Analysis for PhD students (2013)
Assignment 1

Problem 1. Let $0 < \omega_1 \le \omega_2 \le \cdots$ be an increasing sequence of positive numbers satisfying

$$\#\{n \in \mathbb{N} : \omega_n \le T\} = cT^2 + O(T) \qquad \forall T > 0, \tag{1}$$

where $c > 0$ is some constant. Let $\alpha \le 2$. Determine an asymptotic formula for $\sum_{\omega_n < T} \omega_n^{-\alpha}$ as $T \to \infty$.

(10p)

Problem 2. Let $n \in \mathbb{N}$ and $\kappa \in \mathbb{R}$. A vector $x \in \mathbb{R}^n$ is said to be of *Diophantine type* κ if there exists some $c > 0$ such that for all $k \in \mathbb{Z}^n$ and $q \in \mathbb{N}$ we have $|x - q^{-1}k| > cq^{-\kappa}$. Prove that if $\kappa > 1 + n^{-1}$ then almost every $x \in \mathbb{R}^n$ (w.r.t. Lebesgue measure) is of Diophantine type κ.

(15p)

Problem 3. Let f be a bounded real-valued function on $[a, b]$. Prove that f is Riemann integrable iff $\{x \in [a, b] : f \text{ is discontinuous at } x\}$ has Lebesgue measure zero. (Hint: This is Folland's exercise 2:23; note that Folland gives an outline of a proof in his formulation of the exercise.)

(15p)

Problem 4. Let μ be the Borel measure on $\mathbb{R}$ which is given by $\mu = \delta + m_1$, where δ is the Dirac measure at 0 and m_1 is the Lebesgue measure restricted to $[0, 1]$ (viz., $\delta(E) = I(0 \in E)$ and $m_1(E) = m(E \cap [0, 1])$ for any Borel subset $E \subset \mathbb{R}$, where $I(\cdot)$ is the indicator function.) Let μ_2 be the product measure $\mu \times \mu$ on $\mathbb{R}^2$.

(a). Find the Lebesgue decomposition of μ_2 with respect to Lebesgue measure on $\mathbb{R}^2$.

(b). Give a formula for $\widehat{\mu}_2(\xi)$, $\xi \in \mathbb{R}^2$.

(c). Describe $\mu * \mu$ (which is a Borel measure on $\mathbb{R}$) explicitly.

(15p)

Problem 5. For the the following two sequences $\{\mu_N\}$ in $M(\mathbb{R}^2)$, prove that there is a measure $\mu \in M(\mathbb{R}^2)$ such that $\mu_N \to \mu$ in the weak* topology on $M(\mathbb{R}^2) = C_0(\mathbb{R}^2)^*$, and describe μ explicitly.

(a). μ_N given by $\int_{\mathbb{R}^2} f \, d\mu_N = N^{-1} \sum_{k=1}^{N} f(\frac{k}{N}, 0)$, $\forall f \in C_0(\mathbb{R}^2)$.

(b). μ_N given by $\int_{\mathbb{R}^2} f \, d\mu_N = N^{-2} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} f(\frac{k}{N}, \frac{km}{N^2})$, $\forall f \in C_0(\mathbb{R}^2)$.

(10p)

The next few exercises concern the rate of decay of $\widehat{\chi}_E$ for various sets $E \subset \mathbb{R}^n$.

Problem 6. Prove that if $E = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is any box in $\mathbb{R}^n$ then along certain rays $\widehat{\chi}_E(\xi)$ does not decay faster than $|\xi|^{-1}$, while along other rays $\widehat{\chi}_E(\xi)$ decays as fast as $|\xi|^{-n}$.
(In more precise terms: Prove that there exist some $\xi \in \mathbb{R}^n \setminus \{0\}$, $c > 0$ and a sequence $0 < u_1 < u_2 < \ldots$ such that $\lim_{m\to\infty} u_m = \infty$ and $|\widehat{\chi}_E(u_m\xi)| > c u_m^{-1}$ for all m. Also prove that there exist some $\xi \in \mathbb{R}^n \setminus \{0\}$ and $C > 0$ such that $|\widehat{\chi}_E(u\xi)| < C u^{-n}$ for all $u \geq 1$.)

(10p)

Problem 7. For $E \subset \mathbb{R}^n$ and $\delta > 0$ we define $\partial_\delta E$ as the (open) set of all points in $\mathbb{R}^n$ which have distance $< \delta$ to some point in ∂E. Now assume that there exist $C > 0$ and $0 < a \leq 1$ such that $m(\partial_\delta E) < C\delta^a$ for all $0 < \delta \leq 1$. Then prove that E is Lebesgue measurable, and if furthermore E is bounded then there is a constant $K > 0$ such that $|\widehat{\chi}_E(\xi)| \leq K|\xi|^{-a}$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. [Hint: Consider the Fourier transform of $\chi_E - \chi_{\eta+E}$, where η is a suitably chosen vector in $\mathbb{R}^n$.]

(10p)

Problem 8. Let B be a fixed ball in $\mathbb{R}^n$. Prove that there is a constant $C > 0$ such that $|\widehat{\chi}_B(\xi)| \leq C(1 + |\xi|)^{-\frac{1}{2}(n+1)}$ for all $\xi \in \mathbb{R}^n$.

[Hint: We here give an outline of a possible proof. Partial credit will be given for carrying out one or some of these steps. (1) Using invariance under rotations and Fubini's theorem, prove that $\widehat{\chi}_B(\xi) = \int_{-1}^{1} f(x) e^{-2\pi i |\xi| x}\, dx$, where $f(x)$ is the volume of a ball of radius $\sqrt{1-x^2}$ in $\mathbb{R}^{n-1}$. Explicitly, $f(x) = V_{n-1}(1-x^2)^{\frac{1}{2}(n-1)}$ where V_{n-1} is the volume of the $(n-1)$-dimensional unit ball. (2) Prove that for $k < \frac{1}{2}(n+1)$ the kth derivative of $f(x)$ equals $\sum_{j=1}^{k} P_{j,k,n}(x)(1-x^2)^{\frac{1}{2}(n-1)-j}$, where $P_{j,k,n}(x)$ is a polynomial of degree $\leq j$. (3) Now integrate by parts repeatedly in the formula for $\widehat{\chi}_B(\xi)$. For n *odd* there are no problems to integrate by parts $\frac{1}{2}(n+1)$ times and this leads to the desired bound. (4) For n *even* we integrate by parts $\frac{1}{2}n$ times; then split the range of integration into $(-1, -1+h)$, $(-1+h, 1-h)$, $(1-h, 1)$ for an appropriate $h \in (0, 1)$, and apply integration by parts once more for the *middle* region; the desired bound can now be deduced.]

(15p)

Submission deadline: 18 February, 10.15 (i.e. before the problem discussion starts).