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Prov i matematik
Funktionalanalys
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Tillåtna hjälpmedel: Manuella skrivdon, Kreyszigs bok *Introductory Functional Analysis with Applications* och Strömbergssons häften *Spectral theorem for compact, self-adjoint operators* and *Mathematical statements and proofs.*

1. Let X be a normed space and let $g_1, g_2, g_3, \cdots$ be elements in X' with $||g_n|| \leq n^{-2}$ for all $n = 1, 2, 3, \cdots$. We define:

$$T(x) = \Big(g_1(x), g_2(x), g_3(x), \cdots \Big).$$

Prove that this gives a bounded linear operator $T : X \to \ell^1$.

(6p)

2. Let H_1 and H_2 be Hilbert spaces and let $T : H_1 \to H_2$ be a bounded linear operator. Prove that

$$T^*(H_2) \subset \mathcal{N}(T)^{\perp}.$$

(5p)

3. Define $T : \ell^\infty \to \ell^\infty$ by

$$T((\xi_1, \xi_2, \xi_3, \cdots)) = \big(\xi_1,\ \tfrac{1}{2}\sum_{j=1}^{2} \xi_j,\ \tfrac{1}{3}\sum_{j=1}^{3} \xi_j,\ \tfrac{1}{4}\sum_{j=1}^{4} \xi_j,\ \cdots\big).$$

Prove that T is a bounded linear operator $T : \ell^\infty \to \ell^\infty$ and compute the norm $||T||$.

(5p)

4. Let

$$Y = \big\{ (\xi_j) \in \ell^\infty \mid \text{at most finitely many } \xi_j \neq 0 \big\}.$$

Show that Y is not complete. Also prove that the closure of Y in ℓ^∞ equals

$$\overline{Y} = \Big\{ (\xi_j) \in l^\infty \mid \forall r > 0 : \exists N \in \mathbb{Z}^+ : \forall j \geqq N : |\xi_j| < r \Big\}.$$

(In words: $\overline{Y}$ consists of all sequences $(\xi_j) \in l^\infty$ such that for every $r > 0$, there are at most *finitely* many elements ξ_j with $|\xi_j| \geqq r$.)

(6p)

5. Let $e_1, e_2, e_3, \cdots$ be a total and orthonormal sequence in a Hilbert space H and let $\alpha_1, \alpha_2, \alpha_3, \cdots$ be a sequence of real numbers. Define the operator $T : H \to H$ by the formula

$$Tx = \sum_{n=1}^{\infty} \alpha_n \langle x, e_n \rangle e_n.$$

Prove that if T is a compact operator then $\lim_{n\to\infty} \alpha_n = 0$.

(6p)

6. Let $C[0,1]$ be the Banach space of all complex valued continuous functions on $[0,1]$, with the norm $||x|| := \sup_{t\in[0,1]} |x(t)|$. Let $T : C[0,1] \to C[0,1]$ be the bounded linear operator given by $[Tx](t) = (1+t^2)x(t)$. Determine the four sets $\rho(T)$, $\sigma_p(T)$, $\sigma_c(T)$, $\sigma_r(T)$.

(6p)

7. Let X be a Banach space and let $T_1, T_2, T_3, \cdots$ and $S_1, S_2, S_3, \cdots$ be bounded linear operators from X to X such that $S_n T_m = T_m S_n$ for all $n, m \geqq 1$. Assume that (T_n) is strongly operator convergent with limit T and (S_n) is strongly operator convergent with limit S. Prove that $TS = ST$.

(6p)

GOOD LUCK!

Solutions

1. We know that $\sum_{n=1}^{\infty} n^{-2}$ is a convergent sum; define $C = \sum_{n=1}^{\infty} n^{-2}$ (in fact we know $C = \pi^2/6$, but we won't need this). For any $x \in X$ we have

$$||T(x)||_{\ell^1} = \sum_{n=1}^{\infty} |g_n(x)| \leq \sum_{n=1}^{\infty} ||g_n|| \cdot ||x|| \leq \sum_{n=1}^{\infty} n^{-2} \cdot ||x|| = C \cdot ||x||.$$

This proves that $T(x) \in \ell^1$ for all $x \in X$. Also for any $a, b \in \mathbb{C}$ and $x, y \in X$ we have

$$\begin{aligned} T(ax+by) &= \Big(g_1(ax+by), g_2(ax+by), g_3(ax+by), \cdots\Big) \\ &= \Big(ag_1(x)+bg_1(y), ag_2(x)+bg_2(y), ag_3(x)+bg_3(y), \cdots\Big) \\ &= a\Big(g_1(x), g_2(x), g_3(x), \cdots\Big) + b\Big(g_1(y), g_2(y), g_3(y), \cdots\Big) \\ &= aT(x) + bT(y). \end{aligned}$$

Hence T is a linear operator. Finally, our first computation shows that $||T(x)|| \leq C \cdot ||x||$ for all $x \in X$; hence T is bounded.

2. Let v be an arbitrary vector in H_2. Then for every $w \in \mathcal{N}(T)$ we have

$$\langle T^*(v), w \rangle = \langle v, T(w) \rangle = \langle v, 0 \rangle = 0.$$

Hence $T^*(v) \in \mathcal{N}(T)^\perp$. This is true for every $v \in H_2$. Hence $T^*(H_2) \subset \mathcal{N}(T)^\perp$.

3. Take an arbitrary vector $(\xi_j) \in \ell^\infty$, and let $(\eta_k) = T((\xi_j))$. Then, for each $k = 1, 2, 3, \cdots$,

$$|\eta_k| = \frac{1}{k}\left|\sum_{j=1}^{k} \xi_j\right| \leq \frac{1}{k}\sum_{j=1}^{k} |\xi_j| \leq \frac{1}{k}\sum_{j=1}^{k} ||(\xi_j)|| = ||(\xi_j)||.$$

Since this is true for all $k = 1, 2, 3, \cdots$ we have $T((\xi_j)) = (\eta_k) \in \ell^\infty$. This is true for all $(\xi_j) \in \ell^\infty$, hence T is a map from ℓ^∞ to ℓ^∞. Also for any $a, b \in \mathbb{C}$ and $(\xi_j), (\nu_j) \in \ell^\infty$ we have

$$\begin{aligned} T\big(a(\xi_j) + b(\nu_j)\big) &= T\big(a\xi_1 + b\nu_1, a\xi_2 + b\nu_2, a\xi_3 + b\nu_3, \cdots\big) \\ &= \Big(a\xi_1 + b\nu_1, \tfrac{1}{2}\sum_{k=1}^{2}(a\xi_k + b\nu_k), \tfrac{1}{3}\sum_{k=1}^{3}(a\xi_k + b\nu_k), \cdots\Big) \\ &= a\Big(\xi_1, \tfrac{1}{2}\sum_{k=1}^{2}\xi_k, \tfrac{1}{3}\sum_{k=1}^{3}\xi_k, \cdots\Big) + b\Big(\nu_1, \tfrac{1}{2}\sum_{k=1}^{2}\nu_k, \tfrac{1}{3}\sum_{k=1}^{3}\nu_k, \cdots\Big) \\ &= aT\big((\xi_j)\big) + bT\big((\nu_j)\big). \end{aligned}$$

Hence T is a linear operator. It follows from our inequality above that, for all $(\xi_j) \in \ell^\infty$,

$$||T((\xi_j))|| = ||(\eta_k)|| = \sup_k |\eta_k| \le ||(\xi_j)||.$$

Since this is true for all $(\xi_j) \in \ell^\infty$, T is a bounded, and $||T|| \le 1$.

On the other hand, taking $(\xi_j) = (1, 1, 1, \cdots)$ we obtain $(\eta_k) = T((\xi_j)) = (1, 1, 1, \cdots)$, and $||T((\xi_j))|| = ||(\xi_j)|| = 1$; hence $||T|| \ge 1$.

Answer: $||T|| = 1$.

4. Let

$$x_n = (2^{-1}, 2^{-2}, 2^{-3}, \cdots, 2^{-n}, 0, 0, 0, \cdots).$$

Then $x_1, x_2, ... \in Y$. We also define

$$x = (2^{-1}, 2^{-2}, 2^{-3}, \cdots) \in \ell^\infty.$$

We now claim that $x_n \to x$ as $n \to \infty$; this follows from

$$\begin{aligned} ||x - x_n|| &= ||(0, 0, \cdots, 0, 2^{-n-1}, 2^{-n-2}, \cdots)|| \\ &= \sup_{k \geqq n+1} 2^{-k} = 2^{-n-1} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Note that $x \notin Y$, by the definition of Y. Hence by Theorem 1.4-6(b), Y is not closed (as a subspace of ℓ^∞). Hence by Theorem 1.4-7, Y is not complete.

Next, we prove

$$(*) \qquad \overline{Y} = \Big\{(\xi_j) \in l^\infty \mid \forall r > 0 : \exists N \in \mathbb{Z}^+ : \forall j \geqq N : |\xi_j| < r\Big\}.$$

First take an arbitrary vector $x = (\xi_j)$ in the right-hand set. Thus we assume $(\xi_j) \in l^\infty$ and that for every $r > 0$ there is an $N \in \mathbb{Z}^+$ such that $|\xi_j| < r$ holds for all $j \geqq N$. Now form the sequence $x_1, x_2, ...$ where

$$x_n = (\xi_1, \xi_2, \cdots, \xi_n, 0, 0, 0, \cdots).$$

Then $x_1, x_2, \cdots \in Y$. Note that

$$||x - x_n|| = \sup_{j \geqq n+1} |\xi_j|;$$

hence for any given $r > 0$, if we take $N \in \mathbb{Z}^+$ as above then $||x - x_n|| \leqq r$ holds for all $n \geqq N$. This can be achieved for every $r > 0$, and hence $x_n \to x$ in ℓ^∞. It follows that $x \in \overline{Y}$. Hence we have proved:

$$(**) \qquad \overline{Y} \supset \Big\{ (\xi_j) \in l^\infty \mid \forall r > 0 : \exists N \in \mathbb{Z}^+ : \forall j \geqq N : |\xi_j| < r \Big\}.$$

Conversely, take an arbitrary vector $x = (\xi_j) \in \overline{Y}$. Then there is a sequence $x_1, x_2, x_3, \cdots$ of vectors in Y such that $x_n \to x$ in ℓ^∞. Let $r > 0$ be given. Then there is $n \in \mathbb{Z}^+$ such that $||x_n - x|| < r$. Let us write $x_n = (\eta_j)$. Since $x_n \in Y$ there is some $N \in \mathbb{Z}^+$ such that $\eta_j = 0$ for all $j \geqq N$. Also, $||x_n - x|| < r$ implies that $|\eta_j - \xi_j| < r$ for all j. Hence for all $j \geqq N$ we have $|0 - \xi_j| < r$, i.e. $|\xi_j| < r$. Hence we have proved that for every $r > 0$ there is some $N \in \mathbb{Z}^+$ such that $|\xi_j| < r$ holds for all $j \geqq N$. But here x was an arbitrary element in $\overline{Y}$. Hence:

$$(***) \qquad \overline{Y} \subset \Big\{ (\xi_j) \in l^\infty \mid \forall r > 0 : \exists N \in \mathbb{Z}^+ : \forall j \geqq N : |\xi_j| < r \Big\}.$$

Together, (**) and (***) imply (*), Q.E.D.

5. Assume that T is compact and $\lim_{n\to\infty} \alpha_n \neq 0$. We will show that this leads to a contradiction. Since $\lim_{n\to\infty} \alpha_n \neq 0$ there is some $\varepsilon > 0$ and a subsequence $(\alpha_{n_k})_{k=1}^\infty$ with $n_1 < n_2 < n_3 < \cdots$ such that $|\alpha_{n_k}| > \varepsilon$ for all $k = 1, 2, 3, \cdots$. Now $e_{n_1}, e_{n_2}, e_{n_3}, \cdots$ is a bounded sequence in H, hence by Theorem 8.1-3 in Kreyszig's book the sequence $T(e_{n_1}), T(e_{n_2}), T(e_{n_3}), \cdots$ must have a convergent subsequence, say $(T(e_{n_{k_j}}))_{j=1}^\infty$ with $k_1 < k_2 < k_3 < \cdots$. Every convergent sequence is a Cauchy sequence, hence

$$(*) \qquad \Big\| T(e_{n_{k_j}}) - T(e_{n_{k_{j'}}}) \Big\| \to 0 \qquad \text{as } j, j' \to \infty.$$

On the other hand, we have $T(e_n) = \alpha_n e_n$ for each n, and hence for *all* $j \neq j'$:

$$\Big\| T(e_{n_{k_j}}) - T(e_{n_{k_{j'}}}) \Big\| = \Big\| \alpha_{n_{k_j}} e_{n_{k_j}} - \alpha_{n_{k_{j'}}} e_{n_{k_{j'}}} \Big\| = \sqrt{|\alpha_{n_{k_j}}|^2 + |\alpha_{n_{k_{j'}}}|^2}$$

$$(**) \qquad > \sqrt{\varepsilon^2 + \varepsilon^2} > \varepsilon.$$

(We used Pythagoras' theorem; this is ok since $j \neq j'$ implies $n_{k_j} \neq n_{k_{j'}}$ so that $e_{n_{k_j}}$ and $e_{n_{k_{j'}}}$ are orthogonal to each other.)

But $(*)$ and $(**)$ cannot both be true. This is a contradiction.

6. First let λ be an arbitrary complex number which does not belong to the real interval $[1,2]$. Note that the operator $T_\lambda = T - \lambda$ is given by

$$[T_\lambda x](t) = (1 + t^2 - \lambda)x(t).$$

We know that the range of the function $1 + t^2$ on $[0,1]$ is the interval $[1,2]$. Hence $1 + t^2 - \lambda \neq 0$ for all $t \in [0,1]$; hence (since $[0,1]$ is a compact set and $1+t^2-\lambda$ is continuous) there is a constant $r > 0$ such that $|1 + t^2 - \lambda| \geqq r$ for all $t \in [0,1]$. It follows that we can define a new bounded linear operator $A : C[0,1] \to C[0,1]$ by

$$[Ax](t) = (1 + t^2 - \lambda)^{-1}x(t).$$

(We see that $||A|| \leqq r^{-1}$.) It follows immediately from the definition that $T_\lambda A = AT_\lambda = I$, the identity operator on $C[0,1]$. Hence $T_\lambda^{-1} = A$ exists and is a bounded operator. Hence λ belongs to the resolvent set $\rho(T)$. We have thus proved:

$$(\mathbb{C} - [1,2]) \subset \rho(T).$$

Next let λ be an arbitrary number in the interval $[1,2]$. We still have

$$[T_\lambda x](t) = (1 + t^2 - \lambda)x(t).$$

Let $t_0 \in [1,2]$ be the unique number which gives $1 + t_0^2 - \lambda = 0$ (thus, explicitly: $t_0 = \sqrt{\lambda - 1}$). Assume that $x \in C[0,1]$ is a vector with $T_\lambda x = 0$. Then $(1 + t^2 - \lambda)x(t) = 0$ for all $t \in [0,1]$, and this implies $x(t) = 0$ for all $t \neq t_0$ in $[0,1]$. Hence since $x(t)$ is a continuous function we also have $x(t_0) = \lim_{t \to t_0} x(t) = \lim_{t \to t_0} 0 = 0$ (if $t_0 = 0$ then this limit should be interpreted as *right* limit; if $t_0 = 1$ it should be interpreted as *left* limit). Hence $x(t) = 0$ for all $t \in [0,1]$, i.e. $x = 0$ in $C[0,1]$. Hence T_λ is injective and T_λ^{-1} exists (Theorem 2.6-10). Note that if $y = T_\lambda x$ for some $x \in C[0,1]$ then $y(t_0) = (1 + t_0^2 - \lambda)x(t_0) = 0 \cdot x(t_0) = 0$; hence every $y \in \mathcal{R}(T_\lambda)$ satisfies $y(t_0) = 0$. It follows that $\mathcal{R}(T_\lambda)$ is not dense in $C[0,1]$. [Proof: Let $z \in C[0,1]$ be the constant function $z(t) = 1$. Then for every $y \in \mathcal{R}(T_\lambda)$ we have $||y - z|| = \sup_{t \in [0,1]} |y(t) - z(t)| \geqq |y(t_0) - z(t_0)| = |0 - 1| = 1$; hence z is not in the closure of $\mathcal{R}(T_\lambda)$.] But $\mathcal{R}(T_\lambda) = \mathcal{D}(T_\lambda^{-1})$, hence we have proved that $\mathcal{D}(T_\lambda^{-1})$ is not dense in $C[0,1]$. It follows that λ belongs to the residual spectrum, $\lambda \in \sigma_r(T)$. We have thus proved:

$$[1,2] \subset \sigma_r(T).$$

Since the four sets $\rho(T)$, $\sigma_p(T)$, $\sigma_c(T)$, $\sigma_r(T)$ are known to be disjoint it follows that the answer is as follows.
Answer: $\rho(T) = \mathbb{C} - [1,2]$, $\sigma_p(T) = \emptyset$, $\sigma_c(T) = \emptyset$, $\sigma_r(T) = [1,2]$.

7. One can give a fairly short proof by a computation using "limits inside limits", and being very careful about the order of these limits in each step. Note that we make crucial use below of Lemma 4.9.5, and hence of the uniform boundedness theorem!

The fact that (T_n) is strongly operator convergent with limit T means that

$$(*) \qquad \forall y \in X : \quad \lim_{n\to\infty} T_n(y) = T(y).$$

The fact that (S_n) is strongly operator convergent with limit S means that

$$(**) \qquad \forall y \in X : \quad \lim_{n\to\infty} S_n(y) = S(y).$$

Now, for every $x \in X$:

$$\begin{aligned}
& TS(x) && \Big\{\text{Use (*) with } y = S(x).\Big\}\\
&= \lim_{n\to\infty} T_n(S(x)) && \Big\{\text{Use (**) with } y = x.\Big\}\\
&= \lim_{n\to\infty} T_n\big(\lim_{m\to\infty} S_m(x)\big) && \Big\{\text{Each } T_n \text{ is continuous.}\Big\}\\
&= \lim_{n\to\infty} \big(\lim_{m\to\infty} T_n(S_m(x))\big) && \Big\{\text{Use assumption in problem.}\Big\}\\
&= \lim_{n\to\infty} \big(\lim_{m\to\infty} S_m(T_n(x))\big) && \Big\{\text{Use (**) with } y = T_n(x).\Big\}\\
&= \lim_{n\to\infty} \big(S(T_n(x))\big) && \Big\{S \text{ is continuous (by Lemma 4.9.5).}\Big\}\\
&= S\big(\lim_{n\to\infty} T_n(x)\big) && \Big\{\text{Use (*) with } y = x.\Big\}\\
&= S(T(x)).
\end{aligned}$$

Since this holds for every $x \in X$, we have proved $TS = ST$.

Alternative solution, with some comments:
Note that we never claimed "$TS(x) = \lim_{n,m\to\infty} T_n S_m(x)$" in the above solution! We *did* see that "$TS(x) = \lim_{n\to\infty}\big(\lim_{m\to\infty} T_n(S_m(x))\big)$", but this is *not* the same thing.

The statement "$TS(x) = \lim_{n,m\to\infty} T_n S_m(x)$" means in precise terms:

$$\forall \varepsilon > 0 : \exists N, M \in \mathbb{Z}^+ : \forall n \geq N, m \geq M : ||TS(x) - T_n S_m(x)|| < \varepsilon.$$

This statement *is in fact true* (for any fixed $x \in X$), but it is not obvious. In the solution below we give a proof of that statement, along the way to a complete solution.

By Lemma 4.9-5 we have $T, S \in B(X, X)$. Furthermore, since the sequence $(T_n x)$ is bounded for every $x \in X$ (and X is complete), it follows from the uniform boundedness theorem that there is a constant $C > 0$ such that $||T_n|| \leqq C$ for all n (this was also seen in the proof of Lemma 4.9-5). Similarly, there is a constant $D > 0$ such that $||S_n|| \leqq D$ for all n.

Now let x be an arbitrary vector in X. Let $\varepsilon > 0$ be given. Since (T_n) is strongly operator convergent with limit T there is $N \in \mathbb{Z}^+$ such that both

$$||T_n x - Tx|| < \varepsilon \quad \text{and} \quad ||T_n Sx - TSx|| < \varepsilon$$

hold for all $n \geqq N$. Similarly, there is $M \in \mathbb{Z}^+$ such that both

$$||S_m x - Sx|| < \varepsilon \quad \text{and} \quad ||S_m Tx - STx|| < \varepsilon$$

hold for all $m \geqq M$. Hence we have for all $n \geqq N$, $m \geqq M$:

$$\begin{aligned} &||TSx - T_n S_m x|| = \|TSx - T_n Sx + T_n Sx - T_n S_m x\| \\ &\leqq \|TSx - T_n Sx\| + \|T_n(Sx - S_m x)\| \\ &< \varepsilon + ||T_n|| \cdot \|Sx - S_m x\| \\ &< (1 + C)\varepsilon, \end{aligned}$$

and[1]

$$\begin{aligned} &||STx - S_m T_n x|| = \|STx - S_m Tx + S_m Tx - S_m T_n x\| \\ &\leqq \|STx - S_m Tx\| + \|S_m(Tx - T_n x)\| \\ &< \varepsilon + ||S_m|| \cdot \|Tx - T_n x\| \\ &< (1 + D)\varepsilon. \end{aligned}$$

However, by the assumptions of the problem we have $S_m T_n = T_n S_m$ for all n, m; hence by choosing any $n \geqq N$, $m \geqq M$ we obtain:

$$\begin{aligned} &||TSx - STx|| = \|TSx - T_n S_m x + S_m T_n x - STx\| \\ &\leqq ||TSx - T_n S_m x|| + ||STx - S_m T_n x|| \\ &< (1 + C + 1 + D)\varepsilon. \end{aligned}$$

But we are free to choose $\varepsilon > 0$ arbitrarily; hence we can make $(1 + C + 1 + D)\varepsilon$ arbitrarily small. Hence $||TSx - STx|| = 0$, i.e. $TSx = STx$. This holds for all $x \in H$. Hence $TS = ST$, Q.E.D.

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[1] Note that by these two computations, if we note that such M, N can be found for any given $\varepsilon > 0$, we have now proved $TS(x) = \lim_{n,m\to\infty} T_n S_m(x)$ and $ST(x) = \lim_{n,m\to\infty} S_m T_n(x)$! However, we have chosen to complete the solution without using these limits explicitly, instead working with our precise bounds.