Solutions to the Problems (not home assignments)

2.1. (a). The function μ is clearly multiplicative, and hence also the function $f(n) = \frac{\mu(n)}{n^s}$ is multiplicative for every fixed s. Furthermore the series $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ is absolutely convergent when $\sigma > 1$, since $\sum_{n=1}^{\infty} \left|\frac{\mu(n)}{n^s}\right| \leq \sum_{n=1}^{\infty} n^{-\sigma} < \infty$. Hence by Proposition 2.7 we have

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 + \frac{\mu(p)}{p^s} + \frac{\mu(p^2)}{p^{2s}} + \frac{\mu(p^3)}{p^{3s}} + \ldots\right).$$

But for every prime p we have $\mu(p) = -1$ while $\mu(p^2) = \mu(p^3) = \ldots = 0$, hence we get

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 - p^{-s}\right) = \left(\prod_p \left(1 - p^{-s}\right)^{-1}\right)^{-1} = \zeta(s)^{-1}.$$

(b). The function $f(n) = \frac{\mu(n)\chi(n)}{n^s}$ is multiplicative for every fixed s, and the series $\sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s}$ is absolutely convergent when $\sigma > 1$, by the same argument as above. Hence by Proposition 2.7 we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s} &= \prod_p \left(1 + \frac{\mu(p)\chi(p)}{p^s} + \frac{\mu(p^2)\chi(p^2)}{p^{2s}} + \frac{\mu(p^3)\chi(p^3)}{p^{3s}} + \ldots\right) \\ &= \prod_p \left(1 - \chi(p)p^{-s}\right) = \left(\prod_p \left(1 - \chi(p)p^{-s}\right)^{-1}\right)^{-1} = L(s,\chi)^{-1}.\end{aligned}$$

(Note that (a) is a special case of (b), obtained when taking χ to be the trivial character $\chi \equiv 1$.)

2.2. It follows from the formula $\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$ that $\phi(n)$ is a multiplicative function. Hence also the function $f(n) = \phi(n)n^{-s}$ is multiplicative for any fixed s. Next note that if $\sigma > 2$ then the series $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent, since

$$\sum_{n=1}^{\infty} |f(n)| = \sum_{n=1}^{\infty} \phi(n)n^{-\sigma} \leq \sum_{n=1}^{\infty} n^{1-\sigma} < \infty. \tag{613}$$

Hence Proposition 2.7 applies when $\sigma > 2$ and we get

$$\begin{aligned}\sum_{n=1}^{\infty} \phi(n) n^{-s} &= \prod_p \bigl(1 + \phi(p)p^{-s} + \phi(p^2)p^{-2s} + \ldots\bigr)\\
&= \prod_p \Bigl(1 + \sum_{k=1}^{\infty} \Bigl(1 - \frac{1}{p}\Bigr) p^k \cdot p^{-ks}\Bigr)\\
&= \prod_p \Bigl(1 + \Bigl(1 - \frac{1}{p}\Bigr) \sum_{k=1}^{\infty} p^{k(1-s)}\Bigr)\\
&= \prod_p \Bigl(1 + \frac{p-1}{p} \cdot \frac{p^{1-s}}{1-p^{1-s}}\Bigr)\\
&= \prod_p \frac{p(1-p^{1-s}) + (p-1)p^{1-s}}{p(1-p^{1-s})}\\
&= \prod_p \frac{1-p^{-s}}{1-p^{1-s}}\\
&= \frac{\zeta(s-1)}{\zeta(s)}.\end{aligned}$$

2.4. For any s with $\sigma > A+1$ the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ is absolutely convergent, since $\sum_{n=1}^{\infty} |a_n n^{-s}| \ll \sum_{n=1}^{\infty} n^{A-\sigma} < \infty$. Hence for every such s we have, by Proposition 2.7:

$$\sum_{n=1}^{\infty} a_n n^{-s} = \prod_p \bigl(1 + a_p p^{-s} + a_{p^2} p^{-2s} + \ldots\bigr). \tag{614}$$

Here for each prime p the sum $1 + a_p p^{-s} + a_{p^2} p^{-2s} + \ldots$ is absolutely convergent, and hence we may multiply termwise with $(1 - a_p p^{-s} + p^{\alpha} p^{-2s})$ to get

$$\begin{aligned}&\bigl(1 + a_p p^{-s} + a_{p^2} p^{-2s} + \ldots\bigr)(1 - a_p p^{-s} + p^{\alpha} p^{-2s})\\
&= 1 - a_p p^{-s} + a_p p^{-s} + \sum_{k=2}^{\infty} \bigl(a_{p^k} - a_p a_{p^{k-1}} + p^{\alpha} a_{p^{k-2}}\bigr) p^{-ks} = 1 + 0 + \sum_{k=2}^{\infty} 0 p^{-ks} = 1,\end{aligned}$$

where we used our assumption about $\{a_n\}$. Hence for each prime p (and our fixed s with $\sigma > A+1$) we have

$$1 + a_p p^{-s} + a_{p^2} p^{-2s} + \ldots = \frac{1}{1 - a_p p^{-s} + p^{\alpha} p^{-2s}}, \tag{615}$$

and hence from (614) we get

$$\sum_{n=1}^{\infty} a_n n^{-s} = \prod_p \frac{1}{1 - a_p p^{-s} + p^{\alpha} p^{-2s}}. \tag{616}$$

2.6. This is part of the Weierstrass Factorization Theorem, cf. [48, Thm. 15.10].

Let us write $E(w) = (1-w)\exp(w + \frac{1}{2}w^2 + \ldots + \frac{1}{k}w^k)$. This is an analytic function for all $w \in \mathbb{C}$, and we have

$$\begin{aligned} E'(w) &= \Big\{-1 + (1-w)(1 + w + w^2 + \ldots + w^{k-1})\Big\} \exp(w + \frac{1}{2}w^2 + \ldots + \frac{1}{k}w^k) \\ &= -w^k \exp(w + \frac{1}{2}w^2 + \ldots + \frac{1}{k}w^k). \end{aligned} \tag{617}$$

From this we see that $E'(0) = E''(0) = \ldots = E^{(k)}(0) = 0$ and hence the power series expansion of $E(w)$ at $w = 0$ gives $E(w) = E(0) + O(|w|^{k+1}) = 1 + O(|w|^{k+1})$ as $|w| \to 0$. This means that there are constants $A, \delta > 0$ such that

$$\big|E(w) - 1\big| \leq A|w|^{k+1} \qquad \text{for all } w \text{ with } |w| \leq \delta. \tag{618}$$

For fixed z the product which we wish to study is $f(z) = \prod_{n=1}^{\infty} E(z/\rho_n) = \prod_{n=1}^{\infty}(1 + u_n)$ where $u_n := E(z/\rho_n) - 1$. Now (618) says that $|u_n| \leq A|z/\rho_n|^{k+1}$ for all n such that $|z/\rho_n| \leq \delta$. But note that our assumption $\sum_{n=1}^{\infty} |\rho_n|^{-1-k} < \infty$ implies that $\lim_{n\to\infty} |\rho_n| = \infty$; hence there is some N such that $|z/\rho_n| \leq \delta$ holds for all $n > N$, and thus

$$\sum_{n=1}^{\infty} |u_n| \leq \sum_{n=1}^{N} |u_n| + \sum_{n>N} A|z/\rho_n|^{k+1} = \sum_{n=1}^{N} |u_n| + A|z| \sum_{n>N} |\rho_n|^{-1-k} < \infty. \tag{619}$$

Hence the product $f(z) = \prod_{n=1}^{\infty} E(z/\rho_n)$ is indeed absolutely convergent (cf. Theorem 2.2 and Definition 2.2).

Next we prove that the sum $\sum_{n=1}^{\infty} |u_n|$ actually converges uniformly over z in any compact subset. This is clear by inspection in the above proof: It suffices to prove that for any given $R > 0$ the sum $\sum_{n=1}^{\infty} |u_n|$ converges uniformly over $|z| \leq R$. Now there is some N such that $R/|\rho_n| \leq \delta$ holds for all $n > N$, and hence for every z with $|z| \leq R$ we have $|z/\rho_n| \leq \delta$ for all $n > N$, and hence

$$\sum_{n=1}^{\infty} |u_n| \leq \sum_{n=1}^{N} |u_n| + \sum_{n>N} A|z/\rho_n|^{k+1} \leq= \sum_{n=1}^{N} \sup_{|z|\leq R} |u_n| + AR \sum_{n>N} |\rho_n|^{-1-k}. \tag{620}$$

The last expression is a finite positive number, and this concludes the proof that $\sum_{n=1}^{\infty} |u_n|$ converges uniformly on compacta. It now follows from Corollary 2.3 that the product $f(z) = \prod_{n=1}^{\infty} E(z/\rho_n)$ converges uniformly on compacta, and hence by Weierstrass Theorem (cf. footnote 1 on p. 5) $f(z)$ is an analytic function of $z \in \mathbb{C}$.

Finally we prove the statement about zeros: First assume that $\alpha \notin \{\rho_1, \rho_2, \ldots\}$. Then $E(\alpha/\rho_n) \neq 0$ for all n and hence $f(\alpha) \neq 0$ by Theorem 2.2. Next assume that $\alpha \in$

$\{\rho_1, \rho_2, \ldots\}$, and let M be the set of indices j for which $\rho_j = \alpha$. Note that M is finite, since $\sum_{n=1}^{\infty} |\rho_n|^{-1-k} < \infty$. We now have, for all $z \in \mathbb{C}$:

$$f(z) = \prod_{\substack{n \geq 1 \\ n \notin M}} E(z/\rho_n) \prod_{n \in M} E(z/\rho_n) \tag{621}$$

Here the first product is infinite, but the same argument as above shows that it defines an analytic function in the whole plane which is non-zero at $z = \alpha$. The second product is a finite product of analytic functions, and each factor has a zero of order one at $z = \alpha$. (Proof: After a simple rescaling our task is to prove that $E(w)$ has a zero of order one at $w = 1$. This follows from $E(1) = 0$ and $E'(1) = -\exp(1 + \frac{1}{2} + \ldots + \frac{1}{k}) \neq 0$.) Hence $f(z)$ has a zero of order m (exactly) at $z = \alpha$. □

Remark. In fact the following precise version of (618) holds: $\big|E(w) - 1\big| \leq |w|^{k+1}$ for all w with $|w| \leq 1$. This is not too difficult to prove. Cf. [48, Lemma 15.8].

3.1. Theorem 3.3: See (e.g.) [39, Appendix A, Thm. 1].

Proof of Lemma 3.4, "first part": Suppose that g is continuous on $[A, B]$ and that f is locally constant on $[A, B]$. Take $A = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n = B$ so that f is constant on each open interval $(\lambda_j, \lambda_{j+1})$, $j = 0, 1, \ldots, n-1$. Let $\varepsilon > 0$ be given. Then since g is continuous at each point λ_j there is some $\delta > 0$ such that, for each $j \in \{0, 1, \ldots, n\}$, $|g(x) - g(\lambda_j)| < \varepsilon$ holds for each $x \in [A, B]$ with $|x - \lambda_j| \leq \delta$. We also take δ so small that $\lambda_{j+1} - \lambda_j > \delta$ for all $j \in \{0, 1, \ldots, n\}$. Now let $A = x_0 \leq x_1 \leq \ldots \leq x_N = B$ be an arbitrary partition with $\text{mesh}\{x_n\} \leq \delta$, and take arbitrary numbers $\xi_k \in [x_{k-1}, x_k]$ for $k = 1, 2, \ldots, N$. Now there is a unique choice of indices

$$0 \leq k_0 < k_1' \leq k_1 < k_2' \leq k_2 < \ldots < k_{n-1}' \leq k_{n-1} < k_n' < N \tag{622}$$

such that $x_{k_j} \leq \lambda_j < x_{k_j+1}$ for each $j \in \{0, 1, \ldots, n-1\}$ and $x_{k_j'} < \lambda_j \leq x_{k_j'+1}$ for each $j \in \{1, 2, \ldots, n\}$. [Proof: It follows from $A = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n = B$ and $A = x_0 \leq x_1 \leq \ldots \leq x_N = B$ that for each $j \in \{0, 1, \ldots, n-1\}$ there is a unique $k_j \in \{0, 1, \ldots, N-1\}$ with $x_{k_j} \leq \lambda_j < x_{k_j+1}$, and for each $j \in \{1, 2, \ldots, n\}$ there is a unique $k_j' \in \{0, 1, \ldots, N-1\}$ with $x_{k_j'} < \lambda_j \leq x_{k_j'+1}$. Then to see that (622) holds, note that for each $j \in \{1, 2, \ldots, n-1\}$ it follows from $x_{k_j'} < \lambda_j < x_{k_j+1}$ that $k_j' < k_j + 1$, thus $k_j' \leq k_j$. Furthermore, for each $j \in \{0, 1, \ldots, n-1\}$ it follows from $x_{k_j} \leq \lambda_j < x_{k_j+1}$ that $0 < x_{k_j+1} - \lambda_j \leq |x_{k_j+1} - x_{k_j}| \leq \text{mesh}\{x_n\} \leq \delta < \lambda_{j+1} - \lambda_j$; thus $x_{k_j+1} < \lambda_{j+1} \leq x_{k_{j+1}'+1}$, and hence $k_j + 1 < k_{j+1}' + 1$, i.e. $k_j < k_{j+1}'$. Hence (622) holds.]

We will now prove that the Riemann-Stieltjes sum

$$S(\{x_n\}, \{\xi_n\}) = \sum_{k=1}^{N} g(\xi_k)\big(f(x_k) - f(x_{k-1})\big)$$

is close to the sum with which we defined $\int_A^B g(x)\, df(x)$ in (69). We will do this by splitting $\sum_{k=1}^{N} g(\xi_k)\big(f(x_k) - f(x_{k-1})\big)$ into several parts determined by the numbers $k_0, k_1', k_1, \ldots, k_n'$.

First of all, note that for each $j \in \{0, 1, \ldots, n-1\}$ we have

$$\sum_{k_j+1<k\leq k'_{j+1}} g(\xi_k)\big(f(x_k) - f(x_{k-1})\big) = 0. \tag{623}$$

Indeed, we have seen above that $\lambda_j < x_{k_j+1} < \lambda_{j+1}$ and we also know $k_j + 1 \leq k'_{j+1}$ and $x_{k'_{j+1}} < \lambda_{j+1}$; hence $\lambda_j < x_k < \lambda_{j+1}$ holds for all k with $k_j + 1 \leq k \leq k'_{j+1}$. Thus also $f(x_k) = f(\lambda_j+) = f(\lambda_{j+1}-)$ for all these k, and it follows that (623) holds.

Next, for each $j \in \{1, 2, \ldots, n-1\}$ we have seen above that $x_{k_j+1} < \lambda_{j+1}$, and similarly we have $\lambda_{j-1} < x_{k'_j}$; hence $\lambda_{j-1} < x_{k'_j} < \lambda_j < x_{k_j+1} < \lambda_{j+1}$, and this implies that $f(x_{k'_j}) = f(\lambda_j-)$ and $f(x_{k_j+1}) = f(\lambda_j+)$. Hence if $k'_j = k_j$ then

$$\sum_{k=k'_j+1}^{k_j+1} g(\xi_k)\big(f(x_k) - f(x_{k-1})\big) = g(\xi_{k_j+1})\big(f(\lambda_j+) - f(\lambda_j-)\big). \tag{624}$$

On the other hand if $k'_j < k_j$ then $k'_j + 1 \leq k_j$ and $\lambda_j \leq x_{k'_j+1} \leq x_{k_j} \leq \lambda_j$, forcing $x_{k'_j+1} = x_{k_j} = \lambda_j$; thus $x_k = \lambda_j$ for all k with $k'_j + 1 \leq k \leq k_j$, so that

$$\begin{aligned} &\sum_{k=k'_j+1}^{k_j+1} g(\xi_k)\big(f(x_k) - f(x_{k-1})\big) \\ &\quad = g(\xi_{k'_j+1})\big(f(\lambda_j) - f(\lambda_j-)\big) + 0 + \ldots + 0 + g(\xi_{k_j+1})\big(f(\lambda_j+) - f(\lambda_j)\big). \end{aligned} \tag{625}$$

Also, regardless of whether $k'_j = k_j$ or $k'_j < k_j$, for each k with $k'_j + 1 \leq k \leq k_j + 1$ we have $\xi_k \in [x_{k-1}, x_k]$ as well as $\lambda_j \in [x_{k-1}, x_k]$; thus $|\xi_k - \lambda_j| \leq \delta$ and $|g(\xi_k) - g(\lambda_j)| < \varepsilon$. If $k'_j = k_j$ this means that

$$\begin{aligned} &\left| g(\lambda_j)\big(f(\lambda_j+) - f(\lambda_j-)\big) - \sum_{k=k'_j+1}^{k_j+1} g(\xi_k)\big(f(x_k) - f(x_{k-1})\big) \right| \\ &= \left| \big(g(\lambda_j) - g(\xi_{k_j+1})\big)\big(f(\lambda_j+) - f(\lambda_j-)\big) \right| \leq \left| f(\lambda_j+) - f(\lambda_j-) \right| \varepsilon, \end{aligned}$$

while if $k'_j < k_j$ we get

$$\begin{aligned} &\left| g(\lambda_j)\big(f(\lambda_j+) - f(\lambda_j-)\big) - \sum_{k=k'_j+1}^{k_j+1} g(\xi_k)\big(f(x_k) - f(x_{k-1})\big) \right| \\ &= \left| \big(g(\lambda_j) - g(\xi_{k'_j+1})\big)\big(f(\lambda_j) - f(\lambda_j-)\big) + \big(g(\lambda_j) - g(\xi_{k_j+1})\big)\big(f(\lambda_j+) - f(\lambda_j)\big) \right| \\ &\qquad \leq \left| f(\lambda_j) - f(\lambda_j-) \right| \varepsilon + \left| f(\lambda_j+) - f(\lambda_j) \right| \varepsilon. \end{aligned}$$

Note that $\big|f(\lambda_j+)-f(\lambda_j-)\big| \le \big|f(\lambda_j)-f(\lambda_j-)\big| + \big|f(\lambda_j+)-f(\lambda_j)\big|$ and hence in all cases we have

$$\begin{aligned}
&\Big|g(\lambda_j)\big(f(\lambda_j+)-f(\lambda_j-)\big) - \sum_{k=k_j'+1}^{k_j+1} g(\xi_k)\big(f(x_k)-f(x_{k-1})\big)\Big| \\
&\qquad \le \Big|f(\lambda_j)-f(\lambda_j-)\Big|\varepsilon + \Big|f(\lambda_j+)-f(\lambda_j)\Big|\varepsilon, \qquad (626)
\end{aligned}$$

for all $j \in \{1,2,\dots,n-1\}$.

Finally we have $x_k = \lambda_0 (= A)$ for all $k \le k_0$ and $\lambda_0 < x_{k_0+1} < \lambda_1$ so that $f(x_{k_0+1}) = f(\lambda_0+)$ and

$$\sum_{k=1}^{k_0+1} g(\xi_k)\big(f(x_k)-f(x_{k-1})\big) = g(\xi_{k_0+1})\big(f(\lambda_0+)-f(\lambda_0)\big).$$

As above we have $\big|g(\xi_{k_0+1}) - g(\lambda_0)\big| < \varepsilon$, and hence

$$\Big|g(\lambda_0)\big(f(\lambda_0+)-f(\lambda_0)\big) - \sum_{k=1}^{k_0+1} g(\xi_k)\big(f(x_k)-f(x_{k-1})\big)\Big| \le \Big|f(\lambda_0+)-f(\lambda_0)\Big|\varepsilon. \qquad (627)$$

Similarly we have $x_k = \lambda_n (= B)$ for all $k \ge k_n' + 1$ and $\lambda_{n-1} < x_{k_n'} < \lambda_n$ so that $f(x_{k_n'}) = f(\lambda_n-)$ and

$$\sum_{k=k_n'+1}^{N} g(\xi_k)\big(f(x_k)-f(x_{k-1})\big) = g(\xi_{k_n'+1})\big(f(\lambda_n)-f(\lambda_n-)\big).$$

As above we have $\big|g(\xi_{k_n'+1}) - g(\lambda_n)\big| < \varepsilon$, and hence

$$\Big|g(\lambda_n)\big(f(\lambda_n)-f(\lambda_n-)\big) - \sum_{k=k_n'+1}^{N} g(\xi_k)\big(f(x_k)-f(x_{k-1})\big)\Big| \le \Big|f(\lambda_n)-f(\lambda_n-)\Big|\varepsilon. \qquad (628)$$

The total difference which we are interested in is:

$$\Big|\big(f(\lambda_0+)-f(\lambda_0)\big)g(\lambda_0)+\sum_{j=1}^{n-1}\big(f(\lambda_j+)-f(\lambda_j-)\big)g(\lambda_j)+\big(f(\lambda_n)-f(\lambda_n-)\big)g(\lambda_n)$$
$$-S(\{x_n\},\{\xi_n\})\Big|$$
$$=\Big|\big(f(\lambda_0+)-f(\lambda_0)\big)g(\lambda_0)+\sum_{j=1}^{n-1}\big(f(\lambda_j+)-f(\lambda_j-)\big)g(\lambda_j)+\big(f(\lambda_n)-f(\lambda_n-)\big)g(\lambda_n)$$
$$-\sum_{k=1}^{k_0+1}g(\xi_k)\big(f(x_k)-f(x_{k-1})\big)-\sum_{j=0}^{n-1}\sum_{k_j+1<k\leq k'_{j+1}}g(\xi_k)\big(f(x_k)-f(x_{k-1})\big)$$
$$-\sum_{j=1}^{n-1}\sum_{k=k'_j+1}^{k_j+1}g(\xi_k)\big(f(x_k)-f(x_{k-1})\big)-\sum_{k=k'_n+1}^{N}g(\xi_k)\big(f(x_k)-f(x_{k-1})\big)\Big|$$

Using (623), (626), (627) and (628) and the triangle inequality we see that this is

$$\leq\Big|f(\lambda_0+)-f(\lambda_0)\Big|\varepsilon+\sum_{j=1}^{n-1}\Big(\Big|f(\lambda_j)-f(\lambda_j-)\Big|+\Big|f(\lambda_j+)-f(\lambda_j)\Big|\Big)\varepsilon+\Big|f(\lambda_n)-f(\lambda_n-)\Big|\varepsilon.$$

Since f is fixed in our argument, the above can be made arbitrarily small by taking $\varepsilon\to 0$. Hence the value for $\int_A^B g(x)\,df(x)$ given by Definition 3.3 (viz., the limit of $S(\{x_n\},\{\xi_n\})$ as mesh$\{x_n\}\to 0$) agrees with the value for $\int_A^B g(x)\,df(x)$ given by Definition 3.1 (viz., $\big(f(\lambda_0+)-f(\lambda_0)\big)g(\lambda_0)+\sum_{j=1}^{n-1}\big(f(\lambda_j+)-f(\lambda_j-)\big)g(\lambda_j)+\big(f(\lambda_n)-f(\lambda_n-)\big)g(\lambda_n)$). □

Proof of Lemma 3.4, "second part". (Viz: Proof that if $g\in C([A,B])$ and $f\in C^1([A,B])$ then the Riemann-Stieltjes integral $\int_A^B g(x)\,df(x)$ as defined in Definition 3.3.) See (e.g.) [39, Appendix A, Thm. 3].

Proof of the claims in Remark 3.2. Recall that we have defined

(629) $$f(x)=\begin{cases}1 & \text{if } 0\leq x\leq 1\\ 0 & \text{otherwise;}\end{cases}\qquad g(x)=\begin{cases}1 & \text{if } 0< x\leq 1\\ 0 & \text{otherwise.}\end{cases}$$

Proof that $\int_{-1}^0 g\,df$ *exists:* For every partition $-1=x_0\leq x_1\leq\ldots\leq x_N=0$ and every choice of $\xi_j\in[x_{j-1},x_j]$ for $j=1,2,\ldots,N$, using $f(x)=0$ for $x<0$ and $f(0)=1$ we find that we find that $S(\{x_n\},\{\xi_n\})=\sum_{n=1}^N g(\xi_n)\big(f(x_n)-f(x_{n-1})\big)=g(\xi_k)$, where $k\in\{1,\ldots,N\}$ is the unique index for which $x_{k-1}<x_k=x_{k+1}=\ldots=x_N=0$. But from the definition of g we see that $g(\xi_k)=0$, since $\xi_k\in[x_{k-1},x_k]$ and $x_k=0$. Hence $S(\{x_n\},\{\xi_n\})=0$ for *every* admissible choice of $\{x_n\}$ and $\{\xi_n\}$, and hence (by Def. 3.3) $\int_{-1}^0 g(x)\,df(x)=0$.

Proof that $\int_0^1 g\,df$ exists: For every partition $0 = x_0 \leq x_1 \leq \ldots \leq x_N = 1$ and every choice of $\xi_j \in [x_{j-1}, x_j]$ for $j = 1, 2, \ldots, N$, using $f(x) = 1$ for all $x \geq 0$ we find that $S(\{x_n\}, \{\xi_n\}) = \sum_{n=1}^{N} g(\xi_n)\big(f(x_n) - f(x_{n-1})\big) = 0$. Hence $\int_0^1 g(x)\,df(x) = 0$.

Proof that $\int_{-1}^1 g\,df$ does not exist: For an arbitrary partition $-1 = x_0 \leq x_1 \leq \ldots \leq x_N = 1$ and an arbitrary choice of $\xi_j \in [x_{j-1}, x_j]$ for $j = 1, 2, \ldots, N$, using the formula for $f(x)$ we find that $S(\{x_n\}, \{\xi_n\}) = \sum_{n=1}^{N} g(\xi_n)\big(f(x_n) - f(x_{n-1})\big) = g(\xi_k)$, where $k \in \{1, \ldots, N\}$ is the unique index for which $x_{k-1} < 0 \leq x_k$. Hence (by the formula for $g(x)$) $S(\{x_n\}, \{\xi_n\}) = 0$ if $\xi_k \leq 0$ and $S(\{x_n\}, \{\xi_n\}) = 1$ if $\xi_k > 0$. But clearly, for any given $\delta > 0$, there exists a partition $-1 = x_0 \leq x_1 \leq \ldots \leq x_N = 1$ with $\text{mesh}\{x_n\} \leq \delta$ such that (for some k) $x_{k-1} < 0 < x_k$, and then there is an admissible choice of the ξ_j's for which $\xi_k = x_{k-1} < 0$ and thus $S(\{x_n\}, \{\xi_n\}) = 0$, and there is another admissible choice of the ξ_j's for which $\xi_k = x_k \geq 0$ and thus $S(\{x_n\}, \{\xi_n\}) = 1$. Hence $\int_{-1}^1 g\,df$ does not exist.

Proof of the claim just below Remark 3.2. Assume that $A < C < B$, $g \in C([A, B])$, that f is a bounded function on $[A, B]$, and that the two Riemann-Stieltjes integrals $I_1 = \int_A^C g(x)\,df(x)$ and $I_2 = \int_C^B g(x)\,df(x)$ both exist. We then wish to prove that $\int_A^B g(x)\,df(x)$ also exists, and equals $I_1 + I_2$.

Let $\varepsilon > 0$ be given. By our assumptions there is some δ such that $\big|S(\{x_n\}, \{\xi_n\}) - I_1\big| < \varepsilon$ holds for every partition $A = x_0 \leq x_1 \leq \ldots \leq x_N = C$ with $\text{mesh}\{x_n\} < \delta$ and any numbers $\xi_j \in [x_{j-1}, x_j]$ for $j = 1, 2, \ldots, N$, and *also* $\big|S(\{x_n\}, \{\xi_n\}) - I_2\big| < \varepsilon$ holds for every partition $C = x_0 \leq x_1 \leq \ldots \leq x_N = B$ with $\text{mesh}\{x_n\} < \delta$ and any numbers $\xi_j \in [x_{j-1}, x_j]$ for $j = 1, 2, \ldots, N$. Furthermore, since by assumption $\sup_{[A,B]} |f| < \infty$ and g is continuous, we may also assume that δ is so small that $\big|g(x) - g(C)\big| \cdot \sup_{[A,B]} |f| < \varepsilon$ for all $x \in [A, B]$ with $|x - C| \leq \delta$.

Now let $A = x_0 \leq x_1 \leq \ldots \leq x_N = B$ be an arbitrary partition with $\text{mesh}\{x_n\} < \delta$ and assume $\xi_j \in [x_{j-1}, x_j]$ for $j = 1, 2, \ldots, N$. Let $k \in \{1, \ldots, N\}$ be the (unique) index for which $x_{k-1} < C \leq x_k$. Set $x'_j := x_j$ for $j = 1, 2, \ldots, k-1$; $x'_k := C$; $\xi'_j := \xi_j$ for $j = 1, 2, \ldots, k-1$ and $\xi'_k := C$; then the sequence $\{x'_n\}$ is a partition $A \leq x'_0 \leq x'_1 \leq \ldots \leq x'_k = C$ with $\text{mesh}\{x'_n\} \leq \delta$, and $\xi'_j \in [x'_{j-1}, x'_j]$ for all $j = 1, 2, \ldots, k$; hence by our assumptions we have $\big|S(\{x'_n\}, \{\xi'_n\}) - I_1\big| < \varepsilon$, i.e.

$$\Big|\sum_{n=1}^{k-1} g(\xi_n)\big(f(x_n) - f(x_{n-1})\big) + g(C)\big(f(C) - f(x_{k-1})\big) - I_1\Big| < \varepsilon. \tag{630}$$

Similarly, set $x''_0 = C$, $x''_j := x_{j+k-1}$ for $j = 1, 2, \ldots, N-k+1$; $\xi''_1 = C$ and $\xi''_j = \xi_{j+k-1}$ for $j = 2, \ldots, N-k+1$. Then the sequence $\{x''_n\}$ is a partition $C \leq x''_0 \leq x''_1 \leq \ldots \leq x''_{N-k+1} = B$ with $\text{mesh}\{x''_n\} \leq \delta$ and $\xi''_j \in [x''_{j-1}, x''_j]$ for all $j = 1, 2, \ldots, N-k+1$; hence by our assumptions we have $\left|S(\{x_n''\},\{\xi_n''\}) - I_2\right| < \varepsilon$, i.e.

$$\left|g(C)\big(f(x_k)-f(C)\big) + \sum_{n=k+1}^{N} g(\xi_n)\big(f(x_n)-f(x_{n-1})\big) - I_2\right| < \varepsilon. \tag{631}$$

Let us also note that

$$\begin{aligned}&\left|g(\xi_k)\big(f(x_k)-f(x_{k-1})\big) - g(C)\big(f(C)-f(x_{k-1})\big) - g(C)\big(f(x_k)-f(C)\big)\right| \\ &= \left|g(\xi_k)-g(C)\right|\cdot\left|f(x_k)-f(x_{k-1})\right| \le \left|g(\xi_k)-g(C)\right|\cdot 2\sup_{[A,B]}|f| < 2\varepsilon,\end{aligned} \tag{632}$$

where the last inequality follows from our choice of δ and the fact that both $C,\xi_k \in [x_{k-1},x_k]$ and $x_k - x_{k-1} \le \delta$; thus $|\xi_k - C| \le \delta$.

Finally we note that by the triangle inequality, the absolute difference

$$\left|S(\{x_n\},\{\xi_n\}) - (I_1+I_2)\right| = \left|\sum_{n=1}^{N} g(\xi_n)\big(f(x_n)-f(x_{n-1})\big) - (I_1+I_2)\right|$$

is less than or equal to the sum of the absolute differences in (630), (631), (632), and hence

$$< 4\varepsilon.$$

Since this is true for all admissible $\{x_n\},\{\xi_n\}$ with $\text{mesh}\{x_n\} < \delta$, and ε was arbitrarily small, we conclude that $\int_A^B g(x)\,df(x)$ exists and equals $I_1 + I_2$. □

3.3. Write $A(x) = \sum_{1\le n\le x} a_n n^{-\frac{1}{2}}$; then the assumption says that $A(x) \sim x$ as $x \to \infty$, i.e. $\lim_{x\to\infty} \frac{A(x)}{x} = 1$. Hence for any given $\varepsilon > 0$ we can find $X > 1$ so that $\left|\frac{A(x)}{x} - 1\right| < \varepsilon$ for all $x \ge X$, i.e.

$$\left|A(x) - x\right| < \varepsilon x, \qquad \forall x \ge X. \tag{633}$$

Now for $N \ge 1$ we have

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} n^{\frac{1}{2}}(a_n n^{-\frac{1}{2}}) = \int_{1-}^{N} x^{\frac{1}{2}}\,dA(x) = N^{\frac{1}{2}}A(N) - \frac{1}{2}\int_1^N x^{-\frac{1}{2}}A(x)\,dx. \tag{634}$$

Note that *if* $A(x) \equiv x$ then the last expression equals $N^{\frac{1}{2}}N - \frac{1}{2}\int_1^N x^{-\frac{1}{2}}x\,dx = N^{\frac{3}{2}} - \frac{1}{2}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_{x=1}^{x=N} = \frac{2}{3}N^{\frac{3}{2}} + \frac{1}{3}$. Hence

$$\begin{aligned}\left|\sum_{n=1}^{N} a_n - \tfrac{2}{3}N^{\frac{3}{2}}\right| &= \left|N^{\frac{1}{2}}A(N) - \frac{1}{2}\int_1^N x^{-\frac{1}{2}}A(x)\,dx - \Big(N^{\frac{1}{2}}N - \frac{1}{2}\int_1^N x^{-\frac{1}{2}}x\,dx\Big) + \frac{1}{3}\right| \\ &\le N^{\frac{1}{2}}\left|A(N)-N\right| + \frac{1}{2}\int_1^N x^{-\frac{1}{2}}\left|A(x)-x\right|dx + \frac{1}{3}\end{aligned}$$

If $N \geq X$ then this is

$$< N^{\frac{1}{2}}\varepsilon N + \frac{1}{2}\int_1^X x^{-\frac{1}{2}}\big|A(x)-x\big|\,dx + \frac{1}{2}\int_X^N x^{-\frac{1}{2}}\varepsilon x\,dx + \frac{1}{3}$$
$$= \varepsilon\frac{4}{3}N^{\frac{3}{2}} + \Big(\frac{1}{2}\int_1^X x^{-\frac{1}{2}}\big|A(x)-x\big|\,dx - \frac{1}{3}\varepsilon X^{\frac{3}{2}} + \frac{1}{3}\Big)$$

The expression in the big parenthesis does not depend on N and hence is $< \varepsilon\frac{2}{3}N^{\frac{3}{2}}$ (say) for all sufficiently large N. Hence for all sufficiently large N we have $\Big|\sum_{n=1}^N a_n - \frac{2}{3}N^{\frac{3}{2}}\Big| < 2\varepsilon N^{\frac{3}{2}}$, and thus, after division with $\frac{2}{3}N^{\frac{3}{2}}$,

$$\Big|\frac{\sum_{n=1}^N a_n}{\frac{2}{3}N^{\frac{3}{2}}} - 1\Big| < 3\varepsilon.$$

Since ε was arbitrarily small we conclude that $\lim_{N\to\infty}\frac{\sum_{n=1}^N a_n}{\frac{2}{3}N^{\frac{3}{2}}} = 1$, i.e. $\sum_{n=1}^N a_n \sim \frac{2}{3}N^{\frac{3}{2}}$ as $N\to\infty$. □

3.4. (a). Note that $A \geq 0$, since $N(r) > 1$ for all sufficiently large r. If there is some r such that $N(r)=\infty$ then $A=\infty$ and there is nothing to prove. Hence from now on we may assume that $A<\infty$, and thus $N(r)<\infty$ for all r; then N is an increasing function $N:\mathbb{R}\to\mathbb{Z}_{\geq 0}$ with $N(r)=0$ for all $r<0$. Note that for every number $A_1 > A$ there is some $R>0$ such that $\frac{\log N(r)}{\log r} < A_1$ for all $r \geq R$, hence by exponentiating: $N(r) < r^{A_1}$ for all $r\geq R$. This implies that $N(r) \ll (1+r)^{A_1}$ for all $r \geq 0$ (where the implied constant may depend on A_1).

We have for every $\alpha > 0$:

$$\sum_{j=1}^{\infty}(1+|\rho_j|)^{-\alpha} = \int_{0-}^{\infty}(1+x)^{-\alpha}\,dN(x) = \lim_{X\to\infty}\int_{0-}^{X}(1+x)^{-\alpha}\,dN(x) \tag{635}$$
$$= \lim_{X\to\infty}\Big(\Big[(1+x)^{-\alpha}N(x)\Big]_{x=-1}^{x=X} + \alpha\int_0^X (1+x)^{-\alpha-1}N(x)\,dx\Big).$$
$$= \lim_{X\to\infty}\Big((1+X)^{-\alpha}N(X) + \alpha\int_0^X (1+x)^{-\alpha-1}N(x)\,dx\Big).$$

Note that all terms in the last sum are non-negative. Now if $\alpha > A_1$ then for all $X > 0$ we have

$$(636) \qquad (1+X)^{-\alpha}N(X) + \alpha \int_0^X (1+x)^{-\alpha-1}N(x)\,dx$$
$$\ll (1+X)^{-\alpha}(1+X)^{A_1} + \alpha \int_0^X (1+x)^{-\alpha-1}(1+x)^{A_1}\,dx$$
$$= (1+X)^{-\alpha+A_1} + \frac{\alpha}{\alpha - A_1}\Big(1-(1+X)^{-\alpha+A_1}\Big) < \frac{\alpha}{\alpha - A_1}.$$

Since this is true for all $X > 0$ and also the left hand side in (636) is an increasing function of X (being equal to $\int_{0-}^X (1+x)^{-\alpha}\,dN(x)$), it follows that the limit as $X \to \infty$ exists and is finite. Thus: $\sum_{j=1}^\infty (1+|\rho_j|)^{-\alpha}$ converges. We have proved this for every $\alpha > A_1$ and every $A_1 > A$; hence $\sum_{j=1}^\infty (1+|\rho_j|)^{-\alpha}$ converges for every $\alpha > A$, and thus by (120) we have $\tau \le A$. □

(b). Suppose that $\alpha > \tau$ and thus that $\sum_{j=1}^\infty (1+|\rho_j|)^{-\alpha}$ converges, say to the number $C > 0$. Then by the identity (635) (and the fact that $\int_{0-}^X (1+x)^{-\alpha}\,dN(x)$ is an increasing function of X), we have

$$(637) \qquad (1+X)^{-\alpha}N(X) + \alpha \int_0^X (1+x)^{-\alpha-1}N(x)\,dx \le C, \qquad \forall X > 0.$$

Hence $(1+X)^{-\alpha}N(X) \le C$ for all $X > 0$ and this implies $N(X) \le C(1+X)^{\alpha}$ and (by taking the logarithm) $\log N(X) \le \log C + \alpha \log(1+X)$, for all $X > 0$. Hence for $X > 1$ we have

$$(638) \qquad \frac{\log N(X)}{\log X} \le \frac{\log C}{\log X} + \alpha \frac{\log(1+X)}{\log X},$$

and taking $X \to \infty$ this implies

$$(639) \qquad \limsup_{X\to\infty} \frac{\log N(X)}{\log X} \le 0 + \alpha,$$

i.e. $A \le \alpha$. Since this is true for all $\alpha > \tau$ we conclude that $A \le \tau$. □

Remark: The above solution was by "standard & routine but rather stupid use of the Riemann-Stieltjes integral". There are certainly alternative, shorter ways to solve the problem. Here is an example:

If $N(r) = \infty$ for some r then clearly both $A = \infty$ and $\tau = \infty$. Hence from now on we may assume that $N(r)$ is finite for all r. We may then re-order the ρ_j's so that $|\rho_1| \le |\rho_2| \le \ldots$. Now assume that $A_1 > A$; then there is some $R > 0$ such that $N(r) < r^{A_1}$ for all $r \ge R$. Hence for every integer $j > R^{A_1}$, if we set $r := j^{\frac{1}{A_1}} > R$ then $|\rho_j| \le r$ cannot hold, since then $|\rho_1| \le \ldots \le |\rho_j| \le r$ and thus $N(r) \ge j = r^{A_1}$, a contradiction. Hence:

$$|\rho_j| > j^{\frac{1}{A_1}} \qquad \text{for every integer } j > R^{A_1}.$$

Hence for any $\alpha > 0$ we have $\sum_{j>R^{A_1}}(1+|\rho_j|)^{-\alpha} < \sum_{j>R^{A_1}} j^{-\frac{\alpha}{A_1}}$, and thus the sum $\sum_{j=1}^{\infty}(1+|\rho_j|)^{-\alpha}$ converges for every $\alpha > A_1$. Hence $\tau \leq A_1$, and since this is true for every $A_1 > A$ we conclude $\tau \leq A$.

On the other hand, if $A_1 < A$ then there is a sequence of r-values, $0 < r_1 < r_2 < \ldots \to \infty$ such that $\frac{\log N(r_k)}{\log r_k} > A_1$ for all k, and thus $N(r_k) > r_k^{A_1}$. Hence (for any given $\alpha > 0$) $\sum_{j=1}^{\infty}(1+|\rho_j|)^{-\alpha} > N(r_k)(1+r_k)^{-\alpha} > r_k^{A_1}(1+r_k)^{-\alpha}$. If $\alpha < A_1$ then $r_k^{A_1}(1+r_k)^{-\alpha} \to \infty$ as $k \to \infty$, and thus $\sum_{j=1}^{\infty}(1+|\rho_j|)^{-\alpha} = \infty$. Since this is true for every $\alpha < A_1$ it follows that $\tau \geq A_1$. Since this is true for every $A_1 < A$ it follows that $\tau \geq A$.

3.6. Assume that $\vartheta(x) \sim x$ as $x \to \infty$. Let $\varepsilon > 0$ be given. Then there is some $X > 1$ such that

$$\big|\vartheta(x) - x\big| < \varepsilon x, \qquad \forall x \geq X. \tag{640}$$

Now for all $x > 1$ we have

$$\pi(x) = \sum_{p \leq x} 1 = \int_{2-}^{x} \frac{1}{\log y}\, d\vartheta(y) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{1}{y \log^2 y}\, \vartheta(y)\, dy. \tag{641}$$

Here the integral turns out to be small compared with $\frac{\vartheta(x)}{\log x}$: It follows from (640) that $\vartheta(x) \ll x$ for *all* $x \geq 1$, and hence, for all $x \geq 4$:

$$\int_2^x \frac{1}{y\log^2 y}\,\vartheta(y)\,dx \ll \int_2^x \frac{dy}{\log^2 y} \leq \int_2^{\sqrt{x}} \frac{dy}{\log^2 2} + \int_{\sqrt{x}}^x \frac{dy}{\log^2(\sqrt{x})} < \frac{\sqrt{x}}{\log^2 2} + \frac{x}{\log^2(\sqrt{x})} \ll \frac{x}{\log^2 x},$$

and thus for all sufficiently large x we have

$$\int_2^x \frac{1}{y\log^2 y}\,\vartheta(y)\,dx \leq \varepsilon \frac{x}{\log x}.$$

It now follows that for all sufficiently large x we have

$$\Big|\pi(x) - \frac{x}{\log x}\Big| = \Big|\frac{\vartheta(x)}{\log x} + \int_2^x \frac{1}{y\log^2 y}\,\vartheta(y)\,dy - \frac{x}{\log x}\Big| \leq \frac{\big|\vartheta(x) - x\big|}{\log x} + \varepsilon\frac{x}{\log x} < 2\varepsilon\frac{x}{\log x}.$$

Since ε is arbitrarily small this implies that

$$\pi(x) \sim \frac{x}{\log x} \qquad \text{as } x \to \infty.$$

□

3.7. (a). Let σ_a be the abscissa of absolute convergence of $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. We know $\sigma_a < \infty$ by Proposition 3.9. Now for $\sigma > \max(\sigma_a, 1)$ both the Dirichlet series

$\alpha(s) = \sum_{k=1}^{\infty} a_k k^{-s}$ and $\zeta(s) = \sum_{m=1}^{\infty} m^{-s}$ are absolutely convergent, and hence we may multiply the two to get an absolutely convergent double sum:

$$\zeta(s)\alpha(s) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} a_k k^{-s} m^{-s} = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} a_k (km)^{-s}. \tag{642}$$

Substituting $n = km$ we get

$$= \sum_{n=1}^{\infty}\sum_{k|n} a_k n^{-s} = \sum_{n=1}^{\infty} b_n n^{-s} = \beta(s). \tag{643}$$

In particular it follows that $\beta(s)$ has abscissa of absolute convergence $\leq \max(\sigma_a, 1)$. □

(b). For $\sigma > \max(\sigma_a, 1)$ we have both $\zeta(s)\alpha(s) = \beta(s)$ and $\zeta(s) \neq 0$, and hence $\alpha(s) = \zeta(s)^{-1}\beta(s)$. Also for $\sigma > \max(\sigma_a, 1)$ both the Dirichlet series $\beta(s) = \sum_{k=1}^{\infty} b_k k^{-s}$ and $\zeta(s)^{-1} = \sum_{m=1}^{\infty} \mu(m) m^{-s}$ (cf. Problem 2.1 (a)) are absolutely convergent, and hence we may multiply the two to get an absolutely convergent double sum:

$$\alpha(s) = \zeta(s)^{-1}\beta(s) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} b_k k^{-s}\mu(m) m^{-s} = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} b_k \mu(m)(km)^{-s}. \tag{644}$$

Substituting $n = km$ we get $= \sum_{n=1}^{\infty}\left(\sum_{k|n} b_k \mu\left(\frac{n}{k}\right)\right) n^{-s}$, i.e. we conclude that

$$\sum_{n=1}^{\infty} a_n n^{-s} = \alpha(s) = \sum_{n=1}^{\infty}\left(\sum_{k|n} b_k \mu\left(\frac{n}{k}\right)\right) n^{-s} \tag{645}$$

for all s with $\sigma > \max(\sigma_a, 1)$. Hence by Proposition 3.10, $a_n = \sum_{k|n} b_k \mu\left(\frac{n}{k}\right)$ for all $n \in \mathbb{Z}^+$. □

(c). Let $\{a_n\}$ be an arbitrary sequence of complex numbers, and set $b_n = \sum_{d|n} a_d$ for $n = 1, 2, \ldots$.

Now *fix* some $m \in \mathbb{Z}^+$. We wish to prove $a_m = \sum_{d|m} \mu(m/d) b_d$ for *this* m. Form new sequences $\{a'_n\}$ and $\{b'_n\}$ by letting

$$a'_n = \begin{cases} a_n & \text{if } n \mid m \\ 0 & \text{otherwise;} \end{cases} \qquad b'_n = \sum_{d|n} a'_d. \tag{646}$$

Then $a'_n = 0$ holds for all but finitely many n's; hence the Dirichlet series $\sum_{n=1}^{\infty} a'_n n^{-s}$ has abscissa of convergence $-\infty$, and hence by part (b) we have $a'_n = \sum_{d|n} \mu(n/d) b'_d$ for all $n \in \mathbb{Z}^+$. In particular we have

$$a'_m = \sum_{d|m} \mu(m/d) b'_d. \tag{647}$$

Here, by the definition of $\{a'_n\}$ we have $a'_d = a_d$ for all $d \mid m$, and thus by the definition of $\{b'_n\}$ we have $b'_n = \sum_{d|n} a'_d = \sum_{d|n} a_d = b_n$ for all $n \mid m$. Hence (647) gives

$$a_m = \sum_{d|m} \mu(m/d) b_d. \tag{648}$$

□

Alternative: Let us apply part (b) to the sequence $\{a_n\}$ given by $a_1 = 1$ and $a_n = 0$ for all $n \geq 2$. For this sequence we have $b_n = 1$ for all n, and clearly $\sum_{n=1}^{\infty} a_n n^{-s}$ has abscissa of convergence $-\infty$, so that (b) applies; hence we get:

$$\sum_{d|n} \mu(n/d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases} \tag{649}$$

The same formula can also be written as (by substituting $k = n/d$)

$$\sum_{k|n} \mu(k) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2. \end{cases} \tag{650}$$

This is an important formula; note that (distilling the above proof) it follows directly from the identity $\zeta(s)\zeta(s)^{-1} = 1$, using Proposition 3.10.

Using (650) we can easily solve the problem: Let $\{a_n\}$ be an arbitrary sequence of complex numbers, and set $b_n = \sum_{d|n} a_d$ for $n = 1, 2, \ldots$. We then have, for any $n \in \mathbb{Z}^+$:

$$\sum_{d|n} \mu(n/d) b_d = \sum_{d|n} \mu(n/d) \sum_{k|d} a_k = \sum_{k|n} a_k \sum_{\substack{d|n \\ (k|d)}} \mu(n/d). \tag{651}$$

In the inner sum we substitute $j = n/d$; thus j runs through all divisors of n such that $k \mid \frac{n}{j}$, or equivalently $j \mid \frac{n}{k}$. This gives:

$$= \sum_{k|n} a_k \sum_{j|\frac{n}{k}} \mu(j) \tag{652}$$

Applying here (650) we get

$$= \sum_{k|n} a_k \left\{ \begin{matrix} 1 & \text{if } n/k = 1 \\ 0 & \text{if } n/k \geq 2 \end{matrix} \right\} = a_n. \tag{653}$$

□

Remark: The formula (650) can also easily be proved directly from the definition of $\mu(k)$. Namely, if $n \geq 2$ has the prime factorization $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ (with $p_1, \ldots, p_r$ distinct primes and all $\alpha_j \geq 1$, as usual), then

$$\sum_{k|n} \mu(k) = \sum_{j_1=0}^{\alpha_1} \sum_{j_2=0}^{\alpha_2} \cdots \sum_{j_r=0}^{\alpha_r} \mu(p_1^{j_1} p_2^{j_2} \cdots p_r^{j_r}) = \sum_{j_1=0}^{1} \sum_{j_2=0}^{1} \cdots \sum_{j_r=0}^{1} (-1)^{j_1+j_2+\ldots+j_r}$$
$$= \Big(\sum_{j_1=0}^{1}(-1)^{j_1}\Big)\Big(\sum_{j_2=0}^{1}(-1)^{j_2}\Big)\cdots\Big(\sum_{j_r=0}^{1}(-1)^{j_r}\Big) = 0 \cdot 0 \cdots 0 = 0. \tag{654}$$

On the other hand for $n = 1$ we of course trivially have $\sum_{k|n} \mu(k) = \mu(1) = 1$, corresponding to the case "$r = 0$" of the above computation. This proves the formula (650).

From this we see that the machinery with Dirichlet series is not at all needed to prove the Möbius inversion formula. On the other hand the methods with Dirichlet series is useful in many similar problems, and in particular when it comes to *finding* the correct inversion formula. (Note that in the computation (651) we *started* with "$\sum_{d|n} \mu(n/d) b_d$" without hinting how one could come up with this formula in the first place.)

3.8. (a). Using Problem 2.2 and Problem 2.1 we see that, when $\sigma > 2$,

$$\sum_{n=1}^{\infty} \phi(n) n^{-s} = \zeta(s-1)\zeta(s)^{-1} = \Big(\sum_{m=1}^{\infty} m^{1-s}\Big)\Big(\sum_{k=1}^{\infty} \mu(k) k^{-s}\Big). \tag{655}$$

Here both sums are absolutely convergent and hence we may multiply termwise to obtain an absolutely convergent double sum:

$$= \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \mu(k) m^{1-s} k^{-s} = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \mu(k) m (mk)^{-s}. \tag{656}$$

Substituting $n = mk$ we get $= \sum_{n=1}^{\infty} \Big(\sum_{m|n} \mu(n/m) m\Big) n^{-s}$, i.e. we have proved that

$$\sum_{n=1}^{\infty} \phi(n) n^{-s} = \sum_{n=1}^{\infty} \Big(\sum_{m|n} \mu(n/m) m\Big) n^{-s} \qquad (\sigma > 2). \tag{657}$$

Hence by Proposition 3.10 we have $\phi(n) = \sum_{m|n} \mu(n/m) m$ for all $n \in \mathbb{Z}^+$. □

(b). We give a combinatorial proof using the inclusion-exclusion principle.

We set $M = \{1, 2, \ldots, n\}$ and let $p_1, p_2, \ldots, p_r$ be the distinct prime divisors of n. For each $j \in \{1, 2, \ldots, r\}$ we let M_j be the subset of those $m \in M$ which are divisible by p_j. Then an element $m \in M$ is coprime to n if and only if $m \notin M_j$ for all $j \in \{1, 2, \ldots, r\}$. Thus

$$\phi(n) = \#\Big(M \setminus \bigcup_{j=1}^{r} M_j\Big) = n - \#\Big(\bigcup_{j=1}^{r} M_j\Big)$$

By the inclusion-exclusion principle we get

$$= n - \sum_{j=1}^{r} \#M_j + \sum_{1\le j_1<j_2\le r} \#(M_{j_1}\cap M_{j_2}) - \sum_{1\le j_1<j_2<j_3\le r} \#(M_{j_1}\cap M_{j_2}\cap M_{j_3}) + \cdots + (-1)^r \#(M_1\cap M_2\cap\ldots\cap M_r).$$

Now for any $1 \le j_1 < j_2 < \ldots < j_s \le r$, the intersection $M_{j_1}\cap M_{j_2}\cap\ldots\cap M_{j_s}$ is exactly the set of those $m \in M$ which are divisible by $d = p_{j_1}p_{j_2}\cdots p_{j_s}$, and thus $\#(M_{j_1}\cap M_{j_2}\cap \ldots\cap M_{j_s}) = n/d$. Note that in the above sums of sums, each squarefree divisor $d > 1$ of n arises exactly ones, and this happens in a sum with a minus sign if d has an odd number of prime factors, but in a sum with a plus sign if d has an even number of prime factors. Hence the contribution corresponding to d is $\mu(d)n/d$. Hence we get

$$\phi(n) = n + \sum_{\substack{d\mid n\\ d>1 \text{ and squarefree}}} \mu(d)\frac{n}{d}.$$

The requirement "d squarefree" can be omitted since $\mu(d) = 0$ whenever d is not squarefree. Furthermore note that $d = 1$ gives $\mu(d)n/d = n$. Hence

$$\phi(n) = \sum_{d\mid n} \mu(d)\frac{n}{d}.$$

This is seen to agree with the desired formula, if we substitute $d = n/d_{new}$. □

3.9. (a). Note that $d(n) = \sigma_0(n)$; hence (a) follows as a special case of (b).
(b). When $\sigma > \max(1, 1+\text{Re}\alpha)$ we have $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ and $\zeta(s-\alpha) = \sum_{m=1}^{\infty} m^{\alpha-s}$, with both sums being absolutely convergent. Hence we may multiply the two sums termwise to get an absolutely convergent double sum:

$$\zeta(s)\zeta(s-\alpha) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} k^{-s}m^{\alpha-s} = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} m^{\alpha}(km)^{-s}. \tag{658}$$

Here substitute $n = km$; then we get

$$= \sum_{n=1}^{\infty}\Big(\sum_{m\mid n} m^{\alpha}\Big)n^{-s} = \sum_{n=1}^{\infty} \sigma_\alpha(n)n^{-s}. \tag{659}$$

□

3.10. The given formula for b_n may be written as $b_n = \sum_{d\mid n} |\mu(d)|a_{n/d}$, and substituting $d_{new} = n/d$ this becomes $b_n = \sum_{d\mid n} |\mu(n/d)|a_d$. To start with, let us assume that the Dirichlet series $\alpha(s) = \sum_{m=1}^{\infty} a_m m^{-s}$ has abscissa of convergence $\sigma_a < \infty$. Using the formula $\frac{\zeta(s)}{\zeta(2s)} = \sum_{k=1}^{\infty} |\mu(k)|k^{-s}$ ($\sigma > 1$, with absolute convergence) from Problem 2.3(b), we get, for all $\sigma > \max(\sigma_a, 1)$:

(660) $$\frac{\zeta(s)}{\zeta(2s)}\alpha(s) = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}|\mu(k)|a_m(km)^{-s},$$

the double sum being absolutely convergent. Substituting $n = km$ this becomes

(661) $$= \sum_{n=1}^{\infty}\Big(\sum_{m|n}|\mu(n/m)|a_m\Big)n^{-s} = \sum_{n=1}^{\infty} b_n n^{-s}.$$

We have thus proved that for any s with $\sigma > \max(\sigma_a, 1)$ the Dirichlet series $\beta(s) := \sum_{n=1}^{\infty} b_n n^{-s}$ is absolutely convergent, and equals $\frac{\zeta(s)}{\zeta(2s)}\alpha(s)$. Hence for these s we have $\alpha(s) = \frac{\zeta(2s)}{\zeta(s)}\beta(s)$, and using Problem 2.3(a) this gives:

(662) $$\sum_{n=1}^{\infty} a_n n^{-s} = \alpha(s) = \frac{\zeta(2s)}{\zeta(s)}\beta(s) = \Big(\sum_{k=1}^{\infty}\lambda(k)k^{-s}\Big)\Big(\sum_{m=1}^{\infty} b_m m^{-s}\Big)$$
$$= \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\lambda(k)b_m(mk)^{-s} = \sum_{n=1}^{\infty}\Big(\sum_{m|n}\lambda(\frac{n}{m})b_m\Big)n^{-s}.$$

Hence by Proposition 3.10,

(663) $$a_n = \sum_{m|n}\lambda(\frac{n}{m})b_m, \qquad \forall n \in \mathbb{Z}^+.$$

This formula has been proved under the assumption that the Dirichlet series $\alpha(s) = \sum_{m=1}^{\infty} a_m m^{-s}$ has abscissa of convergence $\sigma_a < \infty$, but in exactly the same way as in Problem 3.7(c) it can now be extended to arbitrary sequences $\{a_n\}$.

Answer: $a_n = \sum_{m|n}\lambda(\frac{n}{m})b_m$, for all $n \in \mathbb{Z}^+$.

3.11. We have already proved that (i)$\Longrightarrow$(ii) (any $B \geq A + 1$ works), cf. Problem 2.4. We now prove (ii)$\Longrightarrow$(i). Assume that (ii) holds. Then $\sum_{n=1}^{\infty} a_n n^{-s}$ has abscissa of convergence $\leq B$ and hence as in the proof of Proposition 3.9 we know that $|a_n| \ll n^{B+\varepsilon}$ for any fixed $\varepsilon > 0$ and all $n \in \mathbb{Z}^+$; in particular there is a constant $C > 1$ such that $|a_n| \leq Cn^{B+1}$ for all $n \in \mathbb{Z}^+$. Now fix a real constant B_0 so large that $2^{B_0-B-1} > 2C$ (thus $B_0 > B + 1$) and $2B_0 - 2 > \operatorname{Re}\alpha$. Then for all $n \geq 2$ we have $Cn^{B+1} < \frac{1}{2}n^{B_0}$, since $n^{B_0-B-1} \geq 2^{B_0-B-1} > 2C$; hence

(664) $$|a_n| \leq \tfrac{1}{2}n^{B_0}, \qquad \forall n \geq 2.$$

Now let us define a new sequence $b_1, b_2, b_3, \ldots$ by the following recipe: Set $b_1 = 1$; for each prime p set $b_p := a_p$, and define $b_{p^2}, b_{p^3}, \ldots$ recursively by $b_{p^{k+1}} = b_p b_{p^k} - p^{\alpha} b_{p^{k-1}}$ for $k = 1, 2, \ldots$, and finally define b_n for composite n in the unique way which makes $\{b_n\}$ multiplicative. We intend to prove that $b_n = a_n$ for all n; this will clearly complete the proof that (i) holds.

For each prime p, note that the recursion formula gives for b_{p^2}, using (664):

$$\left|b_{p^2}\right| = \left|b_p^2 - p^{\alpha} b_1\right| \leq \left|b_p^2\right| + p^{\operatorname{Re}\alpha}\left|b_1\right| \leq \tfrac{1}{4}p^{2B_0} + p^{\operatorname{Re}\alpha}$$

Using $2B_0 - 2 > \operatorname{Re}\alpha$ we see that this is

$$< \tfrac{1}{4}p^{2B_0} + p^{2B_0-2} \leq \tfrac{1}{4}p^{2B_0} + \tfrac{1}{4}p^{2B_0} = \tfrac{1}{2}p^{2B_0}.$$

Thus: $\left|b_{p^2}\right| \leq \frac{1}{2}p^{2B_0}$. Similarly one proves by induction that

$$\left|b_{p^k}\right| \leq \tfrac{1}{2}p^{B_0 k}, \qquad \forall k \geq 1. \tag{665}$$

[Proof: Already done for $k = 1, 2$. Now take $k \geq 3$ and assume that the inequality is true for $k-1$ and $k-2$. By the recursion formula we have $\left|b_{p^k}\right| = \left|b_p b_{p^{k-1}} - p^{\alpha} b_{p^{k-2}}\right| \leq \frac{1}{4}p^{B_0+B_0(k-1)} + \frac{1}{2}p^{\operatorname{Re}\alpha}p^{B_0(k-2)} < \frac{1}{4}p^{B_0 k} + \frac{1}{2}p^{2B_0-2}p^{B_0(k-2)} \leq \frac{1}{2}p^{B_0 k}$.]

It follows from (665) that $1 + \left|b_p p^{-s}\right| + \left|b_{p^2}p^{-2s}\right| + \left|b_{p^3}p^{-3s}\right| + \ldots < \infty$ for all s with $\sigma > B_0$, and in fact if $\sigma > B_0 + 1$ then

$$\sum_{k=1}^{\infty}\left|b_{p^k}p^{-ks}\right| \leq \frac{1}{2}\sum_{k=1}^{\infty} p^{(B_0-\sigma)k} = \frac{1}{2}\frac{p^{B_0-\sigma}}{1-p^{B_0-\sigma}} < \frac{1}{2}\frac{p^{B_0-\sigma}}{1-2^{-1}} = p^{B_0-\sigma}, \tag{666}$$

and thus for $\sigma > B_0 + 1$ the product

$$\prod_p \Big(1 + \left|b_p p^{-s}\right| + \left|b_{p^2}p^{-2s}\right| + \left|b_{p^3}p^{-3s}\right| + \ldots\Big) \tag{667}$$

is absolutely convergent. Hence by Lemma 2.8 we have

$$\prod_p \Big(1 + b_p p^{-s} + b_{p^2}p^{-2s} + b_{p^3}p^{-3s} + \ldots\Big) = \sum_{n=1}^{\infty} b_n n^{-s} \tag{668}$$

for all s with $\sigma > B_0 + 1$ (the right hand side also being absolutely convergent for these s). But on the other hand, because of the recursion formula for b_{p^k}, we have for each prime p (by the same computation as in the solution to Problem 2.4):

$$1 + b_p p^{-s} + b_{p^2}p^{-2s} + b_{p^3}p^{-3s} + \ldots = \frac{1}{1 - b_p p^{-s} + p^{\alpha}p^{-2s}} = \frac{1}{1 - a_p p^{-s} + p^{\alpha}p^{-2s}}. \tag{669}$$

Hence, using now our assumption that (ii) holds, we get

$$\sum_{n=1}^{\infty} b_n n^{-s} = \prod_p \Big(1 + b_p p^{-s} + b_{p^2}p^{-2s} + \ldots\Big) = \prod_p \frac{1}{1 - a_p p^{-s} + p^{\alpha}p^{-2s}} = \sum_{n=1}^{\infty} a_n n^{-s}, \tag{670}$$

for all s with $\sigma > B_0 + 1$. Hence by Proposition 3.10 we have $a_n = b_n$ for all $n \in \mathbb{Z}^+$. □

3.12. If $\theta_0 > \limsup_{x\to\infty} \frac{\log|A(x)|}{\log x}$ then (by definition of lim sup) there is some $x_0 > 10$ (say) such that $\theta_0 > \frac{\log|A(x)|}{\log x}$ for all $x \geq x_0$, i.e. $|A(x)| < x^{\theta_0}$ for all $x \geq x_0$. Since $A(x)$ is continuous it then follows that there is some constant $C > 0$ such that $|A(x)| < Cx^{\theta_0}$

for all $x \geq 1$, and thus $\theta_0 \geq \inf\{\theta \in \mathbb{R} : |A(x)| \ll x^\theta, \forall x \geq 1\}$. Since this is true for all $\theta_0 > \limsup_{x\to\infty} \frac{\log|A(x)|}{\log x}$ we conclude that

$$\limsup_{x\to\infty} \frac{\log|A(x)|}{\log x} \geq \inf\{\theta \in \mathbb{R} : |A(x)| \ll x^\theta, \forall x \geq 1\}.$$

On the other hand if $\theta_0 < \limsup_{x\to\infty} \frac{\log|A(x)|}{\log x}$ then (by definition of lim sup) there is a sequence $10 < x_1 < x_2 < \ldots \to \infty$ such that $\frac{\log|A(x_n)|}{\log x_n} > \theta_0$ for all n. Hence $|A(x_n)| > x_n^{\theta_0}$ for all n. This implies that $\limsup_{n\to\infty} \frac{|A(x_n)|}{x_n^{\theta_1}} = \infty$ for every $\theta_1 < \theta_0$, and hence for every $\theta_1 < \theta_0$ the statement "$|A(x)| \ll x^{\theta_1}$, $\forall x \geq 1$" is *false*. Hence (by definition of infimum) $\theta_0 \leq \inf\{\theta \in \mathbb{R} : |A(x)| \ll x^\theta, \forall x \geq 1\}$. Since this holds for every $\theta_0 < \limsup_{x\to\infty} \frac{\log|A(x)|}{\log x}$ we conclude that

$$\limsup_{x\to\infty} \frac{\log|A(x)|}{\log x} \leq \inf\{\theta \in \mathbb{R} : |A(x)| \ll x^\theta, \forall x \geq 1\}.$$

□

4.1. (This is similar to arguments given in the proof of Theorem 4.35.)

(a). This follows from (b).

(b). Let $q_1 = \prod_{j=1}^r c(\chi_j)$. First assume $m, n \in \mathbb{Z}$, $(m,q) = (n,q) = 1$ and $m \equiv n \pmod{q_1}$. Then for each $j \in \{1,\ldots,r\}$ we have $(m, p_j^{\alpha_j}) = (n, p_j^{\alpha_j}) = 1$ and hence $m \equiv n \bmod c(\chi_j)$ since $c(\chi_j) \mid q_1$; hence $\chi_j(m) = \chi_j(n)$. Hence $\chi(m) = \prod_{j=1}^r \chi_j(m) = \prod_{j=1}^r \chi_j(n) = \chi(n)$. This proves that [$\chi(n)$ for n restricted by $(n,q) = 1$] has period q_1, and hence by Lemma 4.20 we have $c(\chi) \mid q_1$.

Next for every $k \in \{1,\ldots,r\}$ we can argue as follows. Since $\chi_k \in X_{p_k^{\alpha_k}}$ we have $c(\chi_k) = p^\beta$ for some $\beta \in \{0,1,\ldots,\alpha_k\}$, by Lemma 4.20. Suppose that $\beta > 0$. Then [$\chi_k(n)$ restricted by $(n, p_k^{\alpha_k}) = 1$] does *not* have period $p_k^{\beta-1}$ and hence there are some $m, n \in \mathbb{Z}$ with $(m, p_k) = (n, p_k) = 1$ and $m \equiv n \pmod{p_k^{\beta-1}}$ and $\chi_k(m) \neq \chi_k(n)$. Now by the Chinese Remainder Theorem there exist $m', n' \in \mathbb{Z}$ such that $m' \equiv m \pmod{p_k^{\alpha_k}}$ and $m' \equiv 1 \pmod{p_j^{\alpha_j}}$ for all $j \neq k$, and $n' \equiv n \pmod{p_k^{\alpha_k}}$ and $n' \equiv 1 \pmod{p_j^{\alpha_j}}$ for all $j \neq k$. Now

$$\chi(m') = \prod_{j=1}^r \chi_j(m') = 1\cdots 1 \cdot \chi_k(m') \cdot 1 \cdots 1 = \chi_k(m') = \chi_k(m),$$

and similarly $\chi(n') = \chi_k(n)$; thus $\chi(m') \neq \chi(n')$. But we also have $(m', q) = (n', q) = 1$ and $m' \equiv n' \bmod q_1/p_k$; hence this proves that *[$\chi(n)$ restricted by $(n,q) = 1$] does not have period q_1/p_k, and thus $c(\chi) \nmid \frac{q_1}{p_k}$.*

We thus have $c(\chi) \mid q_1$ but for each prime $p \mid q_1$ we also have $c(\chi) \nmid \frac{q_1}{p}$. This implies that $c(\chi) = q_1$. □

4.2. Let q have the prime factorization $q = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$; then by Lemma 4.12 combined with Problem 4.1(a), we have $\phi^*(q) = \prod_{j=1}^r \phi^*(p_j^{\alpha_j})$. (Thus ϕ^* is multiplicative.)

It now only remains to compute $\phi^*(p^\alpha)$ for any prime p and any $\alpha \geq 1$. First assume $p \geq 3$. Then by Lemma 4.10 (combined with Lemma 4.6) the Dirichlet characters modulo p^α are in 1-1-correspondence with the $\phi(p^\alpha)$th roots of unity ω, i.e. (since $\phi(p^\alpha) = p^{\alpha-1}(p-1)$) the numbers $\omega = e\bigl(\frac{k}{p^{\alpha-1}(p-1)}\bigr)$ for $k = 0, 1, \ldots, p^{\alpha-1}(p-1) - 1$. The Dirichlet character corresponding to ω is given by $\chi(g^j) = \omega^j$ for all $j \geq 0$, where g is our fixed primitive root modulo p^α.

Now χ is primitive if and only if $c(\chi) = p^\alpha$, and we know that $c(\chi) \mid p^\alpha$; hence χ is primitive if and only if $c(\chi) \nmid p^{\alpha-1}$, i.e. if and only if [$\chi(n)$ restricted by $(n, p^\alpha) = 1$] does not have period $p^{\alpha-1}$. In other words, χ is primitive if and only if there are two $j_1, j_2 \geq 0$ such that $g^{j_1} \equiv g^{j_2} \pmod{p^{\alpha-1}}$ and $\chi(g^{j_1}) \neq \chi(g^{j_2})$. If $\alpha = 1$ then this means that χ is primitive if and only if there are *any* two $j_1, j_2 \geq 0$ with $\chi(g^{j_1}) \neq \chi(g^{j_2})$, and this clearly holds if and only if $\omega \neq 1$; hence there are $p-2$ primitive characters modulo p. If $\alpha \geq 2$ then we know that g is also a primitive root modulo $p^{\alpha-1}$ (see Lemma 4.7); hence $g^{j_1} \equiv g^{j_2} \pmod{p^{\alpha-1}}$ holds if and only if $j_1 \equiv j_2 \pmod{p^{\alpha-2}(p-1)}$. Hence χ corresponding to ω is primitive if and only if there are two $j_1, j_2 \geq 0$ with $j_1 \equiv j_2 \pmod{p^{\alpha-2}(p-1)}$ such that $\chi(g^{j_1}) \neq \chi(g^{j_2})$. But $\chi(g^{j_1}) \neq \chi(g^{j_2}) \iff \omega^{j_1} \neq \omega^{j_2} \iff \omega^{j_1 - j_2} \neq 1$. Hence χ is primitive if and only if ω is *not* a $p^{\alpha-2}(p-1)$th root of unity; that is, if ω is of the form

$$\omega = e\Bigl(\frac{k}{p^{\alpha-1}(p-1)}\Bigr), \qquad k \in \{0, 1, \ldots, p^{\alpha-1}(p-1) - 1\}, \quad p^{\alpha-2}(p-1) \nmid k.$$

There are $p^{\alpha-1}(p-1)(1-p^{-1}) = p^{\alpha-2}(p-1)^2 = p^\alpha(1-p^{-1})^2$ distinct such ω's. Hence the number of primitive roots modulo p^α is:

$$\phi^*(p^\alpha) = p^\alpha \cdot \begin{cases} 1 - 2p^{-1} & \text{if } \alpha = 1 \\ (1-p^{-1})^2 & \text{if } \alpha \geq 2. \end{cases} \tag{671}$$

We proved this for $p \geq 3$ but we claim that the same formula also holds for $p = 2$. Indeed one checks directly that there are *no* primitive characters modulo 2, and that there is *exactly one* primitive character modulo 4. Finally, if $q = 2^\alpha$ with $\alpha \geq 3$ then the Dirichlet characters modulo q are in bijective correspondence with pairs $\langle \omega, \omega' \rangle$ as in Lemma 4.11 and one checks by a similar argument as above that $\langle \omega, \omega' \rangle$ corresponds to a *primitive* character if and only if ω is not a $2^{\alpha-3}$th root of unity. There are $2^{\alpha-3} \cdot 2$ distinct such pairs, and this agrees with (671).

Now (184) follows using $\phi^*(q) = \prod_{j=1}^r \phi^*(p_j^{\alpha_j})$ and (671). □

4.3. If q_1 is a period of $\chi(n)$ restricted by $(n, q) = 1$ then for all integers m, n with $(m, q) = (n, q)$ and $m \equiv n \pmod{q_1}$ we have $\chi(m) = \chi(n)$. In particular, taking $m = 1$, it follows that $\chi(n) = 1$ for all integers n satisfying $(n, q) = 1$ and $n \equiv 1 \pmod{q_1}$.

Conversely, suppose that q_1 is a positive integer and that $\chi(n) = 1$ holds for all integers n satisfying $n \equiv 1 \pmod{q_1}$ and $(n, q) = 1$. Let $q_2 = (q, q_1)$; then we know that there are some integers x, y such that $q_2 = xq + yq_1$. Now if n is any integer satisfying $n \equiv 1 \pmod{q_2}$ and $(n, q) = 1$ then we have $n = 1 + hq_2$ for some integer h, and hence $n = 1 + h(xq + yq_1) \equiv 1 + hyq_1 \pmod{q}$ so that $\chi(n) = \chi(1 + hyq_1)$ and $(1 + hyq_1, q) = 1$. Furthermore

$1 + hyq_1 \equiv 1 \pmod{q_1}$ and thus by our assumption $\chi(1 + hyq_1) = 1$. Hence q_2 has exactly the same property as q_1, i.e. $\chi(n) = 1$ holds for all integers n satisfying $n \equiv 1 \pmod{q_2}$ and $(n, q) = 1$. The advantage is that q_2 also divides q, by construction!

Now take any two integers m_1, m_2 with $(m_1, q) = (m_2, q) = 1$ and $m_1 \equiv m_2 \pmod{q_2}$. Then m_1, m_2 correspond to two elements in $(\mathbb{Z}/q\mathbb{Z})^\times$ and hence there is a unique $n \in (\mathbb{Z}/q\mathbb{Z})^\times$ such that $m_1 \equiv nm_2 \pmod q$. Since $q_2 \mid q$ this implies $m_1 \equiv nm_2 \pmod{q_2}$, which forces $n \equiv 1 \pmod{q_2}$ (since $(m_1, q_2) = (m_2, q_2) = 1$). Hence, by what we proved in last paragraph, $\chi(n) = 1$! Hence $\chi(m_1) = \chi(nm_2) = \chi(n)\chi(m_2) = \chi(m_2)$.

This proves that $\chi(n)$ restricted by $(n, q) = 1$ has period q_2. Since $q_2 \mid q_1$, it follows that $\chi(n)$ restricted by $(n, q) = 1$ also has period q_1. □

4.4. (a). Since χ has period q the sum $\frac{1}{q}\sum_{c=0}^{q-1} \chi(ac + b)$ only depends on a and b modulo q, and equals $\frac{1}{q}\sum_{x\in\mathbb{Z}/q\mathbb{Z}} \chi(ax + b)$. For a given $d \in \mathbb{Z}/q\mathbb{Z}$, the congruence equation $ax+b \equiv d \pmod q$ (viewed as an equation in $x \in \mathbb{Z}/q\mathbb{Z}$) is solvable if and only if $(a, q) \mid d-b$, and in this case it is equivalent with $\frac{a}{(a,q)}x \equiv \frac{d-b}{(a,q)} \pmod{\frac{q}{(a,q)}}$, which has a unique solution $x \pmod{\frac{q}{(a,q)}}$ since $(\frac{a}{(a,q)}, \frac{q}{(a,q)}) = 1$. Hence when x runs through $\mathbb{Z}/q\mathbb{Z}$ in our sum, $ax + b$ visits exactly those congruence classes modulo q which are $\equiv b \pmod{(a, q)}$, and each such congruence class is visited exactly (a, q) times. In other words our sum equals

$$S := \frac{1}{q}\sum_{c=0}^{q-1} \chi(ac + b) = \frac{(a, q)}{q} \sum_{\substack{y\in\mathbb{Z}/q\mathbb{Z} \\ y\equiv b \pmod{(a,q)}}} \chi(y). \tag{672}$$

Now fix some $m \in (\mathbb{Z}/q\mathbb{Z})^\times$; we wish to understand the product $\chi(m)S$. For each y as in the above sum we have $\chi(m)\chi(y) = \chi(my)$, and $my \equiv mb \pmod{(a, q)}$. Thus let us from now on assume that $m \equiv 1 \pmod{(a, q)}$! Then $my \equiv b \pmod{(a, q)}$ for all y as in the above sum. Since $m \in (\mathbb{Z}/q\mathbb{Z})^\times$ we *also* know that the map $y \mapsto my$ is a permutation of $\mathbb{Z}/q\mathbb{Z}$; hence it also restricts to a permutation of the set which we are interested in, i.e. $\{y \in \mathbb{Z}/q\mathbb{Z} : y \equiv b \pmod{(a, q)}\}$. Thus

$$\chi(m)S = \frac{(a, q)}{q} \sum_{\substack{y\in\mathbb{Z}/q\mathbb{Z} \\ y\equiv b \pmod{(a,q)}}} \chi(my) = \frac{(a, q)}{q} \sum_{\substack{y\in\mathbb{Z}/q\mathbb{Z} \\ y\equiv b \pmod{(a,q)}}} \chi(y) = S. \tag{673}$$

Hence if $S \neq 0$ then for all $m \in (\mathbb{Z}/q\mathbb{Z})^\times$ with $m \equiv 1 \pmod{(a, q)}$ we must have $\chi(m) = 1$! In other words, for all integers m with $(m, q) = 1$ and $m \equiv 1 \pmod{(a, q)}$ we must have $\chi(m) = 1$. By Problem 4.3 this implies that (a, q) is a period of $\chi(n)$ restricted by $(n, q) = 1$, and thus $(a, q) = q$ since χ is primitive. We have thus proved that $S = 0$ unless $(a, q) = q$, i.e. unless $q \mid a$.

Finally if $q \mid a$ then all terms in the sum equals $\chi(b)$ and hence the whole sum also equals $\chi(b)$. □

(b). For example if χ is the principal character modulo q and $a = 1$, then $\frac{1}{q}\sum_{c=0}^{q-1} \chi(ac + b) = \frac{\phi(q)}{q}$, and this does not agree with the stated formula (for any $q \geq 2$). □

4.5. The cases $4 \mid a$ (giving $\left(\frac{a}{2}\right) = 0$) and $a \not\equiv 0, 1 \pmod 4$ (giving $\left(\frac{a}{2}\right)$ undefined) are immediate from Definition 4.10. On the other hand if $a \equiv 1 \pmod 4$ then we get from Definition 4.10 combined with Theorem 4.25: $\left(\frac{a}{2}\right) = \left(\frac{2}{|a|}\right) = (-1)^{\frac{a^2-1}{8}}$, which agrees with (174). (Proof: Writing $a = 4n+1$ we have $\frac{a^2-1}{8} = 2n^2 + n \equiv n \pmod 2$, thus $\frac{a^2-1}{8}$ is even if $2 \mid n \iff a \equiv 1 \pmod 8$ and odd if $2 \nmid n \iff a \equiv 5 \pmod 8$.) $\square$

4.7. From Definition 4.12 we see that there is exactly one fundamental discriminant $d \in \{-p, p\}$, namely $d = (-1)^{\frac{p-1}{2}} p$. Hence by Theorem 4.12 there is a unique primitive character modulo p, namely $\chi = \left(\frac{d}{\cdot}\right)$ with $d = (-1)^{\frac{p-1}{2}} p$. Note $d \equiv 1 \pmod 4$; hence by Proposition 4.30 we have $\chi(n) = \left(\frac{d}{n}\right) = \left(\frac{n}{|d|}\right) = \left(\frac{n}{p}\right)$ for all $n > 0$. Hence since χ has periodicity p we actually have $\chi(n) = \left(\frac{n}{p}\right)$ for *all* integers n. $\square$

Alternative, direct proof: Let $g \in \mathbb{Z}$ be a primitive root modulo p and let $\nu : (\mathbb{Z}/p\mathbb{Z})^\times \to \{0, 1, \ldots, p-2\}$ be the corresponding index function. Then by Lemma 4.10 every Dirichlet character modulo p is given by

$$\chi(n) = \begin{cases} \omega^{\nu(n)} & \text{if } (n, q) = 1 \\ 0 & \text{if } (n, q) > 1, \end{cases}$$

where ω is some $(p-1)$th root of unity. If χ is primitive then it is not principal, thus $\omega \neq 1$. Furthermore if χ is real then $\omega = \chi(g)$ is real, thus $\omega = -1$. Also recall that $\left(\frac{g}{p}\right) = -1$ (as we noticed e.g. in the proof of Lemma 4.36). Hence for every $v \in \{0, 1, \ldots, p-2\}$ we have $\chi(g^v) = (-1)^v = \left(\frac{g}{p}\right)^v = \left(\frac{g^v}{p}\right)$, i.e. $\chi(n) = \left(\frac{n}{p}\right)$ for all $n \in (\mathbb{Z}/p\mathbb{Z})^\times$. Hence $\chi(n) = \left(\frac{n}{p}\right)$ for all $n \in \mathbb{Z}$. Conversely the function $n \mapsto \left(\frac{n}{p}\right)$ is easily seen to indeed be a real primitive Dirichlet character modulo p. $\square$

5.4. For fixed z we write $A(x)=\sum_{1\leq n\leq x} z^n$. Then for any integers $1\leq M<N$ we have

$$\sum_{M\leq n<N} n^{-1}z^n=\int_{M-}^N x^{-1}\,dA(x)=\Big[x^{-1}A(x)\Big]_{x=M-}^{x=N}+\int_M^N x^{-2}A(x)\,dx$$
$$=\frac{A(N)}{N}-\frac{A(M-1)}{M}+\int_M^N x^{-2}A(x)\,dx.$$

But $A(x)$ is in fact a geometric sum; if $z\neq 1$ then we have $A(x)=\frac{z^{\lfloor x\rfloor+1}-z}{z-1}$ for all $x\geq 0$, and if $|z|\leq 1$ we conclude that

$$\big|A(x)\big|\leq\frac{|z^{\lfloor x\rfloor+1}|+|z|}{|z-1|}\leq\frac{2}{|z-1|},\qquad\forall x\in\mathbb{R}_{\geq 0},$$

and hence from above we get

$$\Big|\sum_{M\leq n<N} n^{-1}z^n\Big|\leq\frac{2}{|z-1|}\Big(\frac{1}{N}+\frac{1}{M}+\int_M^N\frac{dx}{x^2}\Big)=\frac{4}{M|z-1|}.$$

From this we conclude that the series $\sum_{n=1}^\infty n^{-1}z^n$ converges for all z with $|z|\leq 1$, $z\neq 1$, and in fact that this convergence is *uniform* over $z\in C$ where C is any compact subset of $\{|z|\leq 1\}\setminus\{1\}$. (For if C is such a compact subset then there is some $r>0$ such that $|z-1|\geq r$ for all $z\in C$, and then the above bound shows that $\Big|\sum_{n=M}^\infty n^{-1}z^n\Big|\leq\frac{4}{rM}$ for all $M\geq 1$ and $z\in C$; this implies the stated uniform convergence.)

It follows that the limit function $f(z):=\sum_{n=1}^\infty n^{-1}z^n$ is *continuous* for $z\in C$, for any such compact subset C. Hence in fact $f(z)$ is continuous in the whole set $\{|z|\leq 1\}\setminus\{1\}$. But we know from the Taylor expansion of $-\log(1-z)$ (principal branch) that $f(z)=-\log(1-z)$ for all z with $|z|<1$. Hence since both $f(z)$ and $-\log(1-z)$ are continuous in the set $\{|z|\leq 1\}\setminus\{1\}$ we conclude that $f(z)=-\log(1-z)$ for all z in this set! In other words: (236) holds!

Remark: For a general Taylor series with finite radius of convergence, *Abel's Limit Theorem* gives information on the values of the Taylor series *on* the boundary circle of the disc of the convergence. Cf., e.g., Ahlfors, [1, §2.5].

6.1. It follows from (246) combined with Lemma 6.1 that, for any fixed integer $q \geq 0$,

$$\pi(x) = \frac{x}{\log x} + \frac{1!x}{\log^2 x} + \ldots + \frac{q!x}{(\log x)^{q+1}} + O\Big(\frac{x}{(\log x)^{q+2}}\Big) \qquad \text{as } x \to \infty.$$

In particular

$$\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + O\Big(\frac{x}{(\log x)^3}\Big) \qquad \text{as } x \to \infty.$$

Let us now define $A(x)$ by the relation $\pi(x) = \frac{x}{\log x - A(x)}$. It then follows that $\frac{x}{\log x - A(x)} = \frac{x}{\log x} + \frac{x}{(\log x)^2} + O\Big(\frac{x}{(\log x)^3}\Big)$ as $x \to \infty$. Dividing with $\frac{x}{\log x}$ we get $\frac{\log x}{\log x - A(x)} = 1 + \frac{1}{\log x} + O\Big(\frac{1}{(\log x)^2}\Big)$, hence $\frac{\log x - A(x)}{\log x} = \Big(1 + \frac{1}{\log x} + O\Big(\frac{1}{(\log x)^2}\Big)\Big)^{-1} = 1 - \frac{1}{\log x} - O\Big(\frac{1}{(\log x)^2}\Big) + O\Big(\Big(\frac{1}{\log x} - O\Big(\frac{1}{(\log x)^2}\Big)\Big)^2\Big) = 1 - \frac{1}{\log x} + O\Big(\frac{1}{(\log x)^2}\Big)$. Subtracting 1 and multiplying with $-\log x$ this gives $A(x) = 1 + O\Big(\frac{1}{\log x}\Big)$. In particular $\lim_{x\to\infty} A(x) = 1$. □

6.2. Recall that $-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$ for $\sigma > 1$ (cf. (113)). Multiplication with $\zeta(s)$ gives (for $\sigma > 1$): $-\zeta'(s) = \Big(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}\Big)\Big(\sum_{m=1}^{\infty} m^{-s}\Big) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\Lambda(n)}{(nm)^s} = \sum_{k=1}^{\infty}\sum_{n\mid k}^{\infty} \frac{\Lambda(n)}{k^s}$. That is: $\sum_{k=1}^{\infty} \frac{\log k}{k^s} = \sum_{k=1}^{\infty}\sum_{n\mid k}^{\infty} \frac{\Lambda(n)}{k^s}$ (true, with absolute convergence, for all $\sigma > 1$). Hence by comparison of coefficients (cf. Proposition 3.10) we get:

$$\sum_{n\mid k} \Lambda(n) = \log k, \qquad \forall k \in \mathbb{Z}^+. \tag{674}$$

(The same formula can of course also be proved directly: If $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ then the left hand side equals, since $\Lambda(m) = 0$ unless m is a prime power: $\Big(\sum_{j_1=1}^{\alpha_1} \log p_1\Big) + \Big(\sum_{j_2=1}^{\alpha_2} \log p_2\Big) + \ldots + \Big(\sum_{j_r=1}^{\alpha_r} \log p_r\Big) = \alpha_1 \log p_1 + \alpha_2 \log p_2 + \ldots + \alpha_r \log p_r = \log n$.)

If we sum (674) over positive integers $k \leq x$, we obtain $\sum_{k\leq x}\sum_{n\mid k} \Lambda(n) = \sum_{k\leq x} \log k$, i.e.

$$T(x) := \sum_{n\leq x} \Big\lfloor \frac{x}{n} \Big\rfloor \Lambda(n) = \log\big(\lfloor x \rfloor !\big).$$

□

6.3. From (258) we see that, for all $x \geq 60$, $T(x) - T(\frac{1}{2}x) - T(\frac{1}{3}x) - T(\frac{1}{5}x) + T(\frac{1}{30}x) = -\frac{1}{2}x\log\frac{1}{2} - \frac{1}{3}x\log\frac{1}{3} - \frac{1}{5}x\log\frac{1}{5} + \frac{1}{30}x\log\frac{1}{30} + O\big(\log x\big) = (\frac{1}{2}\log 2 + \frac{1}{3}\log 3 + \frac{1}{5}\log 5 - \frac{1}{30}\log 30)x + O\big(\log x\big) = Kx + O\big(\log x\big)$, where we compute $K = 0.9212920229...$ Next note that, from (257),

$$T(x) - T(\tfrac{1}{2}x) - T(\tfrac{1}{3}x) - T(\tfrac{1}{5}x) + T(\tfrac{1}{30}x) = \sum_{m\leq x} g(x/m)\Lambda(m),$$

where

$$g(a) = \lfloor a \rfloor - \lfloor \tfrac{1}{2}a \rfloor - \lfloor \tfrac{1}{3}a \rfloor - \lfloor \tfrac{1}{5}a \rfloor + \lfloor \tfrac{1}{30}a \rfloor.$$

This function is periodic with period 30, and we find that

$$g(a) = 0 \quad \text{for } a \in [0,1) \cup [6,7) \cup [10,11) \cup [12,13) \cup [15,17) \cup [18,19) \cup [20,23) \cup [24,29);$$
$$g(a) = 1 \quad \text{for } a \in [1,6) \cup [7,10) \cup [11,12) \cup [13,15) \cup [17,18) \cup [19,20) \cup [23,24) \cup [29,30).$$

In particular $0 \leq g(a) \leq 1$ for all $a \in \mathbb{R}$ and hence

$$T(x) - T(\tfrac{1}{2}x) - T(\tfrac{1}{3}x) - T(\tfrac{1}{5}x) + T(\tfrac{1}{30}x) \leq \sum_{m \leq x} \Lambda(m) = \psi(x);$$

whence

$$Kx + O(\log x) \leq \psi(x), \qquad \forall x \geq 60.$$

This implies $\liminf_{x\to\infty} \frac{\psi(x)}{x} \geq K > 0.9212$.

On the other hand we also get

$$T(x) - T(\tfrac{1}{2}x) - T(\tfrac{1}{3}x) - T(\tfrac{1}{5}x) + T(\tfrac{1}{30}x) \geq \sum_{\frac{1}{6}x < m \leq x} \Lambda(m) = \psi(x) - \psi(\tfrac{1}{6}x),$$

and hence

$$\psi(x) - \psi(\tfrac{1}{6}x) \leq Kx + O(\log x), \qquad \forall x \geq 60.$$

Adding this applied to $x, 6^{-1}x, 6^{-2}x, \ldots, 6^{-(k-1)}x$ where k is determined so that $10 \leq 6^{-k}x < 60$, we get

$$\begin{aligned} \psi(x) &= \sum_{j=0}^{k-1} \big(\psi(6^{-j}x) - \psi(6^{-j-1}x)\big) + \psi(6^{-k}x) \\ &\leq K \sum_{j=0}^{k-1} 6^{-j}x + O\big(k \log x + 1\big) \leq \frac{6}{5}K + O\big((\log x)^2\big), \qquad \forall x \geq 60. \end{aligned}$$

Here we have $\frac{6}{5}K = 1.1055504....$ This implies that $\limsup_{x\to\infty} \frac{\psi(x)}{x} \leq \frac{6}{5}K < 1.1056$.

Finally using Proposition 6.2 we conclude that $\liminf_{x\to\infty} \frac{\pi(x)}{x/\log x} > 0.9212$ and $\limsup_{x\to\infty} \frac{\pi(x)}{x/\log x} < 1.1056$. □

6.4. It is equivalent to prove that the sum $-\sum_p \log(1-\chi(p)p^{-1})$ converges to $\log L(1,\chi)$. (Recall that $L(1,\chi) \neq 0$, and $\log L(s,\chi)$ for $\sigma > 1$ is defined by (21), and we now define $\log L(1,\chi)$ as $\lim_{s\to 1^+} \log L(s,\chi)$.) Inserting the Taylor expansion we have

$$-\sum_p \log(1 - \chi(p)p^{-1}) = \sum_p \Big(\sum_{m=1}^{\infty} \chi(p)^m p^{-m}\Big). \tag{675}$$

But here note that the double sum

$$\sum_p \sum_{m=2}^{\infty} \chi(p)^m p^{-m}$$

is *absolutely* convergent, by the same computation as in (9) (which works for $s = 1$). Hence the convergence of (675) is equivalent to the convergence of (the $m = 1$ contribution) $\sum_p \chi(p)p^{-1}$, and this series is convergent by Proposition 6.8.

We have thus proved that the infinite product $\prod_p (1 - \chi(p)p^{-1})^{-1}$ is convergent, if we multiply over the primes p in increasing order.

To compute the value of the sum, we will extend the above proof to show that the sum $-\sum_p \log(1 - \chi(p)p^{-s})$ is *uniformly* convergent for $s \geq 1$. Indeed, inserting the Taylor expansion we have

$$-\sum_p \log(1 - \chi(p)p^{-s}) = \sum_p \Big(\sum_{m=1}^{\infty} \chi(p)^m p^{-ms}\Big), \tag{676}$$

and here the double sum

$$\sum_p \sum_{m=2}^{\infty} \chi(p)^m p^{-ms}$$

is uniformly absolutely convergent for all $s \geq 1$, by the same computation as in (9). Hence the uniform convergence of (676) is equivalent with the uniform convergence of $\sum_p \chi(p)p^{-s}$ for $s \geq 1$, and *this is a consequence of Proposition 6.8 combined with Theorem 3.6!*

We have thus proved that the sum $-\sum_p \log(1 - \chi(p)p^{-s})$ is uniformly convergent for all $s \geq 1$. Since each term is a continuous function of s, it follows that the sum must be a continuous function of s (for $s \geq 1$). Hence in particular,

$$-\sum_p \log(1 - \chi(p)p^{-1}) = \lim_{s \to 1^+} -\sum_p \log(1 - \chi(p)p^{-s}).$$

Using now (21) (cf. Example 2.1) we get

$$= \lim_{s \to 1^+} \log L(s, \chi) = \log L(1, \chi).$$

□

6.6. $\log(n!) = \sum_{m=1}^{n} \log m = \int_{1-}^{n} \log x \, d\lfloor x \rfloor = \Big[(\log x)\lfloor x \rfloor\Big]_{x=1-}^{x=n} - \int_1^n \frac{\lfloor x \rfloor}{x} \, dx = n \log n - \int_1^n \frac{\lfloor x \rfloor}{x} \, dx$. Here we trivially have $\int_1^n \frac{\lfloor x \rfloor}{x} \, dx \leq \int_1^n \frac{x}{x} \, dx = n - 1$ and $\int_1^n \frac{\lfloor x \rfloor}{x} \, dx \geq \int_1^n \frac{x-1}{x} \, dx = n - 1 - \log n$. These two inequalities together prove that $\int_1^n \frac{\lfloor x \rfloor}{x} \, dx = n + O(\log n)$ for all $n \geq 2$. This gives the desired statement. □

7.1. If $y = \pi(x)$ then $\lim_{x\to\infty} \frac{y\log x}{x} = 1$, by the prime number theorem. Hence $\lim_{x\to\infty}\big(\log y + \log\log x - \log x\big) = 0$ and thus $\lim_{x\to\infty} \frac{\log y}{\log x} = 1$. This combined with $\lim_{x\to\infty} \frac{y\log x}{x} = 1$ gives $\lim_{x\to\infty} \frac{y\log y}{x} = 1$, whence the result follows on taking $x = p_n$, since $\pi(p_n) = n$. □

7.2. We use the formula (278), i.e.

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \frac{(x)}{x^{s+1}}\,dx \qquad (\sigma > 0). \tag{677}$$

Recall that the last integral is uniformly absolutely convergent in any compact subset of $\{\sigma > 0\}$. Hence by Weierstrass Theorem we have for every $n \geq 0$:

$$\begin{aligned}\frac{d^n}{ds^n}\Big(s\int_1^\infty \frac{(x)}{x^{s+1}}\,dx\Big) &= \int_1^\infty \frac{d^n}{ds^n}\big(s\cdot(x)\cdot x^{-s-1}\big)\,dx \\ &= (-1)^n s\int_1^\infty \frac{(\log x)^n\cdot(x)}{x^{s+1}}\,dx + (-1)^{n-1}n\int_1^\infty \frac{(\log x)^{n-1}\cdot(x)}{x^{s+1}}\,dx\end{aligned}$$

and hence

$$\frac{d^n}{ds^n}\Big(s\int_1^\infty \frac{(x)}{x^{s+1}}\,dx\Big)_{|s=1} = (-1)^n\Big(\int_1^\infty \frac{(\log x)^n\cdot(x)}{x^2}\,dx - n\int_1^\infty \frac{(\log x)^{n-1}\cdot(x)}{x^2}\,dx\Big)$$

This gives *a* formula for the sought coefficients. (Note that both integrals are absolutely convergent.)

To get a different and perhaps in some sense more explicit formula we rewrite the above as (note: the following computation is valid also for $n = 0$ if we make the special interpretation

$(\log 1)^0 = 1)$:

$$= (-1)^n \int_1^\infty \frac{\big((\log x)^n - n(\log x)^{n-1}\big) \cdot (x - \lfloor x \rfloor)}{x^2}\, dx$$

$$= (-1)^n \lim_{N\to\infty} \sum_{k=1}^{N-1} \int_k^{k+1} \frac{\big((\log x)^n - n(\log x)^{n-1}\big) \cdot (x - k)}{x^2}\, dx$$

$$= (-1)^n \lim_{N\to\infty} \Big(\int_1^N \frac{(\log x)^n - n(\log x)^{n-1}}{x}\, dx - \sum_{k=1}^{N-1} k \int_k^{k+1} \frac{(\log x)^n - n(\log x)^{n-1}}{x^2}\, dx \Big)$$

$$= (-1)^n \lim_{N\to\infty} \Big(\Big[\tfrac{1}{n+1} (\log x)^{n+1} - (\log x)^n \Big]_{x=1}^{x=N} - \sum_{k=1}^{N-1} k \Big[- \frac{(\log x)^n}{x} \Big]_{x=k}^{x=k+1} \Big)$$

$$= (-1)^n \lim_{N\to\infty} \Big(\tfrac{1}{n+1} (\log N)^{n+1} - (\log N)^n - \frac{0}{n+1} + (\log 1)^n + \sum_{k=1}^{N-1} k \Big(\frac{(\log(k+1))^n}{k+1} - \frac{(\log k)^n}{k} \Big) \Big)$$

$$= (-1)^n \lim_{N\to\infty} \Big(- \sum_{k=1}^{N-1} \frac{(\log k)^n}{k} - \frac{(\log N)^n}{N} + \tfrac{1}{n+1} (\log N)^{n+1} \Big) + \begin{Bmatrix} 1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{Bmatrix}$$

$$= (-1)^{n-1} \lim_{N\to\infty} \Big(\sum_{k=1}^{N} \frac{(\log k)^n}{k} - \frac{(\log N)^{n+1}}{n+1} \Big) + \begin{Bmatrix} 1 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{Bmatrix}.$$

Note that this is a natural generalization of Euler's constant $\gamma = \lim_{N\to\infty} \Big(\sum_{k=1}^N \frac{1}{k} - \log N \Big) = 0.577...$! Thus let us define

$$\gamma_n := \lim_{N\to\infty} \Big(\sum_{k=1}^{N} \frac{(\log k)^n}{k} - \frac{(\log N)^{n+1}}{n+1} \Big). \tag{678}$$

(The above computation shows that the limit exists.) Then $\gamma_0 = \gamma$, Euler's constant, and we have proved that the Laurent expansion of $\zeta(s)$ at $s = 1$ is

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} (-1)^n \gamma_n (s-1)^n. \tag{679}$$

□

7.3. Recall from (287) that there exists an absolute constant $A_1 > 0$ such that

$$\big|\zeta(\sigma + ti)\big| \geq \frac{(\sigma - 1)^{\frac{3}{4}}}{A_1 (\log t)^{\frac{1}{4}}} \qquad \text{for all } s \text{ with } 1 \leq \sigma \leq 2,\ t \geq 2. \tag{680}$$

Now to continue we need a bound on $\zeta'(s)$ for s slightly to the left of $\sigma < 1$.

Let us note that the proof of (280) can be extended to give this. Indeed, let us fix any constant $A > 1$, and focus on points s in the region $t \geq 10$, $\sigma \geq \max(\frac{1}{2}, 1 - A(\log t)^{-1})$. Let s_0 be any point in this region, and again consider a circle C with center s_0 and radius $\rho < \frac{1}{2}$. Then for all points s on C we have $\sigma \geq \sigma_0 - \rho \geq 1 - A(\log t)^{-1} - \rho$ and also $\sigma \geq \frac{1}{2} - \rho$. Let us take again $\rho = \frac{1}{2+\log t_0}$; then for all points s on C we have $\sigma \geq \sigma_0 - \rho \geq 1 - A'(\log t)^{-1} - \rho$ (where A' is some constant, $A' > A$) and also $\sigma \geq \frac{1}{2} - \rho$. Hence by (283) we have

$$\left|\zeta(s)\right| \ll \frac{t^{A'(\log t)^{-1}}}{(\log t)^{-1}} \ll \log t$$

at all points s on the circle. Hence since the radius is $\rho = \frac{1}{2+\log t_0}$ we obtain $\left|\zeta'(s_0)\right| \ll (2 + \log t_0)^2$. This has now been proved for all s_0 in the region $t \geq 10$, $\sigma \geq \max(\frac{1}{2}, 1 - A(\log t)^{-1})$!

Hence we can now extend the discussion below (287) as follows: For any $t \geq 10$ and $\sigma \geq 1 - A(\log t)^{-1}$ and $\eta \geq \max(\sigma, 1)$ we have

$$\left|\zeta(\sigma + ti) - \zeta(\eta + ti)\right| \leq A_2(\eta - \sigma)(\log t)^2$$

for some (new) absolute constant; thus we deduce $\left|\zeta(\sigma + ti)\right| > 0$ whenever we can find $\eta \geq \max(\sigma, 1)$ with $\frac{(\eta-1)^{\frac{3}{4}}}{A_1(\log t)^{\frac{1}{4}}} > A_2(\eta - \sigma)(\log t)^2$, viz. $\frac{(\eta-1)^3}{(\eta-\sigma)^4} > (A_1 A_2)^4(\log t)^9$.

Given σ, which η gives maximum?? $f(\eta) = 3\log(\eta - 1) - 4\log(\eta - \sigma)$; thus $f'(\eta) = \frac{3}{\eta-1} - \frac{4}{\eta-\sigma}$; this is > 0 iff $3(\eta - \sigma) > 4(\eta - 1)$, viz. $4 - 3\sigma > \eta$; hence (for $\sigma < 1$) the maximum is obtained at $\eta = 4 - 3\sigma$! This maximum is: $\frac{(\eta-1)^3}{(\eta-\sigma)^4} = \frac{(3-3\sigma)^3}{(4-4\sigma)^4} = \frac{3^3}{4^4}(1 - \sigma)^{-1}$, and thus the condition is that $(1 - \sigma)^{-1} \geq A_3(\log t)^9$ for some absolute constant $A_3 > 0$, i.e. $1 - \sigma \leq A_3^{-1}(\log t)^{-9}$, i.e. $\sigma \geq 1 - A_3^{-1}(\log t)^{-9}$. □

7.6. (Cf. Ingham [28, Thm. C].) We solve (b) directly, since (a) is a special case of (b).

Let $0 < \alpha < 1 < \beta$. Since $A(u)$ is an increasing function of u we have

$$A(x) \leq \frac{1}{\beta x - x} \int_x^{\beta x} A(u)\, du = \frac{A_1(\beta x) - A_1(x)}{(\beta - 1)x},$$

and hence

$$\frac{A(x)}{x^{a-1}(\log x)^b} \leq \frac{1}{\beta - 1}\Big(\frac{A_1(\beta x)}{(\beta x)^a(\log x)^b}\beta^a - \frac{A_1(x)}{x^a(\log x)^b}\Big)$$

Let $x \to \infty$, keeping β fixed. Then since $\lim_{x\to\infty} \frac{A_1(x)}{x^a(\log x)^b} = C$ and $\lim_{x\to\infty} \frac{\log(\beta x)}{\log x} = \lim_{x\to\infty} \frac{\log\beta+\log x}{\log x} = 1$, we get

$$\limsup_{x\to\infty} \frac{A(x)}{x^{a-1}(\log x)^b} \leq C\frac{\beta^a - 1}{\beta - 1}.$$

Similarly, by considering $\int_{\alpha x}^{x} \psi(u)\,du$, we prove that

$$\liminf_{x\to\infty} \frac{A(x)}{x^{a-1}(\log x)^b} \geq C\frac{1-\alpha^a}{1-\alpha}.$$

But $\lim_{t\to 1} \frac{t^a-1}{t-1} = a$ (two-sided limit); hence by taking α and β near enough to 1 we can make both $\frac{\beta^a-1}{\beta-1}$ and $\frac{1-\alpha^a}{1-\alpha}$ be as near as we please to a; hence

$$\lim_{x\to\infty} \frac{A(x)}{x^{a-1}(\log x)^b} = Ca.$$

□

8.1. By Lemma 8.14 we have $\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma - 1 - \sum_{n=1}^{\infty}\Big(\frac{1}{1+n} - \frac{1}{n}\Big) = -\gamma - 1 - \frac{1}{2} + 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \ldots = -\gamma$. Also $\Gamma(1) = 1$ by Lemma 8.12. Hence $\Gamma'(1) = -\gamma$. □

8.2. This is "well-known", with many proofs.

By Proposition 8.18 we have $\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t}t^{-\frac{1}{2}}\,dt = 2\int_0^\infty e^{-u^2}\,du$, where we substituted $t = u^2$ in the last step. One way to compute this (well-known) integral is by considering its *square*, and then using polar coordinates ($x = r\cos\varphi$, $y = r\sin\varphi$):

$$\Big(\int_0^\infty e^{-u^2}\,du\Big)^2 = \int_0^\infty\int_0^\infty e^{-x^2-y^2}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2} r\,dr\,d\varphi = \frac{\pi}{2}\cdot\frac{1}{2}\int_0^\infty e^{-t}\,dt = \frac{\pi}{4}$$

(in the last step we substituted $r = \sqrt{t}$). Hence $\int_0^\infty e^{-u^2}\,du = \frac{1}{2}\sqrt{\pi}$ and thus $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. □

8.3. (Note: The following computation can be seen as a generalization of the proof of $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ which we gave in Problem 8.2.) Fix any $a, b \in \mathbb{C}$ with $\operatorname{Re} a > 0$, $\operatorname{Re} b > 0$. Then

$$\begin{aligned}\Gamma(a)\Gamma(b) &= \int_0^\infty e^{-t}t^{a-1}\,dx \int_0^\infty e^{-u}u^{b-1}\,dy = 4\int_0^\infty e^{-x^2}x^{2a-1}\,dx\int_0^\infty e^{-y^2}y^{2b-1}\,dy \\ &= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}x^{2a-1}y^{2b-1}\,dx\,dy,\end{aligned}$$

where we substituted $t = x^2$ and $u = y^2$. Hence using polar coordinates we get

$$\begin{aligned}\Gamma(a)\Gamma(b) &= 4\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}r^{2a+2b-1}(\cos\varphi)^{2a-1}(\sin\varphi)^{2b-1}\,dr\,d\varphi \\ &= 4\int_0^{\frac{\pi}{2}}(\cos\varphi)^{2a-1}(\sin\varphi)^{2b-1}\,d\varphi\int_0^\infty e^{-r^2}r^{2a+2b-1}\,dr \\ &= \int_0^1 (1-v)^{a-1}v^{b-1}\,dv\int_0^\infty e^{-w}w^{a+b-1}\,dr \\ &= \Gamma(a+b)\int_0^1(1-v)^{a-1}v^{b-1}\,dv,\end{aligned}$$

where in the next to last step we substituted $v = \sin^2\varphi$ and $r = \sqrt{w}$. This gives the stated formula. □

8.4. We first prove a formula for the logarithmic derivative. By Lemma 8.14 we have for every $z \in \mathbb{C} \setminus \{0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots\}$:

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})} = -2\gamma - \frac{1}{z} - \frac{1}{z+\frac{1}{2}} - \sum_{n=1}^{\infty}\Big(\frac{1}{z+n} - \frac{1}{n}\Big) - \sum_{n=1}^{\infty}\Big(\frac{1}{z+\frac{1}{2}+n} - \frac{1}{n}\Big)$$

$$= 2\Big(-\gamma - \frac{1}{2z} - \frac{1}{2z+1} - \sum_{n=1}^{\infty}\Big(\frac{1}{2z+2n} - \frac{1}{2n}\Big) - \sum_{n=1}^{\infty}\Big(\frac{1}{2z+1+2n} - \frac{1}{1+2n}\Big) + \sum_{n=1}^{\infty}\Big(\frac{1}{2n} - \frac{1}{1+2n}\Big)\Big),$$

where the last step is justified since $\sum_{n=1}^{\infty}\big(\frac{1}{2n} - \frac{1}{1+2n}\big)$ is convergent. In fact by (236) (which we proved in Problem 5.4) we have

$$\sum_{n=1}^{\infty}\Big(\frac{1}{2n} - \frac{1}{1+2n}\Big) = 1 + \sum_{m=1}^{\infty} m^{-1}(-1)^m = 1 - \log 2.$$

[Alternative: We have

$$\sum_{n=1}^{\infty}\Big(\frac{1}{2n} - \frac{1}{1+2n}\Big) = \lim_{N\to\infty}\Big(\sum_{n=1}^{N}\frac{1}{2n} - \sum_{n=1}^{N}\frac{1}{1+2n}\Big) = \lim_{N\to\infty}\Big(2\sum_{n=1}^{N}\frac{1}{2n} - \sum_{n=1}^{N}\frac{1}{1+2n} - \sum_{n=1}^{N}\frac{1}{2n}\Big)$$

$$= \lim_{N\to\infty}\Big(\sum_{n=1}^{N}\frac{1}{n} - \sum_{m=2}^{2N+1}\frac{1}{m}\Big),$$

and using Lemma 8.13 this is

$$= \lim_{N\to\infty}\Big(\gamma + \log N - (\gamma - 1 + \log(2N+1))\Big) = 1 - \log 2.]$$

Hence from our previous computation we conclude

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})} = 2\Big(-\gamma - \frac{1}{2z} - \frac{1}{2z+1} - \sum_{m=2}^{\infty}\Big(\frac{1}{2z+m} - \frac{1}{m}\Big) + 1 - \log 2\Big)$$

$$= 2\Big(-\gamma - \log 2 - \frac{1}{2z} - \sum_{m=1}^{\infty}\Big(\frac{1}{2z+m} - \frac{1}{m}\Big)\Big)$$

$$= 2\Big(-\log 2 + \frac{\Gamma'(2z)}{\Gamma(2z)}\Big)$$

Hence we have proved (cf. Definition 8.3 and let's keep $z \in \mathbb{C} \setminus (-\infty, 0]$)

$$\frac{d}{dz}\big(\log\Gamma(z) + \log\Gamma(z+\tfrac{1}{2}) - \log\Gamma(2z) + 2(\log 2)z\big) = 0.$$

Thus the function inside the parenthesis is *constant* throughout $z \in \mathbb{C} \setminus (-\infty, 0]$; exponentiating we conclude that $\Gamma(z)\Gamma(z+\frac{1}{2})\Gamma(2z)^{-1}2^{2z}$ is also constant throughout $z \in \mathbb{C} \setminus (-\infty, 0]$. We can compute the constant e.g. by taking $z = \frac{1}{2}$ (and using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$); this gives that the constant is $= 2\sqrt{\pi}$. Hence $\Gamma(z)\Gamma(z+\frac{1}{2})\Gamma(2z)^{-1}2^{2z} = 2\sqrt{\pi}$ for all $z \in \mathbb{C} \setminus (-\infty, 0]$, and by continuity this must in fact hold for all $z \in \mathbb{C} \setminus \{0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots\}$. This proves the claimed formula. $\square$

Alternative solution: One can get an even quicker solution by working with the *derivative* of the logarithmic derivative of $\Gamma(z)$; cf. Ahlfors [1, p. 200]!

8.5. By Theorem 8.17 we have, when $x \in [a, b]$ and $y \geq 1$,

$$\begin{aligned}
\log\bigl|\Gamma(x \pm iy)\bigr| &= \text{Re } \log \Gamma(x + iy) \\
&= \text{Re } \bigl((x - \tfrac{1}{2} + iy) \log(x + iy)\bigr) - x + \log\sqrt{2\pi} + O\bigl(y^{-1}\bigr) \\
&= (x - \tfrac{1}{2}) \log|x + iy| - y \arg(x + iy) - x + \log\sqrt{2\pi} + O\bigl(y^{-1}\bigr) \\
&= (x - \tfrac{1}{2})\tfrac{1}{2}\Bigl(\log(y^2) + \log\Bigl(1 + \tfrac{x^2}{y^2}\Bigr)\Bigr) - y\Bigl(\tfrac{\pi}{2} - \arctan\tfrac{x}{y}\Bigr) - x + \log\sqrt{2\pi} + O\bigl(y^{-1}\bigr) \\
&= (x - \tfrac{1}{2}) \log y + (x - \tfrac{1}{2})\tfrac{1}{2} \cdot O\bigl(y^{-2}\bigr) - \tfrac{\pi}{2}y + y\bigl(\tfrac{x}{y} + O(y^{-2})\bigr) - x + \log\sqrt{2\pi} + O\bigl(y^{-1}\bigr) \\
&= (x - \tfrac{1}{2}) \log y - \tfrac{\pi}{2}y + \log\sqrt{2\pi} + O\bigl(y^{-1}\bigr)
\end{aligned}$$

Exponentiation of this gives the stated formula (since $e^{O(y^{-1})} = 1 + O(y^{-1})$ for $y \geq 1$). $\square$

8.6. By Stirling's formula, Theorem 8.17, we have

$$\log \Gamma(z + \alpha) = \bigl(z + \alpha - \tfrac{1}{2}\bigr) \log(z + \alpha) - (z + \alpha) + \log\sqrt{2\pi} + O\bigl(|z + \alpha|^{-1}\bigr), \tag{681}$$

for all z with $|z + \alpha| \geq 1$ and $\bigl|\arg(z + \alpha)\bigr| \leq \pi - \varepsilon$. Here and below, for definiteness, we consider the argument function to take its values in $(-\pi, \pi]$, i.e. $\arg : \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$.

Let us fix a constant $C > 1$ so large that $\bigl|\arg(1 + w)\bigr| < \frac{1}{2}\varepsilon$ for all $w \in \mathbb{C}$ with $|w| \leq C$. Then note that if $|z| \geq C|\alpha|$ and $|z| \geq 1$ then $\arg(z + \alpha) = \arg(z(1 + \alpha/z)) \equiv \arg(z) + \arg(1 + \alpha/z) \pmod{2\pi}$ together with $\bigl|\arg(z + \alpha)\bigr| \leq \pi - \varepsilon$ and $\bigl|\arg(1 + \alpha/z)\bigr| < \frac{1}{2}\varepsilon$ and imply that $\bigl|\arg(z)\bigr| \leq \pi - \frac{1}{2}\varepsilon$ and $\arg(z + \alpha) = \arg(z) + \arg(1 + \alpha/z)$. Hence

$$\log(z + \alpha) = \log z + \log\Bigl(1 + \frac{\alpha}{z}\Bigr),$$

where in all three places we use the principal branch of the logarithm function. Since $|\alpha/z| \leq C^{-1} < 1$ we can continue:

$$\log(z + \alpha) = \log z + \frac{\alpha}{z} + O\Bigl(\frac{\alpha^2}{z^2}\Bigr) = \log z + \frac{\alpha}{z} + O\bigl(|z|^{-2}\bigr)$$

(since we allow the implied constant to depend on α). Using this in (681) we get

$$\begin{aligned}\log\Gamma(z+\alpha) &= \big(z+\alpha-\tfrac{1}{2}\big)\Big(\log z+\frac{\alpha}{z}+O\big(|z|^{-2}\big)\Big)-(z+\alpha)+\log\sqrt{2\pi}+O\big(|z+\alpha|^{-1}\big)\\ &= \big(z+\alpha-\tfrac{1}{2}\big)\log z-z+\log\sqrt{2\pi}+O\big(|z|^{-1}\big)+O\big(|z+\alpha|^{-1}\big)\\ &= \big(z+\alpha-\tfrac{1}{2}\big)\log z-z+\log\sqrt{2\pi}+O\big(|z|^{-1}\big),\end{aligned}$$

where in the last step we used the fact that $|z+\alpha|\geq|z|-|\alpha|=|z|\big(1-|\alpha/z|\big)\geq(1-C^{-1})|z|\gg|z|$. Hence we have proved the desired formula for all z satisfying $|z|\geq1$, $|z+\alpha|\geq1$, $|\arg(z+\alpha)|\leq\pi-\varepsilon$ and $|z|\geq C|\alpha|$.

It remains to treat z satisfying $|z|\geq1$, $|z+\alpha|\geq1$, $|\arg(z+\alpha)|\leq\pi-\varepsilon$ and $|z|\leq C|\alpha|$. This is trivial: These set of such z is *compact* and $\log\Gamma(z+\alpha)-\big(z+\alpha-\frac{1}{2}\big)\log z+z-\log\sqrt{2\pi}$ is continuous on this set, hence bounded. Also $|z|$ is bounded on the set; hence $|z|^{-1}$ is bounded from below. Hence by adjusting the implied constant we have $\log\Gamma(z+\alpha)-\big(z+\alpha-\frac{1}{2}\big)\log z+z-\log\sqrt{2\pi}=O\big(|z|^{-1}\big)$ for all z in our compact set, as desired. □

8.7. Recall that we have made a specific choice of the logarithm function $\log\Gamma(z)$ as an analytic function in $\mathbb{C}\setminus(-\infty,0]$, see Definition 8.3. Now let $z\in\mathbb{C}\setminus(-\infty,0]$ be arbitrary, and let C be a circle with centre z and radius $\rho>0$, and assume that ρ is so small that C does not touch $(-\infty,0]$. Then by Cauchy's integral formula,

$$\frac{\Gamma'(z)}{\Gamma(z)}=\frac{d}{dz}\log\Gamma(z)=\frac{1}{2\pi i}\int_C\frac{\log\Gamma(w)}{(w-z)^2}\,dw. \tag{682}$$

Let us now assume $|z|\geq1$ and $\big|\arg z\big|\leq\pi-\varepsilon$. (Without loss of generality we assume that $\varepsilon<\frac{1}{10}$.) Take $\rho=\frac{1}{3}\varepsilon|z|$. Then for any w on C we have $\big|\frac{w}{z}-1\big|=\frac{|w-z|}{|z|}=\frac{\rho}{|z|}=\frac{1}{3}\varepsilon$ and this implies $\arg(\frac{w}{z})<\frac{1}{2}\varepsilon$. Hence $\big|\arg(w)\big|=\big|\arg(z\cdot\frac{w}{z})\big|\leq\pi-\frac{1}{2}\varepsilon$. This implies in particular that C does not touch $(-\infty,0]$, so that (682) holds. By applying Theorem 8.17 with $\frac{1}{2}\varepsilon$ in place of ε we also get that $\log\Gamma(w)=(w-\frac{1}{2})\log w-w+\log\sqrt{2\pi}+O\big(|w|^{-1}\big)$ for all w on C, and hence

$$\frac{\Gamma'(z)}{\Gamma(z)}=\frac{d}{dz}\log\Gamma(z)=\frac{1}{2\pi i}\int_C\frac{(w-\frac{1}{2})\log w-w+\log\sqrt{2\pi}}{(w-z)^2}\,dw+O\Big(\int_C\frac{|w|^{-1}}{|w-z|^2}|dw|\Big).$$

Here the integral equals $\frac{d}{dz}\big((z-\frac{1}{2})\log z-z+\log\sqrt{2\pi}\big)=\log z-\frac{1}{2z}$, by Cauchy's integral formula. To treat the error term we note that for all w on C we have $|\frac{w}{z}-1|=\frac{1}{3}\varepsilon<\frac{1}{2}$; hence $|\frac{w}{z}|>\frac{1}{2}$ and $|w|=|z|\cdot\big|\frac{w}{z}\big|>\frac{1}{2}|z|$. Also $|w-z|=\rho=\frac{1}{3}\varepsilon|z|$. Hence we conclude (since we allow the implied constant to depend on ε):

$$\frac{\Gamma'(z)}{\Gamma(z)}=\log z-\frac{1}{2z}+O\big(|z|^{-2}\big).$$

□

9.1 (a). Writing out the relation $\Lambda(s) = \Lambda(1-s)$ from Theorem 9.1 we have:

$$\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s) = \pi^{-\frac{1}{2}+\frac{1}{2}s}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\zeta(1-s).$$

This identity, as well as those below, is an equality between two functions meromorphic in the whole complex plane. It follows that

$$\zeta(1-s) = \pi^{\frac{1}{2}-s}\frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}-\frac{1}{2}s)}\zeta(s).$$

But we have $\Gamma(\frac{1}{2}-\frac{1}{2}s)\Gamma(\frac{1}{2}+\frac{1}{2}s) = \frac{\pi}{\sin(\pi(\frac{1}{2}-\frac{1}{2}s))} = \frac{\pi}{\cos(\frac{\pi}{2}s)}$, by (319) with $z = \frac{1}{2}-\frac{1}{2}s$. Hence

$$\zeta(1-s) = \pi^{-\frac{1}{2}-s}\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)\cos(\tfrac{\pi}{2}s)\zeta(s).$$

Finally using Legendre's duplication formula (cf. Problem 8.4; use this with $z = \frac{1}{2}s$) we get

$$\zeta(1-s) = \pi^{-s}2^{1-s}\Gamma(s)\cos(\tfrac{\pi}{2}s)\zeta(s).$$

□

9.1 (b). Writing out the relation $\xi(1-s,\overline{\chi}) = \frac{i^a q^{\frac{1}{2}}}{\tau(\chi)}\xi(s,\chi)$ from Theorem 9.1 we have:

$$(\pi/q)^{-\frac{1-s+a}{2}}\Gamma\big(\tfrac{1}{2}(1-s+a)\big)L(1-s,\overline{\chi}) = \frac{i^a q^{\frac{1}{2}}}{\tau(\chi)}(\pi/q)^{-\frac{1}{2}(s+a)}\Gamma\big(\tfrac{1}{2}(s+a)\big)L(s,\chi)$$

It follows that

$$L(1-s,\overline{\chi}) = \frac{i^a q^{\frac{1}{2}}}{\tau(\chi)}(\pi/q)^{\frac{1}{2}-s}\frac{\Gamma(\frac{1}{2}(s+a))}{\Gamma(\frac{1}{2}(1-s+a))}L(s,\chi).$$

If $a=0$ then we have just as in (a): $\frac{\Gamma(\frac{1}{2}(s+a))}{\Gamma(\frac{1}{2}(1-s+a))} = \frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}(1-s))} = \pi^{-1}\cos(\frac{\pi}{2}s)\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}+\frac{1}{2}s) = \pi^{-\frac{1}{2}}2^{1-s}\cos(\frac{\pi}{2}s)\Gamma(s)$ and hence

$$L(1-s,\overline{\chi}) = 2\frac{(2\pi/q)^{-s}}{\tau(\chi)}\cos(\tfrac{\pi}{2}s)\Gamma(s)L(s,\chi).$$

On the other hand if $a=1$ then $\frac{\Gamma(\frac{1}{2}(s+a))}{\Gamma(\frac{1}{2}(1-s+a))} = \frac{\Gamma(\frac{1}{2}s+\frac{1}{2})}{\Gamma(1-\frac{1}{2}s)} = \pi^{-1}\sin(\frac{\pi}{2}s)\Gamma(\frac{1}{2}+\frac{1}{2}s)\Gamma(\frac{1}{2}s) = \pi^{-\frac{1}{2}}2^{1-s}\sin(\frac{\pi}{2}s)\Gamma(s)$ and hence

$$L(1-s,\overline{\chi}) = 2i\frac{(2\pi/q)^{-s}}{\tau(\chi)}\sin(\tfrac{\pi}{2}s)\Gamma(s)L(s,\chi).$$

□

9.1 (c). First of all we replace χ with $\overline{\chi}$ in the formula which we proved in (b):

$$L(1-s,\chi) = \frac{2i^a q^s}{(2\pi)^s\tau(\overline{\chi})}\begin{Bmatrix}\cos(\frac{\pi}{2}s) & \text{if } a=0\\ \sin(\frac{\pi}{2}s) & \text{if } a=1\end{Bmatrix}\Gamma(s)L(s,\overline{\chi}).$$

Note here that $2\cos(\frac{\pi}{2}s) = e^{\pi is/2} + e^{-\pi is/2}$ and $2i\sin(\frac{\pi}{2}s) = e^{\pi is/2} - e^{-\pi is/2}$. Hence the above formula can be written as

$$L(1-s,\chi) = \frac{q^s}{(2\pi)^s \tau(\overline{\chi})}\big(e^{\pi is/2} + \chi(-1)e^{-\pi is/2}\big)\Gamma(s)L(s,\overline{\chi}).$$

Finally recall from (355) and (361) that $\tau(\chi)\tau(\overline{\chi}) = \chi(-1)q$; hence we get

$$L(1-s,\chi) = \frac{q^{s-1}}{(2\pi)^s}\tau(\chi)\Big(\chi(-1)e^{\pi is/2} + e^{-\pi is/2}\Big)\Gamma(s)L(s,\overline{\chi}),$$

which agrees with the formula we wanted to prove. □

9.3 (a). One way to solve the problem is to start by expressing the known sum $\sum_{m\in\mathbb{Z}/q\mathbb{Z}} e\big(\frac{nm}{q}\big)$ (cf. footnote 20) as a combination of some $c_{q'}(n')$'s. Thus: For every $n \in \mathbb{Z}/q\mathbb{Z}$ we have

$$\sum_{m\in\mathbb{Z}/q\mathbb{Z}} e\big(\frac{nm}{q}\big) = \sum_{d|q}\sum_{\substack{m\in\mathbb{Z}/q\mathbb{Z}\\(m,q)=d}} e\big(\frac{nm}{q}\big) = \sum_{d|q}\sum_{\substack{m_1\in\mathbb{Z}/\frac{q}{d}\mathbb{Z}\\(m_1,\frac{q}{d})=1}} e\big(\frac{nm_1}{q/d}\big) = \sum_{d|q} c_{q/d}(n),$$

where in the last step we substituted $m = dm_1$. Hence, using the known value of $\sum_{m\in\mathbb{Z}/q\mathbb{Z}} e\big(\frac{nm}{q}\big)$:

$$\sum_{d|q} c_d(n) = \sum_{d|q} c_{q/d}(n) = \begin{cases} q & \text{if } n \equiv 0 \pmod q \\ 0 & \text{if } n \not\equiv 0 \pmod q. \end{cases}$$

Hence by Möbius inversion (cf. Problem 3.7):

$$c_q(n) = \sum_{d|q}\mu\Big(\frac{q}{d}\Big)\left\{\begin{matrix} d & \text{if } n \equiv 0 \pmod d \\ 0 & \text{if } n \not\equiv 0 \pmod d \end{matrix}\right\} = \sum_{d|(q,n)}\mu\Big(\frac{q}{d}\Big)d.$$

This proves the first formula.

To prove the second formula we note that $\sum_{d|(q,n)}\mu\big(\frac{q}{d}\big)d$ very much resembles the right hand side of the known formula $\phi(m) = \sum_{d|m}\mu\big(\frac{m}{d}\big)d$. To get a closer similarity we set $m = (q,n)$ and $f = q/m$; then the above formula says that

$$c_q(n) = \sum_{d|m}\mu\Big(\frac{mf}{d}\Big)d = \sum_{d|m}\mu\Big(\frac{m}{d}\Big)\left\{\begin{matrix} \mu(f) & \text{if } (m/d,f)=1 \\ 0 & \text{if } (m/d,f)>1 \end{matrix}\right\} d,$$

where the last step follows directly from the definition of the Möbius function μ. Note that $(m/d,f)=1$ holds if and only if d is divisible with $m_f := \prod_{p|f} p^{\text{ord}_p m}$. Hence, substituting $d = m_f d_1$, we get

$$c_q(n) = \mu(f)\sum_{d_1|\frac{m}{m_f}}\mu\Big(\frac{m/m_f}{d_1}\Big)m_f d_1 = \mu(f)m_f\phi\big(\frac{m}{m_f}\big),$$

where we could finally use the known formula $\phi(n) = \sum_{d|n} \mu(\frac{n}{d})d$ (cf. Problem 3.8). To clean up the expression, we note that

$$m_f\phi(\frac{m}{m_f}) = \prod_{p|f} p^{\mathrm{ord}_p m} \prod_{p|\frac{m}{m_f}} \Big(p^{\mathrm{ord}_p(\frac{m}{m_f})}(1-p^{-1})\Big) = \prod_{p|(m,f)} p^{\mathrm{ord}_p m} \prod_{\substack{p|m \\ (p\nmid f)}} \big(p^{\mathrm{ord}_p m}(1-p^{-1})\big)$$

$$= \prod_{p|mf} p^{\mathrm{ord}_p m} \prod_{\substack{p|mf \\ (p\nmid f)}} (1-p^{-1}) = \frac{\prod_{p|mf}\big(p^{\mathrm{ord}_p m + \mathrm{ord}_p f}(1-p^{-1})\big)}{\prod_{p|f}\big(p^{\mathrm{ord}_p f}(1-p^{-1})\big)} = \frac{\phi(mf)}{\phi(f)}.$$

Hence we have proved:

$$c_q(n) = \frac{\phi(mf)}{\phi(f)}\mu(f) = \frac{\phi(q)}{\phi(\frac{q}{(q,n)})}\mu\Big(\frac{q}{(q,n)}\Big).$$

□

9.3 (b). Directly from Definition 9.2 we have

$$\big|\tau(\chi)\big|^2 = \sum_{m_1\in\mathbb{Z}/q\mathbb{Z}} \sum_{m_2\in\mathbb{Z}/q\mathbb{Z}} \chi(m_1)\overline{\chi(m_2)}e\Big(\frac{m_1-m_2}{q}\Big).$$

Here the summation may be restricted to $m_1, m_2 \in (\mathbb{Z}/q\mathbb{Z})^\times$, since $\chi(m)$ whenever $m \notin (\mathbb{Z}/q\mathbb{Z})^\times$. Letting now m^{-1} denote the inverse of m in $(\mathbb{Z}/q\mathbb{Z})^\times$ we obtain

$$\big|\tau(\chi)\big|^2 = \sum_{m_1\in(\mathbb{Z}/q\mathbb{Z})^\times} \sum_{m_2\in(\mathbb{Z}/q\mathbb{Z})^\times} \chi(m_1m_2^{-1})e\Big(\frac{m_1-m_2}{q}\Big).$$

Substituting $m_1 = m_2k$ this becomes

$$\big|\tau(\chi)\big|^2 = \sum_{k\in(\mathbb{Z}/q\mathbb{Z})^\times} \chi(k) \sum_{m_2\in(\mathbb{Z}/q\mathbb{Z})^\times} e\Big(\frac{m_2(k-1)}{q}\Big) = \sum_{k\in(\mathbb{Z}/q\mathbb{Z})^\times} \chi(k)c_q(k-1)$$

$$= \sum_{m\in\mathbb{Z}/q\mathbb{Z}} \chi(m+1)c_q(m),$$

where in the last step we took $m = k+1$ and again used the fact that $\chi(k) = 0$ whenever $k \in (\mathbb{Z}/q\mathbb{Z})^\times$. Using the first formula in (367) we get

$$\big|\tau(\chi)\big|^2 = \sum_{m\in\mathbb{Z}/q\mathbb{Z}} \chi(m+1) \sum_{d|(q,m)} \mu\Big(\frac{q}{d}\Big)d = \sum_{d|q} \mu\Big(\frac{q}{d}\Big)d \sum_{\substack{m\in\mathbb{Z}/q\mathbb{Z} \\ m\equiv 0 \ (\mathrm{mod}\ d)}} \chi(m+1)$$

$$= \sum_{d|q} \mu\Big(\frac{q}{d}\Big)d \cdot \frac{1}{d}\sum_{c=0}^{q} \chi(dc+1).$$

Here the inner sum vanishes for all $d \mid q$ except $d = q$, by Problem 4.4. Hence we get

$$\left|\tau(\chi)\right|^2 = \mu\Big(\frac{q}{q}\Big)q \cdot \chi(1) = q.$$

□