#5. Convolutions (Folland Ch. 8.2)

DEF: For $f, g: \mathbb{R}^n \to \mathbb{C}$, (m'ble), $(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$

Prop 8.6: Assuming all integrals in question exist:

a) $f * g = g * f$

b) $(f * g) * h = f * (g * h)$

c) For $z \in \mathbb{R}^n$: $\tau_z(f * g) = (\tau_z f) * g = f * (\tau_z g)$

d) $\operatorname{supp}(f * g) \subset \overline{\operatorname{supp}(f) + \operatorname{supp}(g)}$

Young's inequality (8.7): If $f \in L^1$ and $g \in L^p$ $(1 \leq p \leq \infty)$ then $(f*g)(x)$ exists for a.e. x, and $\|f*g\|_p \leq \|f\|_1 \cdot \|g\|_p$

proof:

For $p=\infty$: $\left|(f*g)(x)\right| = \left|\int f(x-y)g(y)\,dy\right| \leq \int |f(x-y)g(y)|\,dy \leq \|g\|_\infty \cdot \int |f(x-y)|\,dy = \|g\|_\infty \|f\|_1$

For $p=1$: Use $\int\int |f(x-y)g(y)|\,dy\,dx = \int |f(z)|\,dz \int |g(y)|\,dy = \|f\|_1 \|g\|_1$

General p: $(f*g)(x) = \int_{\mathbb{R}^n} f(y)\,g(x-y)\,dy$, i.e. $f*g = \int_{\mathbb{R}^n} f(y)\cdot(\tau_y g)(\cdot)\,dy$

Minkowski's inequality: $\|F_1 + F_2\|_p \leq \|F_1\|_p + \|F_2\|_p$, $\forall F_1, F_2 \in L^p(\mathbb{R}^n)$

$\rightsquigarrow \|c_1 F_1 + c_2 F_2 + \ldots + c_n F_n\|_p \leq |c_1|\,\|F_1\|_p + \ldots + |c_n|\,\|F_n\|_p$, $\forall F_1, \ldots, F_n \in L^p(\mathbb{R}^n)$, $c_1, \ldots, c_n \in \mathbb{C}$

$\rightsquigarrow \left\|\int_Y F_y \,d\mu(y)\right\|_p \leq \int_Y \|F_y\|_p \,d\mu(y)$ if $\underbrace{(Y, \mathcal{N}, \mu) \ni y \mapsto F_y \in L^p(\mathbb{R}^n)}_{\text{"nice"}}$

Apply with $F_y = \tau_y g$ and $\mu = f \cdot m$

Prop 8.8: If $\frac{1}{p}+\frac{1}{q}=1$, $f \in L^p$, $g \in L^q$ then $f*g \in BC(\mathbb{R}^n)$, is uniformly continuous, and $\|f*g\|_u \leq \|f\|_p \|g\|_q$.

If $1<p,q<\infty$ then $f*g \in C_0(\mathbb{R}^n)$.

proof, outline: Existence and $\|f*g\|_u \leq \|f\|_p \|g\|_q$: From Hölder's inequality.

$h = f*g$ uniformly continuous?

$$\Leftrightarrow \lim_{\substack{y \to 0 \\ (y \in \mathbb{R}^n)}} \|\tau_y h - h\|_u = 0$$

But: $\|\tau_y h - h\|_u = \|(\tau_y f - f)*g\|_u \leq \|\tau_y f - f\|_p \cdot \|g\|_q \xrightarrow{(y\to 0)} 0$

translation continuous in L^p - Prop 8.5 For $p<\infty$

If $1<p,q<\infty$: $f*g \in C_0$?

Take $\{f_n\}$ and $\{g_n\}$ in C_c with $f_n \to f$ and $g_n \to g$.

Then $f_n * g_n \in C_c$ $(\forall n)$ and $f_n * g_n \to f*g$ in $(BC, \|\cdot\|_u)$.

Hence $f*g \in \overline{C_c} = C_0$.

Multi-index notation (Ch. 8.1)

On $\mathbb{R}^n$: $x = (x_1, \dots, x_n)$. Write $\partial_j = \dfrac{\partial}{\partial x_j}$ assume these commute

A multi-index is an n-tuple $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$.

DEF: $|\alpha| = \sum_1^n \alpha_j$, $\alpha! = \prod_1^n \alpha_j!$, $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}$

and $x^\alpha = \prod_1^n x_j^{\alpha_j}$

Ex $\partial^\alpha (fg) = \sum_{\beta + \gamma = \alpha} \dfrac{\alpha!}{\beta! \gamma!} (\partial^\beta f)(\partial^\gamma g)$

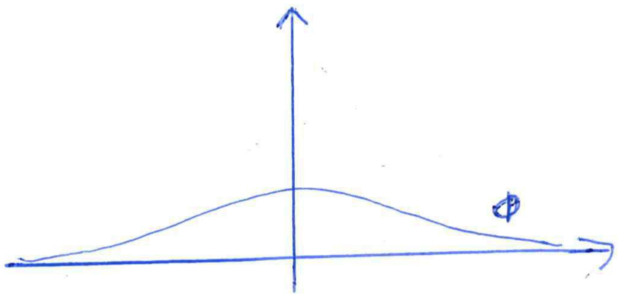
DEF: $C^k(\mathbb{R}^n) = \{ f \in C(\mathbb{R}^n) : \partial_{j_1} \cdots \partial_{j_m} f$ exists and is continuous for all $j_1, \dots, j_m \in \{1, \dots, n\}$, $0 \leq m \leq k \}$.

$\boxed{\begin{array}{l}\text{Prop. 8.10: If } f \in L^1,\ g \in C^k \text{ and } \partial^\alpha g \in BC \text{ for } |\alpha| \leq k, \\ \text{then } f * g \in C^k \text{ and } \partial^\alpha (f * g) = f * (\partial^\alpha g) \text{ for } |\alpha| \leq k.\end{array}}$

Approximate identity & smoothing

DEF: For $\phi : \mathbb{R}^n \to \mathbb{C}$ and $t > 0$, we set $\phi_t(x) = t^{-n}\phi(t^{-1}x)$

Ex



Note: If $\phi \in L^1$ then $\phi_t \in L^1$ and $\int_{\mathbb{R}^n} \phi_t = \int_{\mathbb{R}^n} \phi$, $\forall t > 0$.

Thm 8.14: Assume $\phi \in L^1(\mathbb{R}^n)$. Set $a = \int_{\mathbb{R}^n} \phi \, dx$.
Then for any $f \in L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$, we have $f * \phi_t \xrightarrow{(t\to 0)} af$ in $L^p(\mathbb{R}^n)$.

proof: $(f * \phi_t - af)(x) = \int_{\mathbb{R}^n} \big(f(x-y) - f(x)\big)\,\phi_t(y)\,dy = \int_{\mathbb{R}^n} (\tau_y f - f)(x) \cdot \phi_t(y)\,dy$

Hence by Minkowski's ineq. for integrals:

$$\|f * \Phi_t - af\|_p \leq \int_{\mathbb{R}^n} \|\tau_y f - f\|_p \cdot |\Phi_t(y)|\, dy$$

Now conclude by using the Dominated Convergence Theorem!

Useful fact: $\exists \,\phi \in C_c^\infty(\mathbb{R}^n)$ s.t. $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi = 1$

Ex: $$\psi(x) = \begin{cases} \exp\left((|x|^2-1)^{-1}\right) & \text{if } |x|<1 \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

– then $\psi \geq 0$, $\operatorname{supp}(\psi) \subseteq \overline{B_1^n}$,

and $\psi \in C^\infty$ since $t \mapsto \begin{cases} e^{-1/t} & t>0 \\ 0 & t \leq 0 \end{cases}$ is smooth!

(Exc 3, Ch. 8.1)

Now take $\phi = \left(\int_{\mathbb{R}^n} \psi\right)^{-1} \cdot \psi$.

Consequences:

$$\boxed{\text{Prop 8.17: } C_c^\infty \text{ is dense in } L^p \ (1 \leq p < \infty) \text{ and in } C_0.}$$

The C^∞ Urysohn Lemma (8.18):

If K compact $\subset U$ open $\subset \mathbb{R}^n$, then $\exists f \in C_c^\infty(\mathbb{R})$ s.t.

$\chi_K \leq f \leq \chi_U$ and $\operatorname{supp}(f) \subset U$.

proof: Let $\delta = \frac{1}{3}\operatorname{dist}(K, U^c)$ (> 0).

$V = K + B_\delta$

ϕ – as above.

Take $t > 0$ so small that $\operatorname{supp}(\phi_t) \subset B_\delta$.

Set $f = \chi_V * \phi_t$.