Functional Analysis (2006)
Homework assignment 2

All students should solve the following problems:

1. Define $T : C[0,1] \to C[0,1]$ by $(Tx)(t) = t \int_0^t x(s)\,ds$. Prove that this is a bounded linear operator, and compute $||T||$. Also prove that the inverse $T^{-1} : \mathcal{R}(T) \to C[0,1]$ exists but is not bounded.

2. Let

$$M = \Big\{ x \in L^2[0,1] \;:\; \int_0^1 x(t)\,dt = 0,\; \int_0^1 tx(t)\,dt = 0,\; \int_0^1 t^2 x(t)\,dt = 0 \Big\}.$$

Given $x \in L^2[0,1]$, find a formula for the vector in M which lies closest to x (in the $L^2[0,1]$-norm).

3. (Problem §3.9: 4). Let H_1 and H_2 be two Hilbert spaces and let $T : H_1 \to H_2$ be a bounded linear operator. Suppose that we are given subsets $M_1 \subset H_1$ and $M_2 \subset H_2$ such that $T(M_1) \subset M_2$. Prove that $M_1^\perp \supset T^*(M_2^\perp)$.

4. Let a, b be two positive real numbers. Let x be a vector in a normed space X and assume that $|f(x)| \leqq a$ holds for all $f \in X'$ with $||f|| \leqq b$. Prove that $||x|| \leqq a/b$.

Students taking Functional Analysis as a 6 point course should also solve the following problems:

5. Let $Y_1, Y_2, Y_3, \cdots$ be closed linear subspaces of the Hilbert space H, such that $Y_j \perp Y_k$ for all $1 \leqq j < k$, and $\bigcap_{j=1}^{\infty} Y_j^\perp = \{0\}$. Prove that for every vector $v \in H$ there is a unique choice of vectors $y_1 \in Y_1$, $y_2 \in Y_2$, $y_3 \in Y_3, \cdots$ such that $\sum_{j=1}^{\infty} y_j = v$ in H.

6. Let Y be a subspace of a Banach space X. The *annihilator* Y^a is defined as the subspace $Y^a := \{f \in X' \;:\; f(y) = 0, \forall y \in Y\}$ of X' (cf. §2.10, problem 13). Hence $Y^{aa} = (Y^a)^a$ is a subspace of X''. Let $C : X \to X''$ be the canonical map. Prove that $C(Y) \subset Y^{aa}$. Also prove that if X is reflexive and Y is closed then $C(Y) = Y^{aa}$.

Solutions to problems 1-4 should be handed in by Friday, February 24. Solutions to problems 5-6 should be handed in by Monday, March 13. (Either give the solutions to me directly or put them in my mailbox, third floor, House 3, Polacksbacken.)

Functional Analysis
Solutions to homework assignment 2

1. For all $x_1, x_2 \in C[0,1]$ and all $\alpha, \beta \in \mathbb{R}$ and all $t \in [0,1]$ we have

$$\begin{aligned}(T(\alpha x_1 + \beta x_2))(t) &= t \int_0^t (\alpha x_1 + \beta x_2)(s)\, ds \\ &= \alpha t \int_0^t x_1(s)\, ds + \beta t \int_0^t x_2(s)\, ds \\ &= \alpha T x_1(t) + \beta T x_2(t) \\ &= (\alpha T x_1 + \beta T x_2)(t);\end{aligned}$$

hence

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2,$$

and this shows that T is linear.

Furthermore, for each $x \in C[0,1]$ we have:

$$\begin{aligned}||Tx|| = \max_{t\in[0,1]} \left| t \int_0^t x(s)\, ds \right| &\leqq \max_{t\in[0,1]} |t| \int_0^t |x(s)|\, ds \\ &\leqq \max_{t\in[0,1]} t \int_0^t ||x||\, ds = \max_{t\in[0,1]} t^2 \cdot ||x|| = ||x||.\end{aligned}$$

Hence T is bounded with $||T|| \leqq 1$. In fact if we take x as the constant function $x(t) = 1$ then $||x|| = 1$ and $(Tx)(t) = t \int_0^t x(s)\, ds = t^2$, hence $||Tx|| = \max_{t\in[0,1]} |t^2| = 1$. But $||Tx|| \leqq ||T|| \cdot ||x||$, i.e. $1 \leqq ||T||$. Hence we have proved both $||T|| \leqq 1$ and $||T|| \geqq 1$. It follows that $||T|| = 1$.

Now assume that $x \in C[0,1]$ satisfies $Tx = 0$, i.e. $t \int_0^t x(s)\, ds = 0$ for all $t \in [0,1]$. It then follows that $\int_0^t x(s)\, ds = 0$ for all $t \in (0,1]$, and thus by differentiation with respect to t we get $x(t) = 0$ for all $t \in (0,1]$. Since $x(t)$ is continuous we then also have $x(0) = 0$. Hence $x = 0$. We have thus proved

$$\forall x \in C[0,1] : \quad \Big(Tx = 0 \Longrightarrow x = 0 \Big).$$

Hence by Theorem 2.6-10(a), T^{-1} exists.

Given any $n \in \mathbb{Z}^+$ we let $x_n(t) = t^n$. Then $x_n \in C[0,1]$ and $||x_n|| = \max_{t\in[0,1]} |t^n| = 1$. We let

$$y_n(t) = T x_n(t) = t \int_0^t s^n\, ds = (n+1)^{-1} t^{n+2}.$$

Then $||y_n|| = \max_{t\in[0,1]} |(n+1)^{-1}t^{n+2}| = (n+1)^{-1}$. Also, by construction, $y_n \in \mathcal{R}(T)$ and $T^{-1}y_n = x_n$; thus $||T^{-1}y_n|| = ||x_n|| = 1$. This shows that T^{-1} cannot be bounded. (For if T^{-1} were bounded then we would have $||T^{-1}y_n|| \leqq ||T^{-1}|| \cdot ||y_n||$, i.e. $1 \leqq ||T^{-1}|| \cdot (n+1)^{-1}$, for all $n \in \mathbb{Z}^+$. This is impossible.)

2. Let $f_1, f_2, f_3 \in L^2[0,1]$ be given by $f_1(t) = 1$, $f_2(t) = t$, $f_3(t) = t^2$. The definition of M says that $x \in L^2[0,1]$ belongs to M if and only if x is orthogonal to f_1, f_2, f_3. That is:

$$M = \{f_1, f_2, f_3\}^{\perp} = (\text{Span}\{f_1, f_2, f_3\})^{\perp}.$$

(The last identity holds since $\langle x, f_1\rangle = \langle x, f_2\rangle = \langle x, f_3\rangle = 0$ implies $\langle x, c_1 f_1 + c_2 f_2 + c_3 f_3\rangle = 0$, for all $c_1, c_2, c_3 \in \mathbb{C}$.) Let

$$Y = \text{Span}\{f_1, f_2, f_3\} \qquad (\text{so that } \ M = Y^{\perp}).$$

This is a closed subspace of $L^2[0,1]$ since it is finite dimensional (Theorem 2.4-3), and hence by Theorem 3.3-4, $L^2[0,1]$ decomposes as a direct sum

$$L^2[0,1] = Y \oplus Y^{\perp} = Y \oplus M.$$

This means that given any $x \in L^2[0,1]$ there exist unique vectors $y \in Y$ and $z \in M$ such that $x = y + z$. It is easy to see that in this situation z *is the vector in* M *which lies closest to* x,[1] i.e. $\forall v \in M : ||v - x|| \geqq ||z - x||$. [Proof: If v is an arbitrary vector in M then also $v - z \in M = Y^{\perp}$, and since $y \in Y$ we then have $\langle v - z, y\rangle = 0$; hence we may use Pythagoras theorem: $||v-x||^2 = ||v-z+z-x||^2 = ||v-z-y||^2 = ||v-z||^2 + ||y||^2 \geqq ||y||^2 = ||z-x||^2$, and the proof is complete.]

[1]This is also, in principle, seen in the book in the proof of Theorem 3.3-4. Note that y is *the orthogonal projection of* x *on* Y, and z is *the orthogonal projection of* x *on* M; this concept is discussed in the book on p. 147.

To determine a formula for z as a function of x we first use Gram-Schmidt to find an orthonormal basis in $Y = \mathrm{Span}\{f_1, f_2, f_3\}$:

$\tilde{e}_1 = f_1 = 1;$

$e_1 = \dfrac{\tilde{e}_1}{||\tilde{e}_1||} = 1;$

$\tilde{e}_2 = f_2 - \langle f_2, e_1\rangle e_1 = t - \frac{1}{2}\cdot 1 = t - \frac{1}{2};$

$e_2 = \dfrac{\tilde{e}_2}{||\tilde{e}_2||} = \sqrt{3}(2t-1);$

$\tilde{e}_3 = f_3 - \langle f_3, e_1\rangle e_1 - \langle f_3, e_2\rangle e_2 = t^2 - \frac{1}{3}\cdot 1 - \dfrac{\sqrt{3}}{6}\cdot\sqrt{3}(2t-1) = t^2 - t + \frac{1}{6}$

$e_3 = \dfrac{\tilde{e}_3}{||\tilde{e}_3||} = \sqrt{5}(6t^2 - 6t + 1).$

Now since $y \in Y$ we must have $y = c_1e_1 + c_2e_2 + c_3e_3$ for some constants $c_1, c_2, c_3 \in \mathbb{C}$. We also have $x - y = z \in M = Y^\perp$ and hence for each $j = 1, 2, 3$, since $e_j \in Y$, we have:

$$0 = \langle x - y, e_j\rangle = \langle x, e_j\rangle - \langle c_1e_1 + c_2e_2 + c_3e_3, e_j\rangle = \langle x, e_j\rangle - c_j.$$

Hence $c_j = \langle x, e_j\rangle$. It follows that

$$\begin{aligned} z = x - y &= x - (c_1e_1 + c_2e_2 + c_3e_3) \\ &= x - \langle x, e_1\rangle e_1 - \langle x, e_2\rangle e_2 - \langle x, e_3\rangle e_3. \end{aligned}$$

Answer: The vector $z \in M$ which lies closest to x is

$$z = x - \langle x, e_1\rangle e_1 - \langle x, e_2\rangle e_2 - \langle x, e_3\rangle e_3,$$

i.e.

$$\begin{aligned} z(t) = x(t) - \int_0^1 x(s)\,ds - 3(2t-1)\cdot\int_0^1 x(s)(2s-1)\,ds \\ - 5(6t^2 - 6t + 1)\cdot\int_0^1 x(s)(6s^2 - 6s + 1)\,ds. \end{aligned}$$

3. Let v be an arbitrary vector in $T^*(M_2^\perp)$. Then there is some $w \in M_2^\perp$ such that $v = T^*(w)$. Since $w \in M_2^\perp$ we know that $\langle w, x\rangle = 0$ for every vector $x \in M_2$.

Now let y be an arbitrary vector in M_1. Then

$$\langle v, y\rangle = \langle T^*(w), y\rangle = \langle w, T(y)\rangle.$$

But we have $T(y) \in M_2$, since $y \in M_1$ and $T(M_1) \subset M_2$. Also recall $w \in M_2^\perp$. From these two facts $T(y) \in M_2$ and $w \in M_2^\perp$ it follows that

$\langle w, T(y)\rangle = 0$. Hence from the above computation we see:

$$\langle v, y\rangle = 0.$$

This is true for every $y \in M_1$. Hence $v \in M_1^\perp$.

We have proved that for every $v \in T^*(M_2^\perp)$ we have $v \in M_1^\perp$. Hence $T^*(M_2^\perp) \subset M_1^\perp$, Q.E.D.

4. By Theorem 4.3-3 there exists an $f_0 \in X'$ such that $||f_0|| = 1$ and $f_0(x) = ||x||$. Let $f = bf_0$; then $||f|| = b||f_0|| = b$. In particular $||f|| \leqq b$ and hence by the assumption in the problem we have $|f(x)| \leqq a$. On the other hand $|f(x)| = |bf_0(x)| = b \cdot |f_0(x)| = b \cdot ||x||$. Hence $b \cdot ||x|| \leqq a$, i.e. $||x|| \leqq a/b$, Q.E.D.

Alternative solution:[2]

By Corollary 4.3-4 we have

$$(*) \qquad ||x|| = \sup_{f \in X' - \{0\}} \frac{|f(x)|}{||f||}.$$

Now let f be an arbitrary element in $X' - \{0\}$, as in the above supremum. Set $c = ||f||$; then $c > 0$ since $f \neq 0$. Set $f_0 = (b/c)f \in X'$; then $||f_0|| = (b/c)||f|| = (b/c) \cdot c = b$. Hence *by the assumption in the problem text* we have $|f_0(x)| \leqq a$. But $f = (c/b)f_0$, hence $|f(x)| = (c/b)|f_0(x)| \leqq (c/b)a = ca/b$, and

$$\frac{|f(x)|}{||f||} = \frac{|f(x)|}{c} \leqq \frac{ca/b}{c} = \frac{a}{b}.$$

We have proved that this is true for every $f \in X' - \{0\}$. Hence the supremum in $(*)$ is $\leqq a/b$, i.e. we have proved

$$||x|| \leqq \frac{a}{b},$$

Q.E.D.

[2] In some sense this is actually exactly the same solution as the first one, but in a different language.

5. We first prove uniqueness. Let $v \in H$ be given. Assume that the vectors $y_1 \in Y_1$, $y_2 \in Y_2$, $y_3 \in Y_3, \cdots$ are such that $\sum_{j=1}^{\infty} y_j = v$, i.e. $\lim_{N\to\infty} \sum_{j=1}^{N} y_j = v$. Take any $k \in \{1, 2, 3, \cdots\}$ and any vector $w \in Y_k$; we then have (using Lemma 3.2-2)

$$\langle v, w\rangle = \langle \lim_{N\to\infty} \sum_{j=1}^{N} y_j, w\rangle = \lim_{N\to\infty} \langle \sum_{j=1}^{N} y_j, w\rangle = \lim_{N\to\infty} \sum_{j=1}^{N} \langle y_j, w\rangle.$$

But for each $j \neq k$ we have $\langle y_j, w\rangle = 0$ since $y_j \in Y_j$, $w \in Y_k$ and $Y_j \perp Y_k$. Hence we can continue the computation:

$$= \lim_{N\to\infty} \langle y_k, w\rangle = \langle y_k, w\rangle.$$

Hence we have proved $\langle v, w\rangle = \langle y_k, w\rangle$, i.e. $\langle v - y_k, w\rangle = 0$. This is true for every $w \in Y_k$. Hence $v - y_k \in Y_k^{\perp}$. But Theorem 3.3-4 says that we have a direct sum $H = Y_k \oplus Y_k^{\perp}$, and now from $y_k \in Y_k$, $v - y_k \in Y_k^{\perp}$ we see that $v = y_k + (v - y_k)$ is the unique decomposition of v in this direct sum. Hence y_k is the orthogonal projection of v on Y_k (cf. p. 147). This proves that y_k is uniquely determined from v. This is true for every $k \in \{1, 2, 3, \cdots\}$.

We next prove that every vector can actually be expressed as a sum in the stated way. Let $v \in H$ be given. For each $k \in \{1, 2, 3, \cdots\}$ we let y_k be the orthogonal projection of v on Y_k (this construction is of course suggested by the uniqueness proof above). We now wish to prove $\sum_{j=1}^{\infty} y_j = v$.

For each j with $y_j \neq 0$ we let $e_j = ||y_j||^{-1} \cdot y_j$; then these vectors e_j (where we throw away those indices j for which $y_j = 0$) form an orthonormal sequence, and hence by part (c) of the "main theorem about Hilbert bases" as I formulated it in my lecture, we have $\sum_{j=1}^{\infty} |\langle v, e_j\rangle|^2 \leqq ||v||^2$ (this is Bessel's inequality, Theorem 3.4-6 in the book), and (hence) $\sum_{j=1}^{\infty} \langle v, e_j\rangle \cdot e_j$ is a convergent sum (cf. Theorem 3.5-2(a) in the book). But by definition of orthogonal projection we have $v - y_j \in Y_j^{\perp}$ for each j, and in particular $v - y_j \perp y_j$, thus $\langle v - y_j, y_j\rangle = 0$. This gives $\langle v, y_j\rangle = \langle y_j, y_j\rangle = ||y_j||^2$ and thus if $y_j \neq 0$:

$$\langle v, e_j\rangle \cdot e_j = ||y_j||^{-1} \cdot \langle v, y_j\rangle \cdot e_j = ||y_j||^{-1} \cdot ||y_j||^2 \cdot e_j = y_j.$$

Hence what we have proved is that the sum $\sum_{j=1}^{\infty} y_j$ is convergent!

Let us write

$$v_0 = \sum_{j=1}^{\infty} y_j \in H.$$

We now have for every $k \geqq 1$ and every $w \in Y_k$, by arguing as in the first part of this solution: $\langle v_0, w\rangle = \langle y_k, w\rangle$. Hence $\langle v - v_0, w\rangle = \langle v - y_k, w\rangle = 0$, since $v - y_k \perp Y_k$ because y_k is the orthogonal projection of v on Y_k. This is true for every $w \in Y_k$, hence

$$v - v_0 \in Y_k^{\perp}.$$

This is true for every $k \geqq 1$, hence

$$v - v_0 \in \cap_{k=1}^{\infty} Y_k^{\perp} = \{0\}.$$

Hence

$$v = v_0 = \sum_{j=1}^{\infty} y_j,$$

Q.E.D.

6. Take an arbitrary vector $y \in Y$. Then for every $f \in Y^a$ we have $(C(y))(f) = f(y) = 0$. Hence $C(y) \in Y^{aa}$. This proves that $C(Y) \subset Y^{aa}$.

Next assume that X is reflexive and Y is closed. Take an arbitrary vector $y_0 \in Y^{aa}$. (Thus $y_0 \in X''$.) Since X is reflexive C is surjective, hence there is some vector $x \in X$ such that $y_0 = C(x)$. Since $y_0 \in Y^{aa}$ we have, for all $f \in Y^a$:

$$0 = y_0(f) = (C(x))(f) = f(x).$$

Now assume $x \notin Y$. Then (since Y is closed!) by Lemma 4.6-7 there exists a $g \in X'$ such that $||g|| = 1$, $g(y) = 0$ for all $y \in Y$ (i.e. $g \in Y^a$), and $g(x) = \delta = \inf_{y \in Y} ||y - x|| > 0$. Hence we have both $g \in Y^a$ and $g(x) > 0$; this contradicts the fact from above that $f(x) = 0, \forall f \in Y^a$! Hence the assumption $x \notin Y$ must be discarded. Thus $x \in Y$. Hence from $y_0 = C(x)$ we see $y_0 \subset C(Y)$.

This is true for every $y_0 \in Y^{aa}$. This proves $Y^{aa} \subset C(Y)$.

Together with $C(Y) \subset Y^{aa}$ this proves $C(Y) = Y^{aa}$.