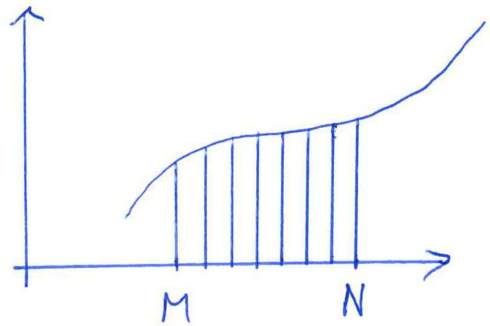
#1. Sums and integrals

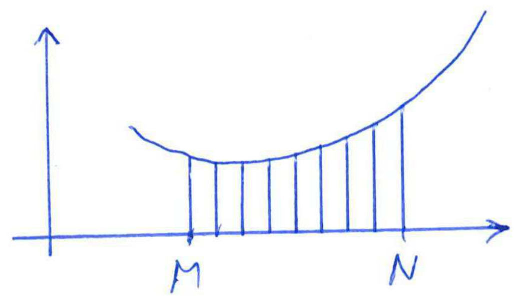
Ex: $M < N$ integers, $f : [M-1, N+1] \longrightarrow \mathbb{R}$

If f increasing:



$$\int_{M-1}^{N} f(x)\,dx \leq \sum_{n=M}^{N} f(n) \leq \int_{M}^{N+1} f(x)\,dx$$

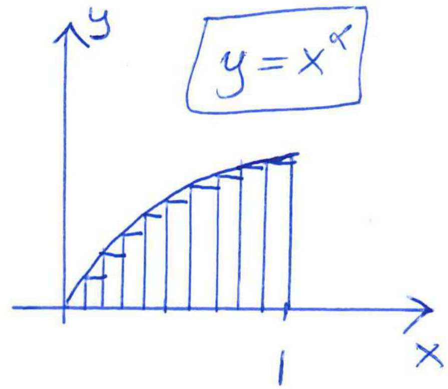
If f convex:



$$\sum_{M}^{N} f(n) \leq \int_{M-\frac{1}{2}}^{N+\frac{1}{2}} f(x)\,dx$$

Other (related) connection: Riemann sum

Ex: Given $\alpha > -1$, study the asymptotic behavior of $\sum_{n=1}^{N} n^\alpha$ as $N \to \infty$!



solution #1: $$\sum_{1}^{N} n^\alpha = N^{\alpha+1} \cdot \underbrace{\sum_{1}^{N} \left(\frac{n}{N}\right)^\alpha \cdot \frac{1}{N}}_{\text{Riemann sum for } \int_0^1 x^\alpha\, dx}$$

Hence $$\sum_{1}^{N} \left(\frac{n}{N}\right)^\alpha \cdot \frac{1}{N} \to \int_0^1 x^\alpha\, dx = \frac{1}{\alpha+1} \quad \text{as } N \to \infty$$

and so $$\boxed{\sum_{n=1}^{N} n^\alpha \sim \frac{N^{\alpha+1}}{\alpha+1} \quad \text{as } N \to +\infty}$$

Def: "$f(x) \sim g(x)$ as $x \to a$" means $\lim_{x \to a} \frac{f(x)}{g(x)} = 1$.

solution #2: Say $\alpha \geq 0$; then $f(x) = x^\alpha$ is $\nearrow$, thus

$$\int_0^N x^\alpha dx \leq \sum_{n=1}^N n^\alpha \leq \int_1^{N+1} x^\alpha dx$$

$$\Rightarrow \frac{N^{\alpha+1}}{\alpha+1} \leq \sum_{n=1}^N n^\alpha \leq \frac{(N+1)^{\alpha+1}-1}{\alpha+1}$$

$$\Rightarrow \boxed{\sum_{n=1}^N n^\alpha = \frac{N^{\alpha+1}}{\alpha+1} + O(N^\alpha), \quad \forall N \in \mathbb{Z}^+}$$

Def: For $a \geq 0$, "$O(a)$" denotes a number $b \in \mathbb{C}$ which ~~is~~ satisfies $|b| \leq Ca$, where C is a "constant".

Ex: Given $0 < w_1 \leqslant w_2 \leqslant \cdots$ satisfying

$$\underline{A(T) := \#\{n \in \mathbb{N} : w_n \leqslant T\} \sim c\,T^2 \quad \text{as} \quad T \to \infty}$$

for some fixed $c > 0$,

(1) For which $\alpha \in \mathbb{R}$ does $\sum_{n=1}^{\infty} w_n^{-\alpha}$ converge?

(2) Then, study the asymptotics of $\sum_{w_n > T} w_n^{-\alpha}$ as $T \to \infty$!

sum over all n with $w_n > T$.

Solution, #(1): Positive sum for order of magnitude, a dyadic decomposition should suffice.

Assume $\alpha > 0$. (We have obvious divergence if $\alpha \leqslant 0$.)

$$\sum_{n=1}^{\infty} w_n^{-\alpha} = \sum_{m=0}^{\infty} \left(\sum_{2^m < w_n \leqslant 2^{m+1}} w_n^{-\alpha} \right) + \sum_{w_n \leqslant 1} w_n^{-\alpha} \qquad \circledast$$

$$\circledast \geqslant \sum_{m=0}^{\infty} \underbrace{\#\{2^m < w_n \leqslant 2^{m+1}\}}_{= A(2^{m+1}) - A(2^m) \sim 3c \cdot 2^{2m}} \cdot 2^{-(m+1)\alpha} \quad \text{and} \quad \circledast \leqslant \sum_{m=0}^{\infty} \#\{2^m < w_n \leqslant 2^{m+1}\} \cdot 2^{-m\alpha} + O(1)$$

$\therefore$ $\circledast$ converges iff $\sum_{m=0}^{\infty} 2^{2m - \alpha m} < \infty$, i.e. iff $\boxed{\alpha > 2.}$

Solution #2: "Summation by parts"

$\boxed{\text{Assume } \alpha > 2}$

$$\sum_{w_n > T} w_n^{-\alpha} = \sum_{w_n > T} \int_{w_n}^{\infty} \alpha x^{-\alpha-1}\,dx = \int_T^{\infty} \sum_{T < w_n \le x} \alpha x^{-\alpha-1}\,dx$$

Change order $\sum \int = \int \sum$.
No problems since $\alpha x^{-\alpha-1} > 0$

$$= \int_T^{\infty} (A(x) - A(T))\, \alpha x^{-\alpha-1}\,dx = \underbrace{\int_T^{\infty} A(x)\, \alpha x^{-\alpha-1}\,dx}_{} - \underbrace{A(T)\,T^{-\alpha}}_{\sim c T^{2-\alpha}}$$

Requires some thought...

$$\sim \int_T^{\infty} c x^2 \cdot \alpha x^{-\alpha-1}\,dx = \frac{c\alpha T^{2-\alpha}}{\alpha - 2}$$

$$\boxed{\therefore \sum_{w_n > T} w_n^{-\alpha} \sim \frac{2c}{\alpha-2} T^{2-\alpha} \quad \text{as} \quad T \to \infty}$$

Alt. presentation:
$$\sum_{w_n > T} w_n^{-\alpha} = \underbrace{\int_T^{\infty} x^{-\alpha}\,dA(x)}_{\text{Riemann-Stieltjes integral}} = \left[x^{-\alpha} A(x)\right]_{x=T}^{x=\infty} + \alpha \int_T^{\infty} x^{-\alpha-1} A(x)\,dx = \ldots\ldots$$

Def: (Riemann integral)

Let $A < B$, $g : [A, B] \to \mathbb{C}$.

A sequence $\{x_n\}_{n=0}^{N}$ with $A = x_0 \leq x_1 \leq \dots \leq x_N = B$ is called a **partition of $[A,B]$**.

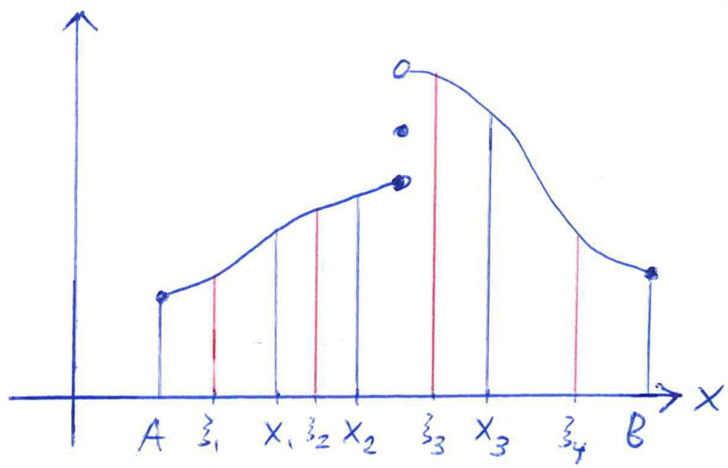
For any partition $\{x_n\}_0^N$ of $[A, B]$ and any points $\xi_n \in [x_{n-1}, x_n]$, $n = 1, \dots, N$,

set $$S(\{x_n\}, \{\xi_n\}) := \sum_{n=1}^{N} g(\xi_n) \cdot (x_n - x_{n-1})$$

"Riemann sum"

Call such $\langle \{x_n\}, \{\xi_n\} \rangle$ a **tagged partition** of $[A, B]$ = **t.p.**

For g real-valued:



$S(\{x_n\}, \{\xi_n\})$ = Area

We say that the Riemann integral $\int_A^B g(x)\,dx$ exists (and g is Riemann integrable)

if

$$\boxed{\begin{array}{l} \exists\, I \in \mathbb{C} : \ \forall\, \varepsilon > 0 : \ \exists\, \delta > 0 : \\ \qquad \forall \text{ t.p } \left\langle \{x_n\}_0^N, \{\xi_n\}_1^N \right\rangle \text{ of } [A,B] : \\ \qquad\qquad \operatorname{mesh}(\{x_n\}) \leqslant \delta \implies \left| S(\{x_n\},\{\xi_n\}) - I \right| < \varepsilon \end{array}}$$

Def: $= \max\limits_{1 \leq n \leq N} (x_n - x_{n-1})$

— and then $\int_A^B g(x)\,dx := I$

(Basic facts: g Riemann integrable on $[A,B]$ $\Rightarrow$ g bounded on $[A,B]$.
g continuous on $[A,B]$ $\Rightarrow$ g Riemann integrable on $[A,B]$.)

Def (Riemann-Stieltjes integral)

Let $A < B$, $f, g : [A,B] \to \mathbb{C}$.

For any t.p. $\langle \{x_n\}_0^N, \{\xi_n\}_1^N \rangle$ of $[A,B]$, set $S(\{x_n\},\{\xi_n\}) := \sum_{n=1}^{N} g(\xi_n) \cdot (f(x_n) - f(x_{n-1}))$

only difference!

We say that $\int_A^B g(x)\, df(x) = I \in \mathbb{C}$ if

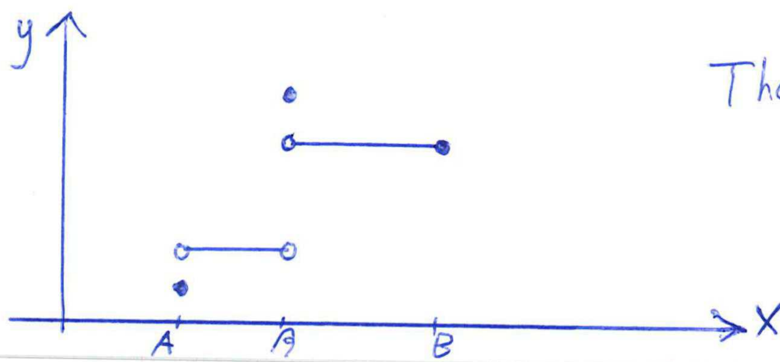
$$\boxed{\begin{array}{l} \forall \varepsilon > 0 : \exists \delta > 0 : \forall \text{ t.p. } \langle \{x_n\}_0^N, \{\xi_n\}_1^N \rangle \text{ of } [A,B] : \\ \quad \text{mesh}(\{x_n\}) \le \delta \Rightarrow |S(\{x_n\},\{\xi_n\}) - I| < \varepsilon \end{array}}$$

Same as before!

See Ex 1.5 in notes

Ex: $g \in C([A,B])$, $f : [A,B] \to \mathbb{C}$ piecewise constant,

say



Then $\int_A^B g\,df = (f(A+) - f(A)) \cdot g(A)$

$+ (f(\mathcal{A}+) - f(\mathcal{A}-)) \cdot g(\mathcal{A})$

$+ (f(B) - f(B-)) \cdot g(B)$.

Basic properties:

- $\underline{g \in C([A,B]) \text{ and } f \in \overbrace{BV([A,B])}^{\text{will define...}}} \Rightarrow \underline{\int_A^B g\,df \text{ exists}}$ (Thm. 1.10)

- For $\underline{\text{any}}$ $f, g : [A,B] \to \mathbb{C}$, $\underline{\underline{\text{if}}}$ $\int_A^B g\,df$ exists, $\underline{\text{then}}$ $\int_A^B f\,dg$ exists and $\underline{\underline{\int_A^B g\,df = \left[g(x)f(x)\right]_A^B - \int_A^B f\,dg}}$ (Thm 1.12 – $\underline{\text{easy}}$!)

- If $f \in C^1([A,B])$ and $g : [A,B] \to \mathbb{C}$ Riemann integrable, then $\underline{\underline{\int_A^B g\,df = \int_A^B g(x) f'(x)\,dx}}$ (both exists). (Thm 1.13)

[Note: Now "alt presentation" on p. 5 is justified!

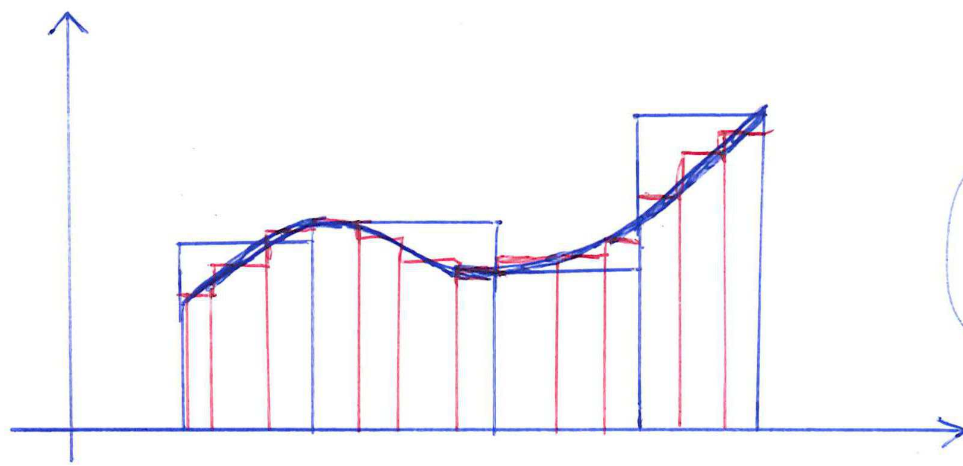
$\underline{\text{Warning}}$: $\int_A^B \neq \int_A^C + \int_C^B$ in general!]

Proof of existence, for $\int_A^B g(x)\,dx$, assuming $g \in C([A,B])$

① One easily verifies (cf. Lemma 1.11): It <u>suffices</u> to prove the following "Cauchy property":

$$\circledast \quad \boxed{\begin{array}{l} \forall \varepsilon > 0 : \exists \delta > 0 : \quad \forall \text{ t.p. } \langle \{x_n\}, \{\xi_n\} \rangle, \langle \{x_n'\}, \{\xi_n'\} \rangle \\ \left[\text{mesh}(\{x_n\}) \leq \delta \text{ and } \text{mesh}(\{x_n'\}) \right] \Rightarrow \left| S(\{x\},\{\xi\}) - S(\{x'\},\{\xi'\}) \right| < \varepsilon \end{array}}$$

② Any two <u>t.p.'s</u> have a <u>common refinement</u>; hence in ⊛ we may assume that $\{x_n'\}$ is a refinement of $\{x_n\}$.



– Must <u>compare</u> the <u>areas!</u>

Use g continuous $\Rightarrow$ uniformly continuous on $[A,B]$!

Thus, given $\varepsilon > 0$ we can take $\delta > 0$ s.t.

$$|g(x) - g(x')| < \varepsilon \quad \text{whenever} \quad |x - x'| < \delta$$

Now if $\text{mesh}(\{x_n\}) \leq \delta$ and $\{x_n'\}$ is a refinement of $\{x_n\}$:

$$S(\{x_n'\},\{\xi_n'\}) - S(\{x_n\},\{\xi_n\}) = \sum_{n=1}^{N'} g(\xi_n')\cdot(x_n' - x_{n-1}') - \sum_{n=1}^{N} g(\xi_n)\cdot(x_n - x_{n-1})$$

$$= \sum_{n=1}^{N} \sum_{\substack{j \\ [x_{j-1}', x_j'] \subset [x_{n-1}, x_n]}} \left(g(\xi_j') - g(\xi_n)\right)\cdot(x_j' - x_{j-1}')$$

$$\text{Abs. value} \leq \sum_{n=1}^{N} \sum_{\substack{j \\ [\ ] \subset [\ ]}} \varepsilon\cdot(x_j' - x_{j-1}') = \varepsilon(B-A)$$

Done! [Replace $\varepsilon \rightsquigarrow \varepsilon(B-A)$]

$\square$

Same approach works in *much* more general situations!

Proof of existence of $\int_A^B g(x)\,df(x)$, for $g \in C([A,B])$, $f \in BV([A,B])$

Def: For $f : [A,B] \to \mathbb{C}$, $\underline{\underline{\mathrm{Var}_{[A,B]}(f)}} = \underbrace{\sup_{\{x_n\}_0^N} \sum_{n=1}^{N} |f(x_n) - f(x_{n-1})|}$

any partition of $[A,B]$

We say $\underline{\underline{f \in BV([A,B])}}$ if $\mathrm{Var}_{[A,B]}(f) < \infty$

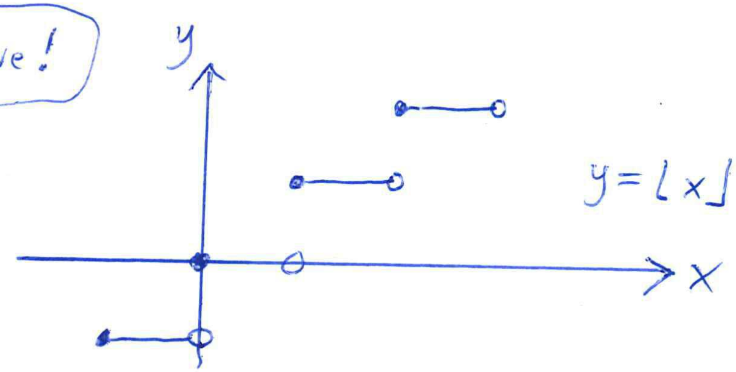
Now (essentially) the SAME existence proof (p. 10–11) works!

Ex: The Euler-Maclaurin summation formula

For $f:[A,B]\to\mathbb{C}$ "slowly varying", estimate $\displaystyle\sum_{\substack{A<n\le B\\(n\in\mathbb{Z})}} f(n)$!?

We will get better estimates than on p. 1-3 above!

Let $\lfloor x\rfloor$ = "floor of x" $= \max\big(\mathbb{Z}\cap(-\infty,x]\big)$



If $f\in C^1([A,B])$

$$\sum_{A<n\le B} f(n) = \int_A^B f(x)\,d\lfloor x\rfloor = \Big[f(x)\cdot\lfloor x\rfloor\Big]_{x=A}^{x=B} - \int_A^B f'(x)\lfloor x\rfloor\,dx$$

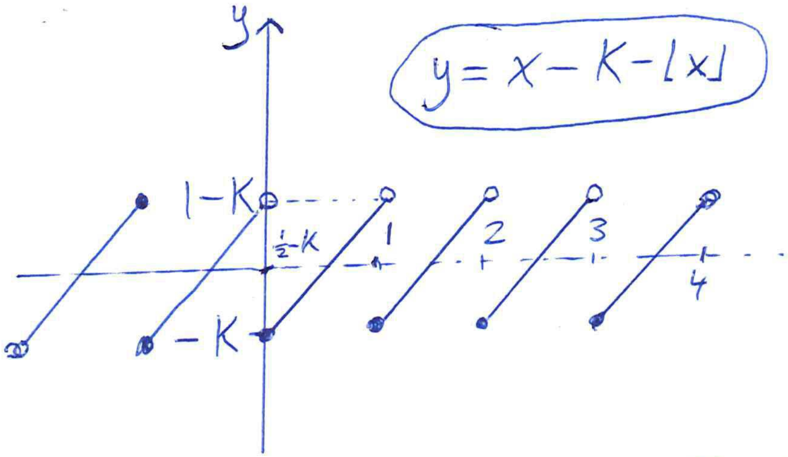
Also, for any $K\in\mathbb{R}$:

$$\int_A^B f(x)\,dx = \Big[f(x)\cdot(x-K)\Big]_{x=A}^{x=B} - \int_A^B f'(x)\,(x-K)\,dx$$

Combine $\Rightarrow$ $$\sum_{A<n\le B} f(n) = \int_A^B f(x)\,dx - \Big[f(x)\cdot(x-K-\lfloor x\rfloor)\Big]_A^B + \int_A^B f'(x)\cdot(x-K-\lfloor x\rfloor)\,dx$$

Remains to understand: $\displaystyle\int_A^B f'(x)\cdot\left(x-K-\lfloor x\rfloor\right)dx$

Here: $y = x - K - \lfloor x \rfloor$ — oscillates around $y = \frac{1}{2} - K$



Principle: An integral of $\underline{[\text{slowly varying}] \times [\text{oscillating around } 0]}$ is $\underline{\text{small}}$, and this can be proved by $\underline{\text{integrating by parts}}$

Thus we choose $K = \frac{1}{2}$ in ⊛, then integrate by parts! <u>Repeat</u>!!

$$\int_A^B f'(x)\cdot\underbrace{\left(x-\lfloor x\rfloor-\tfrac{1}{2}\right)}_{\tilde{x} := x - \lfloor x \rfloor} dx \quad\curvearrowright\quad -\int_A^B f''(x)\cdot\left(\frac{\tilde{x}^2}{2}-\frac{\tilde{x}}{2}+\frac{1}{12}\right)dx \quad\curvearrowright\quad \int_A^B f'''(x)\cdot\left(\frac{\tilde{x}^3}{6}-\frac{\tilde{x}^2}{4}+\frac{\tilde{x}}{12}\right)dx \quad\curvearrowright\quad \cdots$$

Bernoulli polynomials!

Def: The Bernoulli polynomials $B_0(x), B_1(x), B_2(x), \ldots$ are defined by

$$\underline{\underline{B_0(x) \equiv 1}} \quad \text{and} \quad \underline{\underline{\left[B_r'(x) = r \cdot B_{r-1}(x) \quad \text{and} \quad \int_0^1 B_r(x)\,dx = 0\right]}} \quad (r = 1, 2, 3, \ldots)$$

The r:th Bernoulli number is $\underline{\underline{B_r := B_r(0)}}$.

The Euler-Maclaurin summation formula (Thm 1.19)

For $A < B$, $h \in \mathbb{N}$, $f \in C^h([A,B])$ we have

$$\sum_{A < n \leq B} f(n) = \int_A^B f(x)\,dx + \sum_{r=1}^{h} \frac{(-1)^r}{r!} \left[\tilde{B}_r(x) \cdot f^{(r-1)}(x)\right]_{x=A}^{x=B} + (-1)^{h+1} \int_A^B \frac{\tilde{B}_h(x)}{h!} f^{(h)}(x)\,dx$$

where $\underline{\tilde{B}_r(x) := B_r(x - \lfloor x \rfloor)}$

$B_0(x) = 1$

$B_1(x) = x - \frac{1}{2}$

$B_2(x) = x^2 - x + \frac{1}{6}$

$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$

$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$.

Ex. again: Asymptotic behavior of $\underline{\underline{\sum_{n=1}^{N} n^\alpha \quad \text{as } N\to\infty}}$, for $\alpha > -1$ fixed?

(Can now do arbitrary $\alpha \in \mathbb{C}$!)

Apply Euler-Maclaurin with ~~...~~ $A = $ "$1-$" and $B = N$, $f(x) = x^\alpha$, $h \in \mathbb{N}$:

$$\underline{\sum_{n=1}^{N} n^\alpha = \int_1^N x^\alpha\,dx + \sum_{r=1}^{h} \frac{(-1)^r}{r!} B_r \left(f^{(r-1)}(N) - f^{(r-1)}(1)\right) + 1 + (-1)^{h+1} \int_1^N \frac{\widetilde{B}_h(x)}{h!} f^{(h)}(x)\,dx}$$

For $r \geqslant 2$: $\widetilde{B}_r(N) = \widetilde{B}_r(1-) = B_r$.

For $r = 1$: $\widetilde{B}_1(1-) = \frac{1}{2}$ while $B_1 = -\frac{1}{2}$

Assuming $\alpha \neq -1$

$f^{(h)}(x) = \alpha(\alpha-1)\cdots(\alpha-h+1)\,x^{\alpha-h}$

$$\underline{= \frac{N^{\alpha+1}-1}{\alpha+1} + \sum_{r=1}^{h} (-1)^r B_r \cdot \frac{1}{\alpha+1}\binom{\alpha+1}{r}\cdot\left(N^{\alpha-r+1} - 1\right) + 1 + (-1)^{h+1}\binom{\alpha}{h}\cdot\int_1^N \widetilde{B}_h(x)\,x^{\alpha-h}\,dx}$$

$$\underline{\text{Error term}} = \int_1^N O\left(x^{\operatorname{Re}\alpha - h}\right)dx = \begin{cases} O\left(N^{\operatorname{Re}\alpha-h+1}\right) & \operatorname{Re}\alpha > h-1 \\ O(\log N) & \operatorname{Re}\alpha = h-1 \\ O(1) & \operatorname{Re}\alpha < h-1 \end{cases}$$

$\underline{\text{Ex}}$ For $\underline{\alpha = \frac{3}{2},\ h = 3}$: $\underline{\sum_{n=1}^{N} n^{\frac{3}{2}} = \cancel{N}\ \frac{2}{5} N^{\frac{5}{2}} + \frac{1}{2} N^{\frac{3}{2}} + \frac{1}{8} N^{\frac{1}{2}} + \underbrace{O(1)}_{\text{Seems to} \to -0.0254\ldots}}$, $\forall N \geq 1$

Asymptotic expansion!

Can improve "$O(1)$"?

- Write $\int_1^N = \int_1^\infty - \int_N^\infty$, then continue integrating by parts!

$$\text{Get}\quad \boxed{\begin{array}{l} \displaystyle\sum_{n=1}^{N} n^\alpha = \frac{1}{\alpha+1} \sum_{r=0}^{k} (-1)^r B_r \binom{\alpha+1}{r} N^{\alpha-r+1} + C(\alpha) + \underbrace{(-1)^k \binom{\alpha}{k} \cdot \int_N^\infty \widetilde{B}_k(x) \cdot x^{\alpha-k}\, dx}_{\boxed{\text{Error} = O\left(N^{\operatorname{Re}\alpha - k + 1}\right)}} \\ \text{for any } \alpha \in \mathbb{C},\ \alpha \neq -1,\quad k, h \in \mathbb{N},\quad k \geq h > \operatorname{Re}\alpha + 1 \end{array}}$$

Here

$$\underline{C(\alpha) = 1 - \frac{1}{\alpha+1} \sum_{r=0}^{h} (-1)^r B_r \binom{\alpha+1}{r} + (-1)^{h+1} \binom{\alpha}{h} \int_1^\infty \widetilde{B}_h(x) \cdot x^{\alpha-h}\, dx}$$

Actually $\underline{C(\alpha) = \text{"}\sum_{n=1}^{\infty} n^\alpha\text{"}}$ in a sense!