Analysis for PhD students (2013)
Assignment 2

Problem 1. (a). Prove that $L^1(\mathbb{R}^n)$ is vaguely dense in $M(\mathbb{R}^n)$. (Cf. Folland, p. 272, Exercise 40.)
(b). Let $\delta \in M(\mathbb{R}^n)$ be the Dirac measure at 0. Prove that there does not exist any sequence $f_1, f_2, \ldots \in L^1(\mathbb{R}^n)$ which tends to δ in the norm of $M(\mathbb{R}^n)$.

(15p)

Problem 2. Folland p. 255, Exercise 15, all parts a–c (the Sampling Theorem).

(15p)

Problem 3. Let μ be a finite Borel measure on $\mathbb{T}^2$ satisfying

$$\mu(E + (t, 0)) = \mu(E), \qquad \forall E \in \mathcal{B}_{\mathbb{T}^2},\ t \in \mathbb{T}^1.$$

Prove that there is a finite Borel measure μ_1 on $\mathbb{T}^1$ such that

$$\mu(E) = \int_{\mathbb{T}^1} m^1(\{x \in \mathbb{T}^1 \ : \ (x, y) \in E\})\, d\mu_1(y), \qquad \forall E \in \mathcal{B}_{\mathbb{T}^2},$$

where m^1 is the Lebesgue measure on $\mathbb{T}^1$.

(10p)

Problem 4. Prove that for any $m \geq 2$, $a > 0$ and $0 \leq \delta \leq 1$,

$$\int_{\substack{\theta \in (-\pi,\pi) \\ |\sin\theta| < \delta}} \min\Big(1, \big(a^{-1}|\sin\theta|\big)^m\Big) \frac{d\theta}{\sin^2\theta} \asymp_m a^{-1} \min(1, (\delta/a)^{m-1}).$$

(Here "$\asymp$" means "both $\ll$ and $\gg$". The integral is taken over all $\theta \in (-\pi, \pi)$ which satisfy $|\sin\theta| < \delta$.)

(15p)

Problem 5. For $0 \leq r \leq 2$, let $\alpha_n(r)$ be the scaled intersection volume of two n-dimensional unit balls at distance r from each other, i.e. $\alpha_n(r) = \dfrac{m^n(B_1 \cap (B_1 + x))}{m^n(B_1)}$ for any $x \in \mathbb{R}^n$ with $|x| = r$, where B_1 is the unit ball with center at the origin.
(a). Prove that $\alpha_n(r) = c_n \int_{r/2}^1 (1 - t^2)^{\frac{n-1}{2}}\, dt$ where $c_n = \frac{2\Gamma(\frac{n}{2}+1)}{\sqrt{\pi}\Gamma(\frac{n+1}{2})}$.
(b). Prove that $\alpha_n(1) \sim \sqrt{\frac{6}{\pi n}} \left(\frac{3}{4}\right)^{n/2}$ as $n \to \infty$.
(c). Let r_n be such that $\alpha_n(r_n) = \frac{1}{2}$. Determine an asymptotic formula for r_n as $n \to \infty$.

(15p)

Problem 6. Let $C = [-\frac{1}{2}, \frac{1}{2}]^3$, a unit cube in $\mathbb{R}^3$, and set

$$U(w) = \frac{1}{|w|} - \int_C \frac{dx}{|w-x|} \qquad \text{for } w \in \mathbb{R}^3 \setminus \{0\}.$$

Prove that $|U(w)| \ll |w|^{-4}$ as $|w| \to \infty$.

(Hint: Use the Taylor expansion of the function $|w-x|^{-1}$ wrt. x.)

(15p)

Problem 7. Prove that for all $x \in \mathbb{R} \setminus \{0\}$,

$$\int_0^{\pi/2} t^{\frac{1}{2}} \sin(x \cos t)\, dt = \frac{\Gamma(\frac{3}{4}) \sin\big(x - \operatorname{sgn}(x)\frac{3\pi}{8}\big)}{2^{1/4}} |x|^{-\frac{3}{4}} + O(|x|^{-1}).$$

(15p)

Submission deadline: 13 March, 10.15 (i.e. before the problem discussion starts).