Solutions to home assignment 1

2.3. (a). The function $\lambda(n)$ is clearly multiplicative; hence also the function $f(n) = \lambda(n)n^{-s}$ is multiplicative for any fixed s. Furthermore, when $\sigma > 1$ we note that $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent, since $\sum_{n=1}^{\infty} |f(n)| = \sum_{n=1}^{\infty} |\lambda(n)|n^{-\sigma} = \sum_{n=1}^{\infty} n^{-\sigma} < \infty$. Hence by Proposition 2.7 we have, when $\sigma > 1$,

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p \Big(1 + \frac{\lambda(p)}{p^s} + \frac{\lambda(p^2)}{p^{2s}} + \frac{\lambda(p^3)}{p^{3s}} + \ldots\Big) = \prod_p \big(1 - p^{-s} + p^{-2s} - p^{-3s} + \ldots\big)$$

$$= \prod_p \frac{1}{1+p^{-s}} = \prod_p \frac{1-p^{-s}}{1-p^{-2s}} = \frac{\zeta(2s)}{\zeta(s)}.$$

□

(b). The function $|\mu(n)|$ is clearly multiplicative; hence also the function $f(n) = |\mu(n)|n^{-s}$ is multiplicative for any fixed s. Furthermore, when $\sigma > 1$ we note that $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent, since $\sum_{n=1}^{\infty} |f(n)| = \sum_{n=1}^{\infty} |\mu(n)|n^{-\sigma} \leq \sum_{n=1}^{\infty} n^{-\sigma} < \infty$. Hence by Proposition 2.7 we have, when $\sigma > 1$,

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \prod_p \Big(1 + \frac{|\mu(p)|}{p^s} + \frac{|\mu(p^2)|}{p^{2s}} + \frac{|\mu(p^3)|}{p^{3s}} + \ldots\Big) = \prod_p \big(1 + p^{-s}\big)$$

$$= \prod_p \frac{1-p^{-2s}}{1-p^{-s}} = \frac{\zeta(s)}{\zeta(2s)}.$$

2.5. It follows from the assumption in the lemma that $X = \prod_p \big(1+|f(p)|+|f(p^2)|+\ldots\big)$ is a finite real number ($X \geq 1$). We will prove the absolute convergence of $\sum_{n=1}^{\infty} |f(n)|$ by actually proving $\sum_{n=1}^{\infty} |f(n)| \leq X$. To prove this it suffices to prove that $\sum_{n=1}^{N} |f(n)| \leq X$ for each $N \in \mathbb{Z}^+$.

Let N be given, and let $p_1, p_2, \ldots, p_M$ be all prime numbers $\leq N$. Then

$$\prod_{k=1}^{M} \big(1 + |f(p_k)| + |f(p_k^2)| + \ldots\big) \leq \prod_p \big(1 + |f(p)| + |f(p^2)| + \ldots\big) \leq X.$$

On the other hand, by Cauchy's theorem, the finite product $\prod_{k=1}^{M}\big(1+|f(p_k)|+|f(p_k^2)|+\ldots\big)$ may be multiplied out as

$$\prod_{k=1}^{M} \big(1 + |f(p_k)| + |f(p_k^2)| + \ldots\big) = \sum_{v_1=0}^{\infty} \sum_{v_2=0}^{\infty} \cdots \sum_{v_M=0}^{\infty} |f(p_1^{v_1} p_2^{v_2} \cdots p_M^{v_M})|.$$

In the last sum, $p_1^{v_1} p_2^{v_2} \cdots p_M^{v_M}$ runs through exactly those positive integers which only have prime factors $p_1, p_2, \ldots, p_M$ (or a subset of these) in their prime factorization. In particular

$p_1^{v_1} p_2^{v_2} \cdots p_M^{v_M}$ visits all the numbers $1, 2, \ldots, N$. Hence

$$\sum_{n=1}^{N} |f(n)| \leq \prod_{k=1}^{M} \bigl(1 + |f(p_k)| + |f(p_k^2)| + \ldots\bigr) \leq X.$$

This concludes the proof. □

3.2. Write $A(x) = \sum_{1 \leq n \leq x} a_n n^{\beta}$; then we know that $A(x) = x + O(x^{\frac{1}{2}})$ as $x \to \infty$. (In detail: $A(x) = A(\lfloor x \rfloor) = \lfloor x \rfloor + O(\lfloor x \rfloor^{\frac{1}{2}}) = x + O(1) + O(x^{\frac{1}{2}}) = x + O(x^{\frac{1}{2}})$ as $x \to \infty$.) Hence (by possibly taking a larger implied constant) we have

$$A(x) = x + O(x^{\frac{1}{2}}), \qquad \forall x \geq 1. \tag{584}$$

Now for $N \geq 1$ we have

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} n^{-\beta} (a_n n^{\beta}) = \int_{1-}^{N} x^{-\beta} \, dA(x). \tag{585}$$

By Lemma 3.2 (together with a similar type of limit identity as in (85)) this is

$$= N^{-\beta} A(N) - \frac{1}{2} \int_{1}^{N} x^{-\beta} A(x) \, dx. \tag{586}$$

Using now (584) we get

$$\begin{aligned} &= N^{-\beta}\bigl(N + O(N^{\frac{1}{2}})\bigr) + \beta \int_{1}^{N} x^{-\beta-1}\bigl(x + O(x^{\frac{1}{2}})\bigr) \, dx \\ &= N^{1-\beta} + O(N^{\frac{1}{2}-\beta}) + \beta \int_{1}^{N} x^{-\beta} \, dx + O(1) \int_{1}^{N} x^{-\frac{1}{2}-\beta} \, dx \end{aligned} \tag{587}$$

To treat the last integral (the second error term) we use the following fact, which follows directly from the explicit formula for $\int_1^N x^{\alpha} \, dx$:

$$\int_{1}^{N} x^{\alpha} \, dx \ll \begin{cases} N^{\alpha+1} & \text{if } \alpha > -1 \\ \log N & \text{if } \alpha = -1 \\ 1 & \text{if } \alpha < -1 \end{cases} \tag{588}$$

(the implied constant depends on α). Hence we get:

$$\begin{aligned} \sum_{n=1}^{N} a_n &= N^{1-\beta} + O(N^{\frac{1}{2}-\beta}) + \frac{\beta}{1-\beta}\bigl(N^{1-\beta} - 1\bigr) + O(1) \begin{Bmatrix} N^{\frac{1}{2}-\beta} & \text{if } \beta < \frac{1}{2} \\ \log N & \text{if } \beta = \frac{1}{2} \\ 1 & \text{if } \beta > \frac{1}{2} \end{Bmatrix} \\ &= \frac{1}{1-\beta} N^{1-\beta} + O(1) \begin{Bmatrix} N^{\frac{1}{2}-\beta} & \text{if } \beta < \frac{1}{2} \\ \log N & \text{if } \beta = \frac{1}{2} \\ 1 & \text{if } \beta > \frac{1}{2} \end{Bmatrix}. \end{aligned} \tag{589}$$

$\square$

3.5. Set $A(x) = \sum_{1 \leq n \leq x} a_n$; then the assumption says that $A(x) \sim x^2$ as $x \to \infty$, and hence for any given $\varepsilon > 0$ there is some $X > 1$ such that

$$\bigl|A(x) - x^2\bigr| < \varepsilon x^2, \qquad \forall x \geq X. \tag{590}$$

Now for each $N \in \mathbb{Z}^+$ we have

$$\sum_{n=1}^{N} a_n (N-n)^2 = \int_{1-}^{N} (N-x)^2 \, dA(x) = 0 + 2\int_1^N (N-x)A(x)\,dx$$

If $A(x) \equiv x^2$ then the last expression equals

$$2\int_1^N (N-x)x^2\,dx = 2\Bigl[\frac{N}{3}x^3 - \frac{1}{4}x^4\Bigr]_{x=1}^{x=N} = \frac{1}{6}N^4 - \frac{2}{3}N + \frac{1}{2}.$$

Hence for our general $A(x) = \sum_{1 \leq n \leq x} a_n$ we have, for each integer $N > X$:

$$\begin{aligned}
&\Bigl|\sum_{n=1}^{N} a_n (N-n)^2 - \frac{1}{6}N^4\Bigr| = \Bigl|2\int_1^N (N-x)A(x)\,dx - 2\int_1^N (N-x)x^2\,dx - \frac{2}{3}N + \frac{1}{2}\Bigr| \\
&\leq 2\int_1^N (N-x)\bigl|A(x) - x^2\bigr|\,dx + \frac{2}{3}N + \frac{1}{2} \\
&\leq 2\int_1^X N\bigl|A(x) - x^2\bigr|\,dx + 2\int_X^N (N-x)\varepsilon x^2\,dx + \frac{2}{3}N + \frac{1}{2} \\
&\leq 2N\int_1^X \bigl|A(x) - x^2\bigr|\,dx + 2\varepsilon\int_X^N N^3\,dx + \frac{2}{3}N + \frac{1}{2} \\
&\leq 2\varepsilon N^4 + \Bigl(2\int_1^X \bigl|A(x) - x^2\bigr|\,dx + \frac{2}{3}\Bigr)N + \frac{1}{2}
\end{aligned}$$

The expression inside the parenthesis does not depend on N, and hence for all sufficiently large N the above is $< 3\varepsilon N^4$, i.e. we have proved that for all sufficiently large N we have

$$\Bigl|\sum_{n=1}^{N} a_n (N-n)^2 - \frac{1}{6}N^4\Bigr| < 3\varepsilon N^4.$$

Since ε was arbitrarily small this implies that

$$\sum_{n=1}^{N} a_n (N-n)^2 \sim \frac{1}{6}N^4 \qquad \text{as } N \to \infty.$$

$\square$

4.6. The first line in the table on p. 70 is clear from Proposition 4.27. Regarding the three remaining lines, note that for any even positive integer n we have $\bigl(\frac{-4}{n}\bigr) = \bigl(\frac{8}{n}\bigr) = \bigl(\frac{-8}{n}\bigr) = 0$ by Proposition 4.26. Hence from now on we may assume that n is an odd positive integer.

Now $\left(\frac{-4}{n}\right) = \left(\frac{-1}{n}\right)\left(\frac{2}{n}\right)^2 = \left(\frac{-1}{n}\right) = (-1)^{\frac{n-1}{2}}$ by (164) and Theorem 4.25, and this shows that the second line in the table is correct.

Also $\left(\frac{8}{n}\right) = \left(\frac{2}{n}\right)^3 = \left(\frac{2}{n}\right) = (-1)^{\frac{n^2-1}{8}}$ by (164) and Theorem 4.25, and thus the third line in the table is correct.

Finally $\left(\frac{-8}{n}\right) = \left(\frac{-1}{n}\right)\left(\frac{2}{n}\right)^3 = \left(\frac{-1}{n}\right)\left(\frac{2}{n}\right) = (-1)^{\frac{n-1}{2}+\frac{n^2-1}{8}}$ by (164) and Theorem 4.25, and hence the fourth line in the table is correct. □

4.8. Let q have the prime factorization $q = 2^\alpha \prod_{j=1}^r p_j^{\alpha_j}$. From the proof of Theorem 4.34 we see that all characters modulo q are real if and only if $\alpha \leq 3$ and for each $j \in \{1, \ldots, r\}$ all the $\phi(p_j^{\alpha_j})$th roots of unity are real, i.e. $\phi(p_j^{\alpha_j}) \in \{1, 2\}$. Clearly $\phi(p^\alpha) = p^{\alpha-1}(p-1) > 2$ for each prime $p \geq 5$ and any $\alpha \geq 1$, and also $\phi(3^\alpha) > 2$ for all $\alpha \geq 2$. Hence in the prime factorization of q we must have $r = 0$ or $r = 1$ and $p_1 = 3$, $\alpha_1 = 1$.

Conversely, if $\alpha \leq 3$ and $[r = 0$ or $r = 1$ and $p_1 = 3$, $\alpha_1 = 1]$ then indeed all the characters modulo q are real. There are 4 choices of $\alpha \leq 3$; hence if $r = 0$ then we get the following possible q's: $1, 2, 4, 8$, and if $[r = 1,\ p_1 = 3,\ \alpha_1 = 1]$ then we get those same numbers multiplied with 3, viz.: $3, 6, 12, 24$. □

6.5. We fix q and assume that for each a with $(a, q) = 1$ we have $\pi(x; q, a) = \frac{1}{\phi(q)} \operatorname{Li} x + O\big(xe^{-c\sqrt{\log x}}\big)$ as $x \to \infty$ (this is in fact a theorem which will be seen later in the course). Then by possibly increasing the implied constant we actually have $\pi(x; q, a) = \frac{1}{\phi(q)} \operatorname{Li} x + O\big(xe^{-c\sqrt{\log x}}\big)$ for all $x \geq 2$ (and all a with $(a, q) = 1$). Now for any $2 \leq M < N$ we have

$$\begin{aligned}
\sum_{M<p\leq N} \frac{\chi(p)}{p} &= \sum_{\substack{a \bmod q \\ (a,q)=1}} \sum_{\substack{M<p\leq N \\ p\equiv a \pmod q}} \frac{\chi(a)}{p} = \sum_{\substack{a \bmod q \\ (a,q)=1}} \chi(a) \int_M^N \frac{1}{x}\, d\pi(x; q, a) \\
&= \sum_{\substack{a \bmod q \\ (a,q)=1}} \chi(a) \Big(\frac{\pi(N; q, a)}{N} - \frac{\pi(M; q, a)}{M} + \int_M^N \frac{1}{x^2} \pi(x; q, a)\, dx \Big) \\
&= \sum_{\substack{a \bmod q \\ (a,q)=1}} \chi(a) \Big(\frac{\operatorname{Li} N}{N\phi(q)} - \frac{\operatorname{Li} M}{M\phi(q)} + \int_M^N \frac{\operatorname{Li} x}{\phi(q) x^2}\, dx \Big) \\
&\qquad\qquad + \sum_{\substack{a \bmod q \\ (a,q)=1}} O\Big(e^{-c\sqrt{\log N}} + e^{-c\sqrt{\log M}} + \int_M^N \frac{e^{-c\sqrt{\log x}}}{x}\, dx \Big)
\end{aligned}$$

Here the first sum vanishes, since $\sum_{\substack{a \bmod q \\ (a,q)=1}} \chi(a) = 0$, cf. Lemma 3.13. Hence we get (the implied constant may depend on q):

$$= O\Big(e^{-c\sqrt{\log M}}\Big) + O\Big(\int_M^\infty \frac{e^{-c\sqrt{\log x}}}{x}\, dx \Big)$$

Substituting $x = e^{y^2/c^2}$ (thus $y = c\sqrt{\log x}$) we find that

$$\int_M^\infty \frac{e^{-c\sqrt{\log x}}}{x}\,dx = \frac{2}{c^2}\int_{c\sqrt{\log M}}^\infty e^{-y}y\,dy = \frac{2}{c^2}\Big[-(1+y)e^{-y}\Big]_{y=c\sqrt{\log M}}^{y=\infty} = O\Big(\sqrt{\log M}e^{-c\sqrt{\log M}}\Big)$$

Hence we conclude

$$\sum_{M<p\leq N}\frac{\chi(p)}{p} = O\Big(\sqrt{\log M}e^{-c\sqrt{\log M}}\Big),$$

for all $2 \leq M < N$. Since the bound in the right hand side is independent of N it follows that the same bound holds for the infinite sum:

$$\sum_{p>M}\frac{\chi(p)}{p} = O\Big(\sqrt{\log M}e^{-c\sqrt{\log M}}\Big).$$

This proves the first assertion.

Next let us assume the much stronger asymptotic relation $\pi(x;q,a) = \frac{1}{\phi(q)}\operatorname{Li} x + O\big(\sqrt{x}\log x\big)$ as $x \to \infty$, for all a with $(a,q) = 1$. By possibly increasing the implied constant it follows that this relation holds for all $x \geq 2$. Now by a very similar computation as above it follows that for any $2 \leq M < N$ we have

$$\begin{aligned}\sum_{M<p\leq N}\frac{\chi(p)}{p} &= O\big(M^{-\frac12}\log M\big) + O\Big(\int_M^\infty x^{-\frac32}(\log x)\,dx\Big)\\ &= O\big(M^{-\frac12}\log M\big) + O\Big(\int_{\log M}^\infty e^{-\frac12 y}y\,dy\Big) = O\big(M^{-\frac12}\log M\big).\end{aligned}$$

As before we can here take "$N \to \infty$" and hence we obtain the second assertion. □