \in \ell^1$ then $\sum_{j=1}^{\infty} \xi_j$ is absolutely convergent, and hence all series $\sum_{j=n}^{\infty} \xi_j$ $(n = 1, 2, 3, ...)$ are also absolutely convergent. Hence also $\left|\sum_{j=n}^{\infty} \xi_j\right| \leq \sum_{j=n}^{\infty} |\xi_j| \leq \sum_{j=1}^{\infty} |\xi_j| = ||(\xi_j)||$, so that $T((\xi_j))$ is indeed a well-defined element in ℓ^∞.

T is linear, for if $(\xi_n), (\eta_n) \in \ell^1$ and $\alpha, \beta \in K$ then

$$T(\alpha(\xi_n) + \beta(\eta_n)) = (\nu_n),$$

where

$$\nu_n = \sum_{j=n}^{\infty} (\alpha\xi_j + \beta\eta_j) = \alpha \sum_{j=n}^{\infty} \xi_j + \beta \sum_{j=n}^{\infty} \eta_j,$$

and thus

$$T(\alpha(\xi_n) + \beta(\eta_n)) = (\nu_n) = \alpha T((\xi_n)) + \beta T((\eta_n)).$$

(The manipulations are permitted since all sums involved are absolutely convergent.)

T is bounded since

$$||T((\xi_n))|| = \sup_{n\geq 1}\left|\sum_{j=n}^{\infty}\xi_j\right| \leq \sup_{n\geq 1}\sum_{j=n}^{\infty}|\xi_j| = \sum_{j=1}^{\infty}|\xi_j| = ||(\xi_n)||$$

for all $(\xi_n) \in \ell^1$. This also proves $||T|| \leq 1$. Let $e_1 = (1,0,0,0,...)$ (vector in ℓ^1 or in ℓ^∞). Then $T(e_1) = e_1$, and $||e_1|| = 1$ both in ℓ^1 and in ℓ^∞. Hence $||T|| = 1$.

5. Let $T : \ell^1 \to \ell^1$ be the bounded linear operator given by

$$T((\xi_1,\xi_2,\xi_3,...)) = (\alpha_1\xi_1,\alpha_2\xi_2,\alpha_3\xi_3,...),$$

We claim that (T_n) is strongly operator convergent to T. Let $x = (\xi_m)$ be an arbitrary vector in ℓ^1. Then

$$\begin{aligned}||T_nx - Tx|| &= \left\|\big((\alpha_{n,1}-\alpha_1)\xi_1,(\alpha_{n,2}-\alpha_2)\xi_2,(\alpha_{n,3}-\alpha_3)\xi_3,...\big)\right\| \\ &= \sum_{m=1}^{\infty}|\alpha_{n,m}-\alpha_m|\cdot|\xi_m|.\end{aligned}$$

Let $\varepsilon > 0$. Then since $(\xi_m) \in \ell^1$ there is some M such that $\sum_{m=M+1}^{\infty}|\xi_m| < \frac{\varepsilon}{10}$. Furthermore, for each m there is some $N_m \geq 1$ such that $|\alpha_{n,m} - \alpha_m| < \frac{\varepsilon}{10M(1+|\xi_m|)}$ for all $n \geq N_m$, since $\lim_{n\to\infty}\alpha_{n,m} = \alpha_m$. Hence, for all $n \geq \max(N_1,N_2,...,N_M)$ we have:

$$\begin{aligned}||T_nx - Tx|| &= \sum_{m=1}^{M}|\alpha_{n,m}-\alpha_m|\cdot|\xi_m| + \sum_{m=M+1}^{\infty}|\alpha_{n,m}-\alpha_m|\cdot|\xi_m| \\ &\leq \sum_{m=1}^{M}\frac{\varepsilon}{10M(1+|\xi_m|)}\cdot|\xi_m| + \sum_{m=M+1}^{\infty}2\cdot|\xi_m| \\ &\leq \sum_{m=1}^{M}\frac{\varepsilon}{10M} + 2\frac{\varepsilon}{10} < \varepsilon.\end{aligned}$$

This proves that (T_n) is strongly operator convergent to T.

We now give an example to show that (T_n) is not necessarily uniformly operator convergent to T. Let

$$\alpha_{n,m} = \begin{cases}1 & \text{if } n \leq m \\ 0 & \text{if } n > m.\end{cases}$$

Then $\alpha_m = \lim_{n\to\infty}\alpha_{n,m} = 0$ for all $m \geq 1$. Hence $T = 0$ above. Hence if (T_n) is uniformly operator convergent then the limit must be $T = 0$, since (T_n) is strongly operator convergent with limit $T = 0$. We would then have $||T_n - 0|| \to 0$. However,

$$T_n((\xi_m)) = (0, 0, ..., 0, \xi_n, \xi_{n+1}, \xi_{n+2}, ...),$$

and in particular, for $e_n = (0, ..., 0, 1, 0, ...)$ (the "1" in the nth position), $T_n(e_n) = e_n$, and thus

$$||T_n|| \geq \frac{||T_n(e_n)||}{||e_n||} = 1.$$

This shows that $||T_n|| \to 0$ does *not* hold! Hence (T_n) is not uniformly operator convergent in this case.

6. Let $U_N = \{(\xi_n) \in X \mid \forall n \geq N : \xi_n = 0\}$. Then by definition, $X = \cup_{N=1}^{\infty} U_N$. Hence it suffices to prove that each U_N is rare in X. But U_N is finite dimensional, hence U_N is closed in X (Theorem 2.4-3). Hence it only remains to prove that U_N has no interior points. Let $x = (\xi_n) \in U_N$ and $r > 0$. Then the vector

$$v \in (\xi_1, \xi_2, ..., \xi_N, r/2, 0, 0, 0, ...) \in X$$

has distance $r/2$ from (ξ_n) (since $\xi_n = 0$ for all $n > N$), and thus $v \in B(x, r)$. We also have $v \notin U_N$. Hence $B(x, r) \not\subset U_N$. This is true for every $x \in U_N$ and every $r > 0$. Hence U_N has no interior points.

7. Let $\lambda = \frac{1}{2}$. Note that

$$T_\lambda((\xi_n)) = (-\lambda\xi_1, \xi_1 - \lambda\xi_2, \xi_2 - \lambda\xi_3, ...) = (-\tfrac{1}{2}\xi_1, \xi_1 - \tfrac{1}{2}\xi_2, \xi_2 - \tfrac{1}{2}\xi_3, ...).$$

Hence if $(\eta_n) = T_\lambda((\xi_n))$ for $(\xi_n) \in \ell^\infty$ then

$$\begin{cases} \xi_1 = -2\eta_1 \\ \xi_2 = -2\eta_2 - 4\eta_1 \\ \xi_3 = -2\eta_3 - 4\eta_2 - 8\eta_1 \\ ... \\ \xi_n = -\sum_{j=1}^{n} 2^j \eta_{n+1-j} \\ ... \end{cases}$$

This proves that T_λ is injective, i.e. T_λ^{-1} exists. It follows from the above computation that

$$(*) \qquad \mathcal{D}(T_\lambda^{-1}) \subset \left\{ (\eta_n) \in \ell^\infty \;\middle|\; (\xi_n) \in \ell^\infty \text{ for } \xi_n = -\sum_{j=1}^{n} 2^j \eta_{n+1-j} \right\}.$$

(In fact we have

$$\mathcal{D}(T_\lambda^{-1}) = \left\{ (\eta_n) \in \ell^\infty \;\middle|\; (\xi_n) \in \ell^\infty \text{ for } \xi_n = -\sum_{j=1}^{n} 2^j \eta_{n+1-j} \right\},$$

for if $(\eta_n) \in \ell^\infty$ and $\xi_n = -\sum_{j=1}^n 2^j \eta_{n+1-j}$ then one checks $T_\lambda((\xi_n)) = (\eta_n)$, and hence if $(\xi_n) \in \ell^\infty$ then $(\eta_n) \in \mathcal{D}(T_\lambda^{-1})$. However, we only need (*) for our discussion.)

We prove that $\mathcal{D}(T_\lambda^{-1})$ is not dense in ℓ^∞ by proving $e_1 = (1,0,0,...) \notin \overline{\mathcal{D}(T_\lambda^{-1})}$. Assume to the contrary that $e_1 = (1,0,0,...) \in \overline{\mathcal{D}(T_\lambda^{-1})}$. Then there is some $(\eta_n) \in \mathcal{D}(T_\lambda^{-1})$ such that $||(\eta_n) - e_1|| < \frac{1}{10}$. Define $\xi_n = -\sum_{j=1}^n 2^j \eta_{n+1-j}$; then since $(\eta_n) \in \mathcal{D}(T_\lambda^{-1})$ we have $(\xi_n) \in \ell^\infty$, by (*) above. But $|\eta_1 - 1| < \frac{1}{10}$ and $|\eta_n| < \frac{1}{10}$ for all $n \geq 1$ and hence, for each $n \geq 2$,

$$\begin{aligned} |\xi_n| &= \left| -\sum_{j=1}^{n} 2^j \eta_{n+1-j} \right| = \left| 2^n \eta_1 + \sum_{j=1}^{n-1} 2^j \eta_{n+1-j} \right| \\ &\geq 2^n |\eta_1| - \sum_{j=1}^{n-1} 2^j |\eta_{n+1-j}| > 2^n (1 - \tfrac{1}{10}) - \sum_{j=1}^{n-1} 2^j \tfrac{1}{10} \\ &= 2^n(1 - \tfrac{1}{10}) - \tfrac{1}{10} \cdot (2^n - 2) > 2^n (1 - \tfrac{1}{5}) > 2^{n-1}. \end{aligned}$$

This proves that $(\xi_n) \notin \ell^\infty$, a contradiction.

Alternative (inspired by the solution of Patrik Thunström): We prove that $\mathcal{D}(T_\lambda^{-1})$ is not dense in ℓ^∞ by proving $(1,1,1,...) \notin \overline{\mathcal{D}(T_\lambda^{-1})}$. Assume to the contrary that $(1,1,1,...) \in \overline{\mathcal{D}(T_\lambda^{-1})}$. Then there is some $(\eta_n) \in \mathcal{D}(T_\lambda^{-1})$ such that $||(\eta_n) - (1,1,1,...)|| < 1$, i.e. $|\eta_n - 1| < 1$ for all n. Then $\text{Re } \eta_n > 0$ for all n, and hence defining $\xi_n = -\sum_{j=1}^n 2^j \eta_{n+1-j}$ we have

$$\text{Re } \xi_n = -\sum_{j=1}^{n} 2^j \text{Re } \eta_{n+1-j} < -2^n \text{Re } \eta_1.$$

Since $\text{Re } \eta_1 > 0$ this implies that $\text{Re } \xi_n \to -\infty$ as $n \to \infty$. It follows that $(\xi_n) \notin \ell^\infty$, a contradiction.

Alternative approach, also proving that $\mathcal{D}(T_\lambda^{-1})$ is closed. (Inspired by the solution of Martin Linder.)

Note that if $(\eta_n) = T_\lambda((\xi_n))$, then for each $k \geq 2$:

$$\sum_{j=0}^{\infty} 2^{-j}\eta_{j+k} = \sum_{j=0}^{\infty} 2^{-j}(\xi_{j+k-1} - \tfrac{1}{2}\xi_{j+k})$$
$$= \sum_{j=0}^{\infty} 2^{-j}\xi_{j+k-1} - \sum_{j=0}^{\infty} 2^{-(j+1)}\xi_{j+k} = \xi_{k-1}.$$

(All sums above are clearly absolutely convergent, since $(\eta_n) \in \ell^\infty$ and $(\xi_n) \in \ell^\infty$. Hence the above manipulations are permitted.) In particular, using the above for $k = 2$ it follows that

$$(*) \qquad \eta_1 = -\tfrac{1}{2}\xi_1 = -\tfrac{1}{2}\sum_{j=0}^{\infty} 2^{-j}\eta_{j+2} = -\sum_{j=2}^{\infty} 2^{1-j}\eta_j.$$

Conversely, let (η_n) be *any* vector in ℓ^∞ satisfying (*). Then *define* (ξ_k) through

$$\xi_k = \sum_{j=0}^{\infty} 2^{-j}\eta_{j+k+1}.$$

Then

$$|\xi_k| \leq \sum_{j=0}^{\infty} 2^{-j}|\eta_{j+k+1}| \leq ||(\eta_n)|| \sum_{j=0}^{\infty} 2^{-j} = 2 \cdot ||(\eta_n)||.$$

Hence $(\xi_k) \in \ell^\infty$. We now look at $T_\lambda((\xi_k))$. The *first* entry in $T_\lambda((\xi_k))$ is

$$-\tfrac{1}{2}\xi_1 = -\tfrac{1}{2}\sum_{j=0}^{\infty} 2^{-j}\eta_{j+2} = -\sum_{j=2}^{\infty} 2^{1-j}\eta_j = \eta_1,$$

because of (*). The n:th entry in $T_\lambda((\xi_k))$ is, for $n \geq 2$:

$$\xi_{n-1} - \tfrac{1}{2}\xi_n = \sum_{j=0}^{\infty} 2^{-j}\eta_{j+n} - \tfrac{1}{2}\sum_{j=0}^{\infty} 2^{-j}\eta_{j+n+1}$$
$$= \eta_n + \sum_{j=1}^{\infty} 2^{-j}\eta_{j+n} - \sum_{j=0}^{\infty} 2^{-(j+1)}\eta_{(j+1)+n} = \eta_n.$$

Hence $T_\lambda((\xi_k)) = (\eta_n)$, and thus $(\eta_n) \in \mathcal{R}(T_\lambda)$. We have proved that $(\eta_n) \in \ell^\infty$ belongs to $\mathcal{R}(T_\lambda)$ if and only if (*) holds, i.e.

$$\mathcal{D}(T_\lambda^{-1}) = \mathcal{R}(T_\lambda) = \left\{ (\eta_n) \in \ell^\infty \;\middle|\; \eta_1 = -\sum_{j=2}^{\infty} 2^{1-j} \eta_j \right\}.$$

But note that $f((\eta_n)) := \sum_{n=1}^{\infty} 2^{1-n} \eta_n$ is a bounded linear functional $f \in (\ell^\infty)'$, and by the above formula,

$$\mathcal{D}(T_\lambda^{-1}) = \left\{ (\eta_n) \in \ell^\infty \;\middle|\; f((\eta_n)) = 0 \right\} = \mathcal{N}(f).$$

Hence $\mathcal{D}(T_\lambda^{-1})$ is closed in ℓ^∞ (cf. Cor. 2.7-10), and $\overline{\mathcal{D}(T_\lambda^{-1})} = \mathcal{N}(f) \neq \ell^\infty$, since (e.g.) $f((1,0,0,0,...)) = 1 \neq 0$. Hence $\mathcal{D}(T_\lambda^{-1})$ is not dense in ℓ^∞.