Hints / solution sketches to problems

3.13.
– see my solution sketches for the previous problem session (11 Sept).

4.1. (Similar to arguments appearing in the proof of Theorem 4.35.)
(a). This follows from (b).
(b). Let $q_1 = \prod_{j=1}^{r} c(\chi_j)$. First assume $m, n \in \mathbb{Z}$, $(m, q) = (n, q) = 1$ and $m \equiv n \pmod{q_1}$. Then for each $j \in \{1, \ldots, r\}$ we have $m \equiv n$ mod $c(\chi_j)$ since $c(\chi_j) \mid q_1$; and also $(m, p_j^{\alpha_j}) = (n, p_j^{\alpha_j}) = 1$; therefore $\chi_j(m) = \chi_j(n)$. Hence $\chi(m) = \prod_{j=1}^{r} \chi_j(m) = \prod_{j=1}^{r} \chi_j(n) = \chi(n)$. This proves that [$\chi(n)$ for n restricted by $(n, q) = 1$] has period q_1, and hence by Lemma 4.20 we have $c(\chi) \mid q_1$.

Next for every $k \in \{1, \ldots, r\}$ we can argue as follows. Since $\chi_k \in X_{p_k^{\alpha_k}}$ we have $c(\chi_k) = p^{\beta}$ for some $\beta \in \{0, 1, \ldots, \alpha_k\}$, by Lemma 4.20. Suppose that $\beta > 0$. Then [$\chi_k(n)$ restricted by $(n, p_k^{\alpha_k}) = 1$] does *not* have period $p_k^{\beta-1}$ and hence there are some $m, n \in \mathbb{Z}$ with $(m, p_k) = (n, p_k) = 1$ and $m \equiv n \pmod{p_k^{\beta-1}}$ and $\chi_k(m) \neq \chi_k(n)$. Now by the Chinese Remainder Theorem there exist $m', n' \in \mathbb{Z}$ such that $m' \equiv m \pmod{p_k^{\alpha_k}}$ and $m' \equiv 1 \pmod{p_j^{\alpha_j}}$ for all $j \neq k$, and $n' \equiv n \pmod{p_k^{\alpha_k}}$ and $n' \equiv 1 \pmod{p_j^{\alpha_j}}$ for all $j \neq k$. Now

$$\chi(m') = \prod_{j=1}^{r} \chi_j(m') = 1 \cdots 1 \cdot \chi_k(m') \cdot 1 \cdots 1 = \chi_k(m') = \chi_k(m),$$

and similarly $\chi(n') = \chi_k(n)$; thus $\chi(m') \neq \chi(n')$. But we also have $(m', q) = (n', q) = 1$ and $m' \equiv n' \mod q_1/p_k$; hence this proves that *[$\chi(n)$ restricted by $(n, q) = 1$] does not have period q_1/p_k, and thus* $c(\chi) \nmid \frac{q_1}{p_k}$.

We thus have $c(\chi) \mid q_1$ but for each prime $p \mid q_1$ we also have $c(\chi) \nmid \frac{q_1}{p}$. This implies that $c(\chi) = q_1$. $\square$

4.3. If q_1 is a period of $\chi(n)$ restricted by $(n, q) = 1$ then for all integers m, n with $(m, q) = (n, q)$ and $m \equiv n \pmod{q_1}$ we have $\chi(m) = \chi(n)$. In particular, taking $m = 1$, it follows that $\chi(n) = 1$ for all integers n satisfying $(n, q) = 1$ and $n \equiv 1 \pmod{q_1}$.

Conversely, suppose that q_1 is a positive integer and that $\chi(n) = 1$ holds for all integers n satisfying $n \equiv 1 \pmod{q_1}$ and $(n, q) = 1$. Let $q_2 = (q, q_1)$; then we know that there are some integers x, y such that $q_2 = xq + yq_1$. Now if n is any integer satisfying $n \equiv 1 \pmod{q_2}$ and $(n, q) = 1$ then we have $n = 1 + hq_2$ for some integer h, and hence $n = 1 + h(xq + yq_1) \equiv 1 + hyq_1 \pmod{q}$ so that $\chi(n) = \chi(1 + hyq_1)$ and $(1 + hyq_1, q) = 1$. Furthermore $1 + hyq_1 \equiv 1 \pmod{q_1}$ and thus by our assumption $\chi(1 + hyq_1) = 1$. Hence q_2 has exactly the same property as q_1, i.e. $\chi(n) = 1$ holds for all integers n satisfying $n \equiv 1 \pmod{q_2}$ and $(n, q) = 1$. The advantage is that q_2 also divides q, by construction!

Now take any two integers m_1, m_2 with $(m_1, q) = (m_2, q) = 1$ and $m_1 \equiv m_2 \pmod{q_2}$. Then m_1, m_2 correspond to two elements in $(\mathbb{Z}/q\mathbb{Z})^\times$ and hence there is a unique $n \in (\mathbb{Z}/q\mathbb{Z})^\times$ such that $m_1 \equiv nm_2 \pmod{q}$. Since $q_2 \mid q$ this implies $m_1 \equiv nm_2 \pmod{q_2}$, which forces $n \equiv 1 \pmod{q_2}$ (since $(m_1, q_2) = (m_2, q_2) = 1$). Hence, by what we proved in last paragraph, $\chi(n) = 1$! Hence $\chi(m_1) = \chi(nm_2) = \chi(n)\chi(m_2) = \chi(m_2)$.

This proves that $\chi(n)$ restricted by $(n, q) = 1$ has period q_2. Since $q_2 \mid q_1$, it follows that $\chi(n)$ restricted by $(n, q) = 1$ also has period q_1. $\square$

6.1. It follows from (246) combined with Lemma 6.1 that, for any fixed integer $q \geq 0$,

$$\pi(x) = \frac{x}{\log x} + \frac{1!x}{\log^2 x} + \ldots + \frac{q!x}{(\log x)^{q+1}} + O\Big(\frac{x}{(\log x)^{q+2}}\Big) \qquad \text{as } x \to \infty.$$

In particular

$$\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + O\Big(\frac{x}{(\log x)^3}\Big) \qquad \text{as } x \to \infty.$$

Let us now define $A(x)$ by the relation $\pi(x) = \frac{x}{\log x - A(x)}$. It then follows that $\frac{x}{\log x - A(x)} = \frac{x}{\log x} + \frac{x}{(\log x)^2} + O\Big(\frac{x}{(\log x)^3}\Big)$ as $x \to \infty$. Dividing with $\frac{x}{\log x}$ we get $\frac{\log x}{\log x - A(x)} = 1 + \frac{1}{\log x} + O\Big(\frac{1}{(\log x)^2}\Big)$, hence $\frac{\log x - A(x)}{\log x} = \Big(1 + \frac{1}{\log x} + O\Big(\frac{1}{(\log x)^2}\Big)\Big)^{-1} = 1 - \frac{1}{\log x} - O\Big(\frac{1}{(\log x)^2}\Big) + O\Big(\Big(\frac{1}{\log x} - O\Big(\frac{1}{(\log x)^2}\Big)\Big)^2\Big) = 1 - \frac{1}{\log x} + O\Big(\frac{1}{(\log x)^2}\Big)$. Subtracting 1 and multiplying with $-\log x$ this gives $A(x) = 1 + O\Big(\frac{1}{\log x}\Big)$. In particular $\lim_{x\to\infty} A(x) = 1$. $\square$

6.2. Recall that $-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$ for $\sigma > 1$ (cf. (113)). Multiplication with $\zeta(s)$ gives (for $\sigma > 1$): $-\zeta'(s) = \Big(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}\Big)\Big(\sum_{m=1}^{\infty} m^{-s}\Big) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\Lambda(n)}{(nm)^s} = \sum_{k=1}^{\infty}\sum_{n|k} \frac{\Lambda(n)}{k^s}$. That is: $\sum_{k=1}^{\infty} \frac{\log k}{k^s} = \sum_{k=1}^{\infty}\sum_{n|k} \frac{\Lambda(n)}{k^s}$ (true, with absolute convergence, for all $\sigma > 1$). Hence by comparison of coefficients (cf. Proposition 3.10) we get:

$$\sum_{n|k} \Lambda(n) = \log k, \qquad \forall k \in \mathbb{Z}^+. \tag{1}$$

(The same formula can of course also be proved directly: If $k = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$, then the left hand side equals, since $\Lambda(m) = 0$ unless m is a prime power: $\Big(\sum_{j_1=1}^{\alpha_1} \log p_1\Big) + \Big(\sum_{j_2=1}^{\alpha_2} \log p_2\Big) + \ldots + \Big(\sum_{j_r=1}^{\alpha_r} \log p_r\Big) = \alpha_1 \log p_1 + \alpha_2 \log p_2 + \ldots + \alpha_r \log p_r = \log k$.)

If we add (1) over positive integers $k \leq x$, we obtain $\sum_{k\leq x}\sum_{n|k} \Lambda(n) = \sum_{k\leq x} \log k$, i.e.

$$T(x) := \sum_{n\leq x} \Big\lfloor \frac{x}{n} \Big\rfloor \Lambda(n) = \log\big(\lfloor x \rfloor !\big).$$

$\square$

6.4. We will start by proving that the sum $-\sum_p \log(1-\chi(p)p^{-1})$ [1] converges, where we use the principal branch of the logarithm in each term. Note that $\left|\chi(p)p^{-1}\right| < 1$ for each prime p; hence the Taylor expansion of $\log(1-z)$ applies, and we have:

$$-\sum_p \log\Big(1-\frac{\chi(p)}{p}\Big) = \sum_p \Big(\sum_{m=1}^{\infty} \frac{\chi(p)^m}{mp^m}\Big). \tag{2}$$

But here note that the double sum

$$\sum_p \sum_{m=2}^{\infty} \frac{\chi(p)^m}{mp^m}$$

is *absolutely* convergent, by the same computation as in LN page 7, equation (9) (which works for $s=1$). Hence the convergence of (2) is equivalent to the convergence of (the $m=1$ contribution) $\sum_p \chi(p)p^{-1}$, and this series is convergent *by Proposition 6.8.*

Hence we have now proved that the sum $-\sum_p \log(1-\chi(p)p^{-1})$ is convergent. By exponentiating (and using the fact that the function $z \mapsto e^z$ is continuous), it follows that the product $\prod_p \big(1-\frac{\chi(p)}{p}\big)^{-1}$ is convergent! In order to compute the value of this infinite product, we will prove that

$$-\sum_p \log\Big(1-\frac{\chi(p)}{p}\Big) = \lim_{s\to 1+} -\sum_p \log\Big(1-\frac{\chi(p)}{p^s}\Big), \tag{3}$$

or equivalently that

$$\sum_p \Big(\sum_{m=1}^{\infty} \frac{\chi(p)^m}{mp^m}\Big) = \lim_{s\to 1+} \sum_p \Big(\sum_{m=1}^{\infty} \frac{\chi(p)^m}{mp^{ms}}\Big). \tag{4}$$

The same statement can be expressed as follows:

$$\sum_{n=1}^{\infty} \frac{a_n}{n} = \lim_{s\to 1+} \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \tag{5}$$

where $a_n = \chi(p)^m m^{-1}$ whenever n is a prime power $n=p^m$, and otherwise $a_n = 0$. But we have proved above that the Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is convergent for $s = s_0 := 1$; hence by LN Theorem 3.6 [2] this Dirichlet series converges *uniformly* for all real $s \geq 1$! This implies that

[1] Here and in any sum below, it is understood that we add over the primes in *increasing order.*

[2] We apply LN Theorem 3.6 with an arbitrary $H > 0$ and then use the fact that the real interval $[1,\infty)$ is contained in the sector $\{s = \sigma + it \,:\, \sigma \geq 1,\ |t-0| \leq H(\sigma-1)\}$.

we may change order of limit and summation in the right hand side of (5), and therefore the equality in (5) holds! Thus we have proved (3).

Finally, exponentiating both sides of (3), and using the fact that the function $z \mapsto e^z$ is continuous, it follows that

$$\prod_p \Big(1 - \frac{\chi(p)}{p}\Big)^{-1} = \lim_{s\to 1+} \prod_p \Big(1 - \frac{\chi(p)}{p^s}\Big)^{-1} = \lim_{s\to 1+} L(s,\chi).$$

But we know that since χ is nonprincipal, the function $L(s,\chi)$ is holomorphic in the half place $\Re(s) > 0$ (see LN Example 3.5); in particular $L(s,\chi)$ is continuous at $s = 1$, and hence it follows from the above relation that $\prod_p \big(1 - \frac{\chi(p)}{p}\big)^{-1} = L(1,\chi)$. □

Remark: In the above solution we used LN Proposition 6.8, which has a rather complicated but "elementary" proof. However, it should be noted that LN Proposition 6.8 may alternatively be derived as a *standard* consequence of a more advanced result, namely the *PNT for arithmetic sequences with an error term,* which states that there exists an absolute constant $c_1 > 0$ such that for any $q \in \mathbb{Z}^+$ and any $a \in \mathbb{Z}$ with $(a,q) = 1$,

$$\pi(x;q,a) = \frac{1}{\phi(q)} \operatorname{Li} x + O_q\big(x e^{-c_1\sqrt{\log x}}\big) \qquad \text{as } x \to \infty \tag{6}$$

(see LN Theorem 15.6[3]). Indeed, using (6) one can prove, by mimicking the solution of HW1.2, that there exists a real constant $A(q,a)$ and an absolute constant $c_2 > 0$ such that

$$\sum_{\substack{p<x \\ p\equiv a \mod q}} \frac{1}{p} = \frac{1}{\phi(q)} \log\log x + A(q,a) + O_q\big(e^{-c_2\sqrt{\log x}}\big) \tag{7}$$

as $x \to \infty$. Letting now χ be any non-principal character mod q, we multiply the relation in (7) with $\chi(a)$, and then add over all $a \in (\mathbb{Z}/q\mathbb{Z})^\times$ and use $\sum_{a\in(\mathbb{Z}/q\mathbb{Z})^\times} \chi(a) = 0$. This gives:

$$\sum_{p<x} \frac{\chi(p)}{p} = \sum_{a\in(\mathbb{Z}/q\mathbb{Z})^\times} \chi(a)A(q,a) + O_q\big(e^{-c_2\sqrt{\log x}}\big). \tag{8}$$

Letting $x \to \infty$ in (8) shows that $\sum_p \frac{\chi(p)}{p} = \sum_{a\in(\mathbb{Z}/q\mathbb{Z})^\times} \chi(a)A(q,a)$,[4] and subtracting this formula from (8) gives

$$\sum_{p\geq x} \frac{\chi(p)}{p} = O_q\big(e^{-c_2\sqrt{\log x}}\big). \tag{9}$$

That is, we have proved LN Proposition 6.8 with a *stronger* error term! □

[3]The relation (6) is much easier to prove than Theorem 15.6.

[4]Here and in (9), it is understood that we add over the primes p *in increasing order.*

6.6. This can be proved using the standard method (from basic calculus) of estimating a sum using an integral. Namely, the function $f(x) = \log x$ is increasing for $x > 0$; hence for every real $y > 0$ we have

$$\int_{y-1}^{y} \log x \, dx \leq \log y \leq \int_{y}^{y+1} \log x \, dx.$$

Hence for any $n \geq 2$:

$$\begin{aligned}
\log(n!) &= \sum_{m=1}^{n} \log m \leq \sum_{m=1}^{n} \int_{m}^{m+1} \log x \, dx = \int_{1}^{n+1} \log x \, dx \\
&= \Big[x \log x - x\Big]_{x=1}^{x=n+1} = (n+1)\log(n+1) - (n+1) - (-1) \\
&= (n+1)\big(\log n + O(n^{-1})\big) - n = (n+1)\log n - n + O(1) \\
&= n \log n - n + O(\log n).
\end{aligned}$$

(In the above computation we used $\log(n+1) = \log n + O(n^{-1})$, which can be proved e.g. by the Mean Value Theorem applied to the function $\log x$.) Also for any $n \geq 2$:

$$\begin{aligned}
\log(n!) &= \sum_{m=1}^{n} \log m = \sum_{m=2}^{n} \log m \geq \sum_{m=2}^{n} \int_{m-1}^{m} \log x \, dx = \int_{1}^{n} \log x \, dx \\
&= \Big[x \log x - x\Big]_{x=1}^{x=n} = n \log n - n - (-1) = n \log n - n + O(\log n).
\end{aligned}$$

Together, the above two inequalities prove that $\log(n!) = n \log n - n + O(\log n)$. □

Alternative solution, using integration by parts: We have:

$$\begin{aligned}
\log(n!) = \sum_{m=1}^{n} \log m = \int_{1-}^{n} \log x \, d\lfloor x \rfloor &= \Big[(\log x)\lfloor x \rfloor\Big]_{x=1-}^{x=n} - \int_{1}^{n} \frac{\lfloor x \rfloor}{x} \, dx \\
&= n \log n - \int_{1}^{n} \frac{\lfloor x \rfloor}{x} \, dx.
\end{aligned}$$

Here we trivially have $\int_1^n \frac{\lfloor x \rfloor}{x} \, dx \leq \int_1^n \frac{x}{x} \, dx = n - 1$ and $\int_1^n \frac{\lfloor x \rfloor}{x} \, dx \geq \int_1^n \frac{x-1}{x} \, dx = n - 1 - \log n$. These two inequalities together prove that $\int_1^n \frac{\lfloor x \rfloor}{x} \, dx = n + O(\log n)$ for all $n \geq 2$. This gives the desired statement. □

7.1. If $y = \pi(x)$ then $\lim_{x\to\infty} \frac{y \log x}{x} = 1$, by the prime number theorem. Hence, by taking the logarithm, we have $\lim_{x\to\infty} \big(\log y + \log\log x - \log x\big) = 0$, and thus, after dividing by $\log x$, we also have $\lim_{x\to\infty} \frac{\log y}{\log x} = 1$. This combined with $\lim_{x\to\infty} \frac{y \log x}{x} = 1$ gives $\lim_{x\to\infty} \frac{y \log y}{x} = 1$, whence the result follows on taking $x = p_n$, since $\pi(p_n) = n$. □

7.2. We use the formula (278), i.e.

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \frac{(x)}{x^{s+1}}\,dx \qquad (\sigma > 0). \tag{10}$$

Recall that the last integral is uniformly absolutely convergent in any compact subset of $\{\sigma > 0\}$. Hence by Weierstrass Theorem we have for every $n \geq 0$:

$$\frac{d^n}{ds^n}\Big(s\int_1^\infty \frac{(x)}{x^{s+1}}\,dx\Big) = \int_1^\infty \frac{d^n}{ds^n}\big(s\cdot(x)\cdot x^{-s-1}\big)\,dx$$
$$= (-1)^n s\int_1^\infty \frac{(\log x)^n\cdot(x)}{x^{s+1}}\,dx + (-1)^{n-1}n\int_1^\infty \frac{(\log x)^{n-1}\cdot(x)}{x^{s+1}}\,dx$$

and hence

$$\frac{d^n}{ds^n}\Big(s\int_1^\infty \frac{(x)}{x^{s+1}}\,dx\Big)_{|s=1} = (-1)^n\int_1^\infty \frac{\big((\log x)^n - n(\log x)^{n-1}\big)\cdot(x)}{x^2}\,dx$$

This gives a formula for the sought coefficients. (Note that the integral is absolutely convergent.)

To get a different and perhaps in some sense more explicit formula we rewrite the above as (note that the following computation is valid also for $n = 0$ if we make the special interpretation $(\log 1)^0 = 1$):

$$= (-1)^n\int_1^\infty \frac{\big((\log x)^n - n(\log x)^{n-1}\big)\cdot(x - \lfloor x\rfloor)}{x^2}\,dx$$
$$= (-1)^n\lim_{N\to\infty}\sum_{k=1}^{N-1}\int_k^{k+1}\frac{\big((\log x)^n - n(\log x)^{n-1}\big)\cdot(x-k)}{x^2}\,dx$$
$$= (-1)^n\lim_{N\to\infty}\Big(\int_1^N \frac{(\log x)^n - n(\log x)^{n-1}}{x}\,dx - \sum_{k=1}^{N-1}k\int_k^{k+1}\frac{(\log x)^n - n(\log x)^{n-1}}{x^2}\,dx\Big)$$
$$= (-1)^n\lim_{N\to\infty}\Big(\Big[\tfrac{1}{n+1}(\log x)^{n+1} - (\log x)^n\Big]_{x=1}^{x=N} + \sum_{k=1}^{N-1}k\Big[\frac{(\log x)^n}{x}\Big]_{x=k}^{x=k+1}\Big)$$
$$= (-1)^n\lim_{N\to\infty}\Big(\frac{(\log N)^{n+1}}{n+1} - (\log N)^n - \frac{0}{n+1} + (\log 1)^n + \sum_{k=1}^{N-1}k\Big(\frac{(\log(k+1))^n}{k+1} - \frac{(\log k)^n}{k}\Big)\Big)$$
$$= (-1)^n\lim_{N\to\infty}\Big(-\sum_{k=1}^{N-1}\frac{(\log k)^n}{k} - \frac{(\log N)^n}{N} + \frac{(\log N)^{n+1}}{n+1}\Big) + \begin{Bmatrix}1 & \text{if } n=0\\ 0 & \text{if } n\geq 1\end{Bmatrix}$$
$$= (-1)^{n-1}\lim_{N\to\infty}\Big(\sum_{k=1}^{N}\frac{(\log k)^n}{k} - \frac{(\log N)^{n+1}}{n+1}\Big) + \begin{Bmatrix}1 & \text{if } n=0\\ 0 & \text{if } n\geq 1\end{Bmatrix}.$$

Note that this is a natural generalization of Euler's constant,

$$\gamma = \lim_{N\to\infty}\Big(\sum_{k=1}^{N}\frac{1}{k} - \log N\Big) = 0.577...$$

Thus, let us define

$$\gamma_n := \lim_{N\to\infty}\Big(\sum_{k=1}^{N}\frac{(\log k)^n}{k} - \frac{(\log N)^{n+1}}{n+1}\Big). \tag{11}$$

(The above computation shows that the limit exists.) Thus $\gamma_0 = \gamma$, Euler's constant, and the relation which we have proved above can be summarized as:

$$\frac{d^n}{ds^n}\Big(s\int_1^\infty \frac{(x)}{x^{s+1}}\,dx\Big) = (-1)^{n-1}\gamma_n + \left\{\begin{matrix} 1 & \text{if } n=0 \\ 0 & \text{if } n\geq 1\end{matrix}\right\}.$$

It follows that the Taylor series of the (holomorphic) function $s \mapsto s\int_1^\infty \frac{(x)}{x^{s+1}}\,dx$ at $s=1$ is:

$$s\int_1^\infty \frac{(x)}{x^{s+1}}\,dx = (-\gamma_0+1) + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\gamma_n}{n!}(s-1)^n.$$

Using this in (10), we have proved that the Laurent expansion of $\zeta(s)$ at $s=1$ is

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty}\frac{(-1)^n\gamma_n}{n!}(s-1)^n. \tag{12}$$

The constants γ_n are called the *Stieltjes constants*; see Wikipedia. $\square$

7.4 (a). Using Problem 2.1(a) and integration by parts (cf. (105) in Theorem 3.11) we get for all $s \in \mathbb{C}$ with $\sigma > 1$:

$$\zeta(s)^{-1} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = s \int_1^\infty \frac{M(x)}{x^{s+1}}\, dx.$$

Integrating by parts once more (using the obvious bound $|M(x)| \leq x$) we get:

$$\zeta(s)^{-1} = s(s+1) \int_1^\infty \frac{M_1(x)}{x^{s+2}}\, dx.$$

This holds, with absolute convergence in the integral in the right hand side, for all s with $\sigma > 1$. It also follows from Theorem 7.4 that $\int_{c-i\infty}^{c+i\infty} \big| \frac{1}{s(s+1)} \frac{1}{\zeta(s)} \big| \, |ds| < \infty$ for every $c > 1$. Hence by Mellin inversion (cf. Theorem 7.7 and the proof of (290)) we have

$$M_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \frac{1}{\zeta(s)}\, ds \tag{13}$$

for all $x > 0$ and all $c > 1$. □

7.4 (b). This is very similar to the proof of Theorem 7.9 in LN, the only essential differences being that in place of (295) we use $\frac{M_1(x)}{x^2} = \int_{(c)} g(s) x^{s-1}\, ds$, where $g(s) := \frac{1}{2\pi i} \frac{1}{s(s+1)} \zeta(s)^{-1}$, and the fact that $g(s)$ in analytic *without exceptions* in the closed half-plane $\{\sigma \geq 1\}$, i.e. there is no pole at $s = 1$, and thus (295) is now replaced with $\frac{M_1(x)}{x^2} = \int_L g(s) x^{s-1}\, ds$ (with L chosen as before). The conclusion is that $\lim_{x\to\infty} \frac{M_1(x)}{x^2} = 0$, as desired. □

7.4 (c). For any real numbers $0 < u_1 \leq u_2$ we have

$$\big| M(u_2) - M(u_1) \big| = \Big| \sum_{u_1 < n \leq u_2} \mu(n) \Big| \leq \#\{n \in \mathbb{Z} \,:\, u_1 < n \leq u_2\} \leq 1 + (u_2 - u_1),$$

and hence by symmetry we have for any choice of $u_1, u_2 > 0$:

$$\big| M(u_2) - M(u_1) \big| \leq 1 + |u_2 - u_1|.$$

Now let $\beta > 1$. Using the above we have, for any $x > 0$ and any $u \geq x$: $M(x) \geq M(u) - (u - x) - 1$ and $M(x) \leq M(u) + (u - x) + 1$, and thus

$$M(x) \geq \frac{1}{\beta x - x} \int_x^{\beta x} \big(M(u) - (u - x) - 1 \big)\, du$$

and

$$M(x) \leq \frac{1}{\beta x - x} \int_x^{\beta x} \big(M(u) + (u - x) + 1 \big)\, du.$$

These two can be collected into

$$\Big|M(x) - \frac{1}{\beta x - x}\int_x^{\beta x} M(u)\,du\Big| \leq \frac{1}{\beta x - x}\int_x^{\beta x} \big((u-x)+1\big)\,du$$
$$= \frac{1}{\beta x - x}\big(\tfrac{1}{2}(\beta-1)^2x^2 + (\beta-1)x\big).$$

Hence, dividing with x:

$$\Big|\frac{M(x)}{x} - \frac{M_1(\beta x) - M_1(x)}{(\beta-1)x^2}\Big| \leq \tfrac{1}{2}(\beta-1) + \frac{1}{x}.$$

Letting here $x \to \infty$, using $\lim_{x\to\infty} \frac{M_1(x)}{x^2} = 0$ which we know from part (b), we obtain

$$\limsup_{x\to\infty}\Big|\frac{M(x)}{x} - 0\Big| \leq \tfrac{1}{2}(\beta - 1).$$

By taking β near enough to 1 we can make $\frac{1}{2}(\beta-1)$ be as near as we please to 0; hence

$$\lim_{x\to\infty}\frac{M(x)}{x} = 0.$$

$\square$

7.7. As in the proof of Theorem 7.4, one computes:

$$\log\big|L(\sigma,\chi_0)^3L(\sigma+it,\chi)^4L(\sigma+2it,\chi^2)\big| = \sum_{n=2}^{\infty} a_n n^{-\sigma}\Big(3 + 4\Re\big(\chi(n)n^{-it}\big) + \Re\big(\chi(n)^2n^{-2it}\big)\Big),$$

where now $a_n = m^{-1}$ if $n = p^m$ *and* $p \nmid q$, and $a_n = 0$ otherwise. Here, given any n with $a_n > 0$, set $z := \chi(n)n^{-it}$; and note then that $|z| = 1$; thus $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$, and now

$$3 + 4\Re\big(\chi(n)n^{-it}\big) + \Re\big(\chi(n)^2n^{-2it}\big) = 3 + 4\cos\theta + \cos 2\theta \geq 0.$$

Hence (analogue of LN (286)):

$$\big((\sigma-1)L(\sigma,\chi_0)\big)^3\Big|\frac{L(\sigma+ti,\chi)}{\sigma-1}\Big|^4\big|L(\sigma+2ti,\chi^2)\big| \geq \frac{1}{\sigma-1}, \tag{14}$$

for all $\sigma > 1$ and all $t \in \mathbb{R}$. Hence for any $t \neq 0$, the proof of Theorem 7.4 can be mimicked, giving that $L(1+it,\chi) \neq 0$. Also *if* $t = 0$ *and* χ *is complex*, we can let $\sigma \to 1^+$ in the above relation to conclude that $L(1,\chi) \neq 0$. Indeed, χ complex implies that $\chi^2 \neq \chi_0$, so that $L(s,\chi^2)$ is holomorphic at $s = 1$.

However, if χ is *real*, then $\chi^2 = \chi_0$ and so $L(s,\chi^2)$ has a (simple) pole at $s = 1$; and then we ***cannot*** use the above relation to prove $L(1,\chi) \neq 0$. $\square$