Uppsala Universitet
Matematiska Institutionen
Andreas Strömbergsson

Prov i matematik
Funktionalanalys
Kurs: NVP, Frist.
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Skrivtid: 9–11.30
Tillåtna hjälpmedel: Manuella skrivdon, Kreyszigs bok *Introductory Functional Analysis with Applications* och Strömbergssons häfte *Spectral theorem for compact, self-adjoint operators.*

1. Let X and Y be normed spaces and let $T \in B(X,Y)$.

(a) Prove that if T is surjective then $T^\times$ is injective.
(3p)

(b) Prove that $T^\times$ is surjective if and only if T is injective and $T^{-1} \in B(\mathcal{R}(T), X)$.
(4p)

2. Let $n \geq 2$ be an integer and let $A : H \to H$ be a bounded *positive* self-adjoint operator on the Hilbert space H.
(a) Prove that there exists a bounded positive self-adjoint operator $S : H \to H$ such that $S^n = A$.
(4p)

(b) Prove that S in (a) is unique, i.e. if $S_1, S_2 \in B(H,H)$ are both positive and self-adjoint and $S_1^n = S_2^n = A$, then $S_1 = S_2$.
(3p)

(*Hints:* In Part (a) you may eg. use the theory in §9.9 and §9.10. Part (b) is difficult and gives few scores, hence skip it unless you quickly see an approach, or you have finished the other problems. You may use the fact which I have stated in class, that given T as in Thm. 9.9-1, there exists *only one* spectral family (E_λ) such that the formula $T = \int_{-\infty}^{\infty} \lambda dE_\lambda$ holds.)

3. (a) Prove that there exists an $f \in (\ell^\infty)'$ such that $f((\xi_n)) = \lim_{n\to\infty} \xi_n$ holds for every $(\xi_n) \in c$.
(Recall that $c = \{(\xi_n) \in \ell^\infty \mid \lim_{n\to\infty} \xi_n \text{ exists}\}$).
(2p)

(b) Let $e_n = (0,0,...,1,0,...) \in \ell^\infty$, with 1 in the nth place, and $v_k = \sum_{n=1}^{k} e_n$. Prove that the sequence $v_1, v_2, v_3, ...$ is *not* weakly convergent in ℓ^∞.
(2p)

(*Hint:* You may want to use $f \in (\ell^\infty)'$ from (a), and also, for $j = 1,2,3,...$, use $f_j \in (\ell^\infty)'$ defined by $f_j((\xi_n)) = \xi_j$.)

(c) Prove that the sequence $e_1, e_2, e_3, ...$ is weakly convergent in ℓ^∞ to $(0,0,0,...)$.
(2p)

GOOD LUCK!

Solutions

1.(a) Assume that T is surjective. Let $f \in Y'$ and assume $T^\times f = 0$. Then for all $x \in X$ we have $T^\times f(x) = 0$, that is, $f(Tx) = 0$. Since T is surjective this implies $f(y) = 0$ for all $y \in Y$. Hence $f = 0$. This proves that $T^\times$ is injective.

(b) Assume that T is injective and $T^{-1} \in B(\mathcal{R}(T), X)$. Let $g \in X'$. Then $g \circ T^{-1} \in B(\mathcal{R}(T), K)$, i.e. $g \circ T^{-1}$ is a bounded linear functional on $\mathcal{R}(T)$. Hence by the Hahn-Banach Theorem 4.3-2, there exists some $f \in Y'$ such that $f(y) = g \circ T^{-1}(y)$ for all $y \in \mathcal{R}(T)$. Now, for all $x \in X$ we have $Tx \in \mathcal{R}(T)$, and hence

$$T^\times f(x) = f(Tx) = g(T^{-1}(Tx)) = g(x).$$

Hence $T^\times f = g$. This proves that $T^\times$ is surjective.

Conversely, assume that $T^\times : Y' \to X'$ is surjective. We know that Y' and X' are Banach spaces (cf. Thm. 2.10-4), hence by the Open Mapping Theorem (more precisely the Open unit ball Lemma 4.12-3), there is a constant $r > 0$ such that $B_{X'}(0, r) \subset T^\times(B_{Y'}(0, 1))$, i.e.

$$\forall f \in X' : \; ||f|| < r \Longrightarrow \exists g \in Y' : ||g|| < 1 \text{ and } T^\times g = f.$$

Scaling all the vectors in this statement with a factor $2/r$ we obtain:

$$(*) \qquad \forall f \in X' : \; ||f|| < 2 \Longrightarrow \exists g \in Y' : ||g|| < 2/r \text{ and } T^\times g = f.$$

Now take $x \in X$, $x \neq 0$ arbitrary. By Theorem 4.3-3 there is some $f \in X'$ such that $||f|| = 1$ and $f(x) = ||x||$. Hence, by (*), there is some $g \in Y'$ with $||g|| < 2/r$ such that $T^\times g = f$. Now

$$(**) \qquad ||x|| = f(x) = T^\times g(x) = g(Tx) \leq ||g|| \cdot ||Tx|| < (2/r)||Tx||.$$

In particular, since $x \neq 0$ and $||x|| > 0$ we have $||Tx|| > 0$ and $Tx \neq 0$; this proves that T is injective so that $T^{-1} : \mathcal{R}(T) \to X$ exists. Also note that $y = Tx$ runs through all of $\mathcal{R}(T)$ when x runs through X; hence (**) gives $||T^{-1}y|| < (2/r)||y||$, for all nonzero $y \in \mathcal{R}(T)$. Hence $T^{-1} \in B(\mathcal{R}(T), X)$.

2. (a) By the Spectral Theorem 9.9-1 there exists a spectral family (E_λ) such that

$$A = \int_{-\infty}^{\infty} \lambda \, dE_\lambda.$$

In fact, by that theorem, (E_λ) is a spectral family on $[m, M]$, and $A = \int_{m-0}^{M} \lambda \, dE_\lambda$, where

$$m = \inf_{||x||=1} \langle Tx, x \rangle, \qquad M = \sup_{||x||=1} \langle Tx, x \rangle.$$

Here $m \geq 0$, since A is positive (see definition on p. 470). Set $f(\lambda) = \lambda^{1/n}$; this is a well-defined, continuous and real-valued function on the interval $[0, \infty)$, and in particular on $[m, M]$. Now $S = f(A)$ is a bounded self-adjoint operator on H which is defined on p. 513 (and a formula for $f(A)$ is given in Theorem 9.10-1). Using Theorem 9.10-2(c) repeatedly we find that $S^n = f(T)^n = g(T)$, where $g(\lambda) = f(\lambda)^n = (\lambda^{1/n})^n = \lambda$ on $[m, M]$, hence $g(T) = T$, i.e. we have

$$S^n = A.$$

Finally, note that $\lambda^{1/n} \geq 0$ for all $\lambda \in [m, M]$; hence by Theorem 9.10-2(d) we have $S = f(A) \geqq 0$.

(b) Assume that S has all the properties as stated in (a). By the Spectral Theorem 9.9-1 there exists a spectral family (F_λ) such that

$$S = \int_{-\infty}^{\infty} \lambda \, dF_\lambda.$$

Arguing as in (a) we see that (F_λ) is in fact a spectral family on a finite interval $[m_1, M_1]$ where $m_1 \geq 0$, and $S = \int_{m_1-0}^{M_1} \lambda \, dF_\lambda$. By Theorem 9.9-1,

$$A = S^n = \int_{m_1-0}^{M_1} \lambda^n \, dF_\lambda.$$

Now let (G_λ) be the spectral family which is defined by

$$G_\lambda = \begin{cases} F_{\lambda^{1/n}} & \text{if } \lambda \geq 0 \\ 0 & \text{if } \lambda < 0 \end{cases}$$

(It follows by direct inspection in the definition on p. 495 that (G_λ) is indeed a spectral family.) It follows from the definition of the Riemann-Stieltjes integral over a spectral family that

$$A = S^n = \int_{m_1-0}^{M_1} \lambda^n \, dF_\lambda = \int_{m_1^n-0}^{M_1^n} \lambda \, dG_\lambda.$$

[Proof: If P is any partition of $[m_1, M_1]$, say $m_1 = t_0 < t_1 < \cdots < t_v = M_1$ then we let Q be the partition $m_1^n = t_0^n < t_1^n < \cdots < t_v^n = M_1^n$ of $[m_1^n, M_1^n]$. Then by the definition of G_λ,

$$(*) \qquad s_1(P) = t_0^n F_{t_0} + \sum_{j=1}^{v} t_j^n (F_{t_j} - F_{t_{j-1}}) = t_0^n G_{t_0^n} + \sum_{j=1}^{v} t_j^n (G_{t_j^n} - G_{t_{j-1}^n}) = s_2(Q),$$

that is, the P-Riemann sum $s_1(P)$ for the integral $\int_{m_1-0}^{M_1} \lambda^n \, dF_\lambda$ equals the Q-Riemann sum $s_2(Q)$ for the integral $\int_{m_1^n-0}^{M_1^n} \lambda \, dG_\lambda$. Note also that

$$\begin{aligned}\eta(Q) = \max_{1\le j\le v} (t_j^n - t_{j-1}^n) &= \max_{1\le j\le v} \int_{t_{j-1}}^{t_j} n x^{n-1} \, dx \le \max_{1\le j\le v} n \int_{t_{j-1}}^{t_j} M_1^{n-1} \, dx \\ &= n M_1^{n-1} \max_{1\le j\le v} (t_j - t_{j-1}) = n M_1^{n-1} \eta(P).\end{aligned}$$

Hence if P runs through a sequence of partitions such that $\eta(P) \to 0$, then $\eta(Q) \to 0$, and hence $s_2(Q) \to \int_{m_1^n-0}^{M_1^n} \lambda \, dG_\lambda$ and $s_1(P) \to \int_{m_1-0}^{M_1} \lambda^n \, dF_\lambda$. Using (*) this proves the claim.]

By an almost identical argument, one also proves

$$(**) \qquad S = \int_{m_1-0}^{M_1} \lambda \, dF_\lambda = \int_{m_1-0}^{M_1} \lambda^{1/n} \, dG_\lambda.$$

But we have proved $A = \int_{m_1^n-0}^{M_1^n} \lambda \, dG_\lambda$, and by the uniqueness of the spectral family in Theorem 9.9-1, (G_λ) is the unique spectral family associated with A, i.e. equal to the family (E_λ) which we used in (a). Hence $G_\lambda = E_\lambda$ for all $\lambda \in \mathbb{R}$. Hence by (**), we have $S = \int_{m_1-0}^{M_1} \lambda^{1/n} \, dE_\lambda = \int_{-\infty}^{\infty} \lambda^{1/n} \, dE_\lambda$.

Hence we have proved that there is *only one* operator S which has all the properties stated in (a).

3. (a) We know that c is a subspace of ℓ^∞. We define a bounded linear functional $g \in c'$ by

$$g((\xi_n)) = \lim_{n\to\infty} \xi_n.$$

(g is easily checked to be bounded and linerar.) By the Hahn-Banach theorem there exists a bounded linear functional $f \in (\ell^\infty)'$ such that $f((\xi_n)) = g((\xi_n))$ for all $(\xi_n) \in c$. In other words, we now have $f((\xi_n)) = \lim_{n\to\infty} \xi_n$ for all $(\xi_n) \in c$.

(b) For each n we define $f_j \in (\ell^\infty)'$ by $f_j((\xi_n)) = \xi_j$. Now assume that (v_k) is weakly convergent in ℓ^∞ to $v = (\eta_n) \in \ell^\infty$. Then for each j we have $\lim_{k\to\infty} f_j(v_k) = f_j(v)$. But note that $f_j(v_k) = 1$ for all $k \ge j$, and $f_j(v) = \eta_j$; hence $\lim_{k\to\infty} 1 = \eta_j$, i.e. $\eta_j = 1$. Hence $v = (1, 1, 1, ...)$. However, let $f \in (\ell^\infty)'$ be as in (a). Then $f(v_k) = 0$ for each k, whereas $f(v) = 1$. Hence $\lim_{k\to\infty} f(v_k) \neq f(v)$. This contradicts the fact that v_k is weakly convergent to v.

(c) Assume the contrary, i.e. that (e_n) is not weakly convergent to $(0, 0, 0, ...)$ in ℓ^∞. Then there is some $f \in (\ell^\infty)'$ such that we do *not* have $\lim_{n\to\infty} f(e_n) = f((0, 0, 0, ...)) = 0$. Hence there is some $\varepsilon > 0$ and a sequence $1 \leq n_1 < n_2 < ...$ such that $|f(e_{n_j})| > \varepsilon$ for all j. Now define for each $k \geq 1$:

$$w_k = \sum_{j=1}^{k} \frac{|f(e_{n_j})|}{f(e_{n_j})} \cdot e_{n_j} \in \ell^\infty.$$

Then $||w_k|| = 1$, and

$$f(w_k) = \sum_{j=1}^{k} \frac{|f(e_{n_j})|}{f(e_{n_j})} f(e_{n_j}) = \sum_{j=1}^{k} |f(e_{n_j})| > k\varepsilon.$$

This implies $||f|| > k\varepsilon$. But this *cannot* be true for all $k \geq 1$, i.e. we have arrived at a contradiction.

Hence (e_n) *is* weakly convergent to $(0, 0, 0, ...)$.