THE BOLTZMANN-GRAD LIMIT OF THE LORENTZ GAS IN A UNION OF LATTICES

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Abstract. The Lorentz gas describes an ensemble of noninteracting point particles in an infinite array of spherical scatterers. In a recent paper by Marklof and Strömbergsson [20], a framework was developed for proving, for a given deterministic scatterer configuration $\mathcal{P}$, the convergence of the particle dynamics to a limiting transport process, in the limit of low scatterer density. This framework was proved to apply for a large class of deterministic scatterer configurations, including various types of quasicrystals, and also in the case when $\mathcal{P}$ is a fixed realisation of a Poisson point process. In the present paper we prove that the framework from [20] also applies in the case when the scatterer configuration $\mathcal{P}$ is an arbitrary finite union of (possibly shifted) Euclidean lattices. In the special case when the lattices are pairwise incommensurable, our result on the limiting transport process settles a conjecture from an earlier paper by Marklof and Strömbergsson [19].

Contents

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1. Introduction

The Lorentz gas [13] describes the dynamics of a cloud of non-interacting point particles in an array of fixed spherical scatterers of radius $\rho > 0$, centered at the elements of a given locally finite point set $\mathcal{P} \subset \mathbb{R}^d$. Each particle travels with constant velocity along straight lines, and each time it hits a scatterer it is deflected by elastic reflection or by the force of a spherically symmetric potential.[1] We denote the position and velocity of a point particle at time t by $\boldsymbol{q}(t)$ and $\boldsymbol{v}(t)$. Since the particle speed outside the scatterers is a constant of motion we may without loss of generality consider only point particles having unit speed, viz., $\|\boldsymbol{v}(t)\| = 1$. This means that the particle dynamics takes place in the unit tangent bundle $\mathrm{T}^1(\mathcal{K}_\rho) := \mathcal{K}_\rho \times \mathrm{S}_1^{d-1}$ of the domain

$$\mathcal{K}_\rho := \mathbb{R}^d \setminus (\mathcal{P} + \mathcal{B}_\rho^d),$$

where $\mathcal{B}_\rho^d$ denotes the open ball of radius ρ, centered at the origin. The Liouville measure on $\mathrm{T}^1(\mathcal{K}_\rho)$ is $\mathrm{vol} \times \sigma$, where vol denotes the Lebesgue measure on $\mathbb{R}^d$ and $\sigma := \mathrm{vol}_{\mathrm{S}_1^{d-1}}$ is the Lebesgue measure on S_1^{d-1}.

Since the gas particles are assumed to be non-interacting, to study the evolution of a particle cloud, we may just as well consider the orbit $t \mapsto (\boldsymbol{q}(t), \boldsymbol{v}(t))$ of a *single* point particle starting from a *random* point $(\boldsymbol{q}_0, \boldsymbol{v}_0)$, chosen according to a given probability measure on the phase space $\mathrm{T}^1(\mathcal{K}_\rho)$. Then $t \mapsto (\boldsymbol{q}(t), \boldsymbol{v}(t))$ becomes random flight process, which we call the *Lorentz process.* A central challenge is to determine whether, in the limit of small scatterer density, that is as $\rho \to 0$, the Lorentz process converges to a limiting stochastic process. In order to give a precise formulation of this question, we assume from now on that $\mathcal{P}$ has an asymptotic density, meaning that there exists a constant $c_{\mathcal{P}} > 0$ such that for any bounded set $\mathcal{D} \subset \mathbb{R}^d$ with boundary of Lebesgue measure zero, we have

$$\lim_{R\to\infty} \frac{\#(\mathcal{P} \cap R\mathcal{D})}{R^d} = c_{\mathcal{P}} \, \mathrm{vol}(\mathcal{D}). \tag{1.1}$$

Then a simple heuristic argument shows that the mean free path length, i.e. the mean time between consecutive collisions, should be expected to scale as ρ^{1-d} as $\rho \to 0$. It is therefore natural to consider the so called *Boltzmann-Grad* scaling, in which length and time units are rescaled by a factor of ρ^{1-d}. That is, we consider the macroscopic coordinates

$$\big(\boldsymbol{Q}(t), \boldsymbol{V}(t)\big) = \big(\rho^{d-1}\boldsymbol{q}(\rho^{-(d-1)}t), \boldsymbol{v}(\rho^{-(d-1)}t)\big).$$

The challenge now is to prove that the rescaled random flight process $t \mapsto (\boldsymbol{Q}(t), \boldsymbol{V}(t))$ converges to a limiting random flight process as $\rho \to 0$.

There are two instances where this problem has been fully understood for some years. The first is the case when $\mathcal{P}$ is a fixed realisation of a Poisson point process. Here Boldrighini, Bunimovich and Sinai [2] proved that the Lorentz process converges to a limit that is consistent with the linear Boltzmann equation (cf. also [8] and [27]). In fact the paper [2] is restricted to dimension $d = 2$ and hard sphere scatterers, but it was proved in [20] that the results generalise to general dimensions and soft scattering potentials.

The second instance is when the scatterer configuration $\mathcal{P}$ equals a Euclidean lattice $\mathcal{L}$ of full rank in $\mathbb{R}^d$. For this case, Marklof and Strömbergsson [15, 16, 14, 17] proved convergence of the Lorentz process to a limiting random flight process which in fact only depends on the asymptotic density of $\mathcal{L}$. The limit process is Markovian only on an extended phase space

[1] See also Remark 1.1 below concerning the case of overlapping scatterers.

which, in addition to position and momentum, also includes the impact parameter and distance to the next collision. The corresponding transport equation is in particular not consistent with the linear Boltzmann equation. This new transport equation was obtained independently in dimension $d = 2$ for $\mathcal{P} = \mathbb{Z}^2$ by Caglioti and Golse [3, 4], subject to a heuristic assumption that was proved (in any dimension) in [16]. In the lattice setting, the limit transport process in fact satisfies a superdiffusive central limit theorem [21], with a mean-square displacement proportional to $t \log t$ (where t is time measured in units of the mean collision time), rather than the standard linear scaling which appears in the case of random scatterer configurations.

In a recent paper by Marklof and Strömbergsson [20], a general framework was developed which, under a certain set of hypotheses on the scatterer configuration $\mathcal{P}$, allows the proof of convergence of the rescaled Lorentz process $t \mapsto (\boldsymbol{Q}(t), \boldsymbol{V}(t))$ to a limiting random flight process. This framework was proved to apply when $\mathcal{P}$ belongs to a certain class of quasicrystals (this includes in particular the case when $\mathcal{P}$ is a general *periodic* point set), and also in the case when $\mathcal{P}$ is a fixed realisation of a Poisson point process of constant intensity.

Our main goal in the present paper is to prove that the framework from [20] also applies in the case when the scatterer configuration $\mathcal{P}$ is an arbitrary finite union of *grids.* (By definition, a 'grid' is a translate of a full rank lattice in $\mathbb{R}^d$.) That is, we will assume that

$$\mathcal{P} = \bigcup_{i=1}^{N} \mathcal{L}_i, \tag{1.2}$$

where each $\mathcal{L}_i$ is a grid. In the special case when the $\mathcal{L}_i$ are pairwise incommensurable (see p. 8 below for the definition), such scatterer configurations $\mathcal{P}$ have previously been considered in the paper [19], where among other things a conjectural description of the Boltzmann-Grad limit of the Lorentz process was given. The main result of the present paper settles that conjecture as a special case; see Remark 1.3 below. However, as we will see, the general case of $\mathcal{P}$ as in (1.2), without the pairwise incommensurability assumption, is considerably more difficult to handle and involves interesting new phenomena.

Remark 1.1. In the case when some scatterers in the family $\{\boldsymbol{p} + \mathcal{B}_\rho^d \, : \, \boldsymbol{p} \in \mathcal{P}\}$ *overlap,* there are some technical annoyances appearing in the definition of the Lorentz flow $t \mapsto (\boldsymbol{q}(t), \boldsymbol{v}(t))$. Let us note that overlapping scatterers in general *do* exist in the situation studied in the present paper. Indeed, if $\mathcal{P}$ is any finite union of grids which is not periodic, then for every $\rho > 0$ there exist points $\boldsymbol{p} \neq \boldsymbol{p}'$ in $\mathcal{P}$ with $\|\boldsymbol{p} - \boldsymbol{p}'\| < 2\rho$.

In the classical case when the particles interact with the scatterers through specular reflection, this matter is handled in a standard manner [20, Ch. 1.2]: In this case the Lorentz flow equals the standard billiard flow in the region $\mathcal{K}_\rho$, and this flow is technically defined only on a subset of $\mathrm{T}^1(\mathcal{K}_\rho)$ of full measure with respect to the Liouville measure vol $\times \sigma$; the exceptional points include all points $(\boldsymbol{q}, \boldsymbol{v}) \in \partial\mathcal{K}_\rho \times \mathrm{S}_1^{d-1}$ for which $\boldsymbol{v}$ points into a scatterer, and also, in the case of scatterer overlaps, any initial condition for which the particle at some time point either in the past or in the future collides with an intersection point of two or more scatterer boundaries.

In the case of scatterer interaction through a spherically symmetric potential, a simple way to avoid intricacies in the definition of the Lorentz flow is to *remove,* in an ad hoc manner, scatterer centers in $\mathcal{P}$ causing overlap [20, Ch. 1.3]. Since the probability of the particle hitting a scatterer which is overlapped by another scatterer tends to zero in the Boltzmann-Grad limit, the limiting random flight process becomes the same independently of the precise choice of convention.

1.1. The hypotheses on the scatterer configuration $\mathcal{P}$; statement of main result.

We will now recall the precise formulation of the hypotheses on $\mathcal{P}$ imposed in [20]. We first need to introduce some technical notation.

For any topological space S, we write $P(S)$ for the set of Borel probability measures on S, equipped with the weak topology. We will only consider $P(S)$ when S is separable and metrizable; recall that then also $P(S)$ is metrizable [1, pp. 72-73]. Also, given any locally compact second countable Hausdorff space $\mathcal{X}$, we let $N(\mathcal{X})$ be the family of locally finite counting measures on $\mathcal{X}$, equipped with the vague topology (then $N(\mathcal{X})$ is a Polish space), and let $N_s(\mathcal{X})$ be the subset of *simple* counting measures; this is a Borel subset of $N(\mathcal{X})$. The elements of $N_s(\mathcal{X})$ may be identified with the family of locally finite subsets of $\mathcal{X}$ through $\nu \mapsto \mathrm{supp}(\nu)$. The inverse map is $\{x_i\} \mapsto \sum_i \delta_{x_i}$. We will use this identification between point sets and simple counting measures throughout this work, often using the same notation for point set and counting measure. A *point process* in $\mathcal{X}$ is, by definition, a random element ξ in $N(\mathcal{X})$. It is called *simple* if $\xi \in N_s(\mathcal{X})$ almost surely. We identify $P(N_s(\mathcal{X}))$ with the set of probability measures $\nu \in P(N(\mathcal{X}))$ with $\nu(N_s(\mathcal{X})) = 1$. Then a point process ξ is simple if and only if its law is in $P(N_s(\mathcal{X}))$.

In the present paper we will always have

$$\mathcal{X} := \mathbb{R}^d \times \Sigma$$

for some compact metric space Σ. We will view vectors in $\mathbb{R}^d$ as *row* vectors; thus $\mathrm{GL}_d(\mathbb{R})$ acts on $\mathbb{R}^d$ from the right. We extend the natural actions on $\mathbb{R}^d$ of $\mathbb{R}^d$ (by translation), of $\mathbb{R}^\times$ (by dilation) and of $\mathrm{GL}_d(\mathbb{R})$ (by multiplication from the right) by the trivial action on Σ; thus for any $(\boldsymbol{w}, \varsigma) \in \mathcal{X}$ and any $\boldsymbol{x} \in \mathbb{R}^d$, $T \in \mathbb{R}^\times$ and $A \in \mathrm{GL}_d(\mathbb{R})$ we set $(\boldsymbol{w}, \varsigma) + \boldsymbol{x} := (\boldsymbol{w} + \boldsymbol{x}, \varsigma)$, $T(\boldsymbol{w}, \varsigma) := (T\boldsymbol{w}, \varsigma)$ and $(\boldsymbol{w}, \varsigma)A := (\boldsymbol{w}A, \varsigma)$. These actions also give rise to natural, continuous, actions on $N_s(\mathcal{X})$. For example, for any $A \in \mathrm{GL}_d(\mathbb{R})$ and $\mathcal{Q} \in N_s(\mathcal{X})$ (viewed as a locally finite subset of $\mathcal{X}$), $\mathcal{Q}A := \{xA \,:\, x \in \mathcal{Q}\}$.

For $\rho > 0$ we denote by D_ρ the diagonal matrix

$$D_\rho = \mathrm{diag}(\rho^{d-1}, \rho^{-1}, \cdots, \rho^{-1}) \in \mathrm{SL}_d(\mathbb{R}).$$

We also fix, once and for all, a map $R : \mathrm{S}_1^{d-1} \to \mathrm{SO}(d)$ with the property $\boldsymbol{v}R(\boldsymbol{v}) = \boldsymbol{e}_1$ for all $\boldsymbol{v} \in \mathrm{S}_1^{d-1}$, where $\boldsymbol{e}_1 := (1, 0, \ldots, 0)$. We assume that R is continuous when restricted to S_1^{d-1} minus one point; the choice of R is otherwise arbitrary.

Now let $\mathcal{P}$ be an arbitrary fixed locally finite subset of $\mathbb{R}^d$ with constant asymptotic density $c_{\mathcal{P}}$. We also assume given a compact metric space Σ, a map $\varsigma : \mathcal{P} \to \Sigma$, a Borel probability measure m on Σ, and a continuous map $\varsigma \mapsto \mu_\varsigma$ from Σ to $P(N(\mathcal{X})) = P(N(\mathbb{R}^d \times \Sigma))$. Set

$$\widetilde{\mathcal{P}} = \{(\boldsymbol{p}, \varsigma(\boldsymbol{p})) \,:\, \boldsymbol{p} \in \mathcal{P}\} \subset \mathcal{X}.$$

We refer to ς as a *marking* of $\mathcal{P}$, and Σ as the corresponding *space of marks*. For any $\boldsymbol{q} \in \mathbb{R}^d$, $\boldsymbol{v} \in \mathrm{S}_1^{d-1}$ and $\rho > 0$, we set

$$\widetilde{\mathcal{P}}_{\boldsymbol{q}} = \begin{cases} \widetilde{\mathcal{P}} \setminus \{(\boldsymbol{q}, \varsigma(\boldsymbol{q}))\} & (\boldsymbol{q} \in \mathcal{P}) \\ \widetilde{\mathcal{P}} & (\boldsymbol{q} \notin \mathcal{P}) \end{cases} \tag{1.3}$$

and

$$\mathcal{Q}_\rho(\boldsymbol{q}, \boldsymbol{v}) = (\widetilde{\mathcal{P}}_{\boldsymbol{q}} - \boldsymbol{q})\, R(\boldsymbol{v})\, D_\rho. \tag{1.4}$$

Given any $\boldsymbol{q} \in \mathbb{R}^d$ and $\lambda \in P(\mathrm{S}_1^{d-1})$, if we take $\boldsymbol{v}$ random in $(\mathrm{S}_1^{d-1}, \lambda)$ then $\mathcal{Q}_\rho(\boldsymbol{q}, \boldsymbol{v})$ becomes a point process; we write $\mu_{\boldsymbol{q},\rho}^{(\lambda)} \in P(N_s(\mathcal{X}))$ for its distribution. Finally set $\mu_{\mathcal{X}} = \mathrm{vol} \times \mathsf{m}$.

The assumption on the scatterer configuration $\mathcal{P}$, under which the convergence of the rescaled Lorentz process to a limiting flight process was proved in [20], is that $\mathcal{P}$ can be equipped with data $\varsigma, \Sigma, \mathsf{m}$ and $\varsigma \mapsto \mu_\varsigma$ as above, in such a way that the following six conditions [P1]–[P3] and [Q1]–[Q3] are fulfilled (see [20, Sec. 2.3]):

[P1] *Uniform density:* For any bounded $B \subset \mathcal{X}$ with $\mu_{\mathcal{X}}(\partial B) = 0$, we have

$$\lim_{T \to \infty} \frac{\#(\widetilde{\mathcal{P}} \cap TB)}{T^d} = c_{\mathcal{P}} \mu_{\mathcal{X}}(B). \tag{1.5}$$

[P2] *Spherical equidistribution:* Let $P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$ be the set of $\lambda \in P(\mathrm{S}_1^{d-1})$ which are absolutely continuous with respect to σ. There exists a subset $\mathcal{E} \subset \mathcal{P}$ of density zero[2] such that for any fixed $T \geq 1$ and $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$, we have

$$\mu_{\boldsymbol{q},\rho}^{(\lambda)} \xrightarrow{\mathrm{w}} \mu_{\varsigma(\boldsymbol{q})} \quad \text{as } \rho \to 0, \text{ uniformly for } \boldsymbol{q} \in \mathcal{P}_T(\rho) := \mathcal{P} \cap \mathcal{B}^d_{T\rho^{1-d}} \setminus \mathcal{E}. \tag{1.6}$$ [3]

[P3] *No escape of mass:* For every bounded Borel set $B \subset \mathbb{R}^d$,

$$\lim_{\xi\to\infty} \limsup_{\rho\to 0} \; [\mathrm{vol}\times\sigma]\big(\big\{(\boldsymbol{q},\boldsymbol{v}) \in B \times \mathrm{S}_1^{d-1} \;:\; \mathcal{Q}_\rho(\rho^{1-d}\boldsymbol{q},\boldsymbol{v}) \cap (\mathfrak{Z}_\xi \times \Sigma) = \emptyset\big\}\big) = 0,$$

with the open cylinder $\mathfrak{Z}_\xi = (0,\xi) \times \mathcal{B}_1^{d-1} \subset \mathbb{R}^d$.

[Q1] $\mathrm{SO}(d-1)$*-invariance:* For every $\varsigma \in \Sigma$,

$$\mu_\varsigma \text{ is invariant under the action of } \mathrm{SO}(d-1) := \{k \in \mathrm{SO}(d) \;:\; \boldsymbol{e}_1 k = \boldsymbol{e}_1\}.$$

[Q2] *Coincidence-free first coordinates:* For every $\varsigma \in \Sigma$,

$$\mu_\varsigma(\{\nu \in N(\mathcal{X}) \;:\; \exists x_1 \in \mathbb{R} \text{ s.t. } \nu(\{x_1\} \times \mathbb{R}^{d-1} \times \Sigma) > 1\}) = 0.$$

[Q3] *Small probability of large voids:* For every $\varepsilon > 0$ there exists $R > 0$ such that for all $\varsigma \in \Sigma$ and $\boldsymbol{x} \in \mathbb{R}^d$ we have

$$\mu_\varsigma\big(\big\{\nu \in N(\mathcal{X}) \;:\; \nu\big(\mathcal{B}_R^d(\boldsymbol{x}) \times \Sigma\big) = 0\big\}\big) < \varepsilon. \tag{1.7}$$

Here $\mathcal{B}_R^d(\boldsymbol{x}) := \boldsymbol{x} + \mathcal{B}_R^d$, the open ball of radius R centered at $\boldsymbol{x}$.

We can now state the main result of the present paper:

Theorem 1.1. *Let $\mathcal{P}$ be a union of grids in $\mathbb{R}^d$. Then there exists a compact metric space Σ, a marking $\varsigma : \mathcal{P} \to \Sigma$, a continuous map $\varsigma \mapsto \mu_\varsigma$ from Σ to $P(N(\mathcal{X}))$, and a Borel probability measure m on Σ, such that all the assumptions [P1]–[P3] and [Q1]–[Q3] hold.*

The explicit description of the data $\big[\Sigma, \varsigma, \varsigma \mapsto \mu_\varsigma, \mathsf{m}\big]$ which we use in the proof of Theorem 1.1 is given in Section 5. In fact it turns out that while Σ is in general unavoidably an infinite set, m is actually supported on a *finite* subset $\Sigma' \subset \Sigma$.

Remark 1.2. In the special case when $\mathcal{P}$ is *periodic*, i.e. when $\mathcal{P}$ may be represented as in (1.2) with all the $\mathcal{L}_i$s being translates of one and the same fixed lattice, the result of Theorem 1.1 was proved in [20, Ch. 5.2-3].

1.2. **The limiting flight process.** We now discuss some of the consequences of Theorem 1.1 when combined with the main results of [20]. Let us write $F_{t,\rho}$ for the rescaled Lorentz flow;

$$\big(\boldsymbol{Q}(t), \boldsymbol{V}(t)\big) = F_{t,\rho}\big(\boldsymbol{Q}(0), \boldsymbol{V}(0)\big).$$

For notational reasons we extend the dynamics to the inside of each scatterer trivially, that is, set $F_{t,\rho} = \mathrm{id}$ whenever $\boldsymbol{Q}$ is inside the scatterer. Thus $F_{t,\rho}$ is now a flow defined on all of $\mathrm{T}^1(\mathbb{R}^d)$, the unit tangent bundle of $\mathbb{R}^d$.

It now follows from Theorem 1.1 together with the central results of [20] that if $\mathcal{P}$ is any fixed union of grids, and if Λ is a fixed Borel probability measure on $\mathrm{T}^1(\mathbb{R}^d)$ which is absolutely continuous with respect to Liouville measure, then the random flight process $t \mapsto F_{t,\rho}(\boldsymbol{Q}_0, \boldsymbol{V}_0)$ obtained by taking the initial data $(\boldsymbol{Q}_0, \boldsymbol{V}_0)$ random with respect to Λ, converges, as $\rho \to 0$, to a random flight process $\{\Xi(t) \;:\; t > 0\}$ [20, Theorems 1.1 and 1.3]. This random flight process is defined as the flow with unit speed along a random piecewise linear curve, whose

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[2] We say that a subset $\mathcal{E}$ of $\mathbb{R}^d$ has *density zero* if it has asymptotic density zero in the sense of (1.1); note that this holds if and only if it holds for $\mathcal{D} = \mathcal{B}_1^d$, viz., if and only if $\lim_{R\to\infty} R^{-d}\#(\mathcal{E} \cap \mathcal{B}_R^d) = 0$.

[3] The statement in (1.6) means by definition: $\big[\forall\varepsilon > 0\text{: } \exists\rho_0 > 0\text{: } \forall\rho \in (0,\rho_0)\text{: } \forall\boldsymbol{q} \in \mathcal{P}_T(\rho)\text{: } d(\mu_{\boldsymbol{q},\rho}^{(\lambda)}, \mu_{\varsigma(\boldsymbol{q})}) < \varepsilon\big]$, where d is some metric on $P(N(\mathcal{X}))$ realizing the weak topology. This definition is independent of the choice of d; indeed see [20, Lemma 2.1] and note that $\{\mu_\varsigma \;:\; \varsigma \in \Sigma\}$ is a compact subset of $P(N(\mathcal{X}))$, since it is the continuous image of the compact set Σ.

path segments, when considered in combination with the *marks* of the scatterers involved in the collisions, are generated by a Markov process with memory two. Specifically, if we let the random trajectory $t \mapsto F_{t,\rho}(\boldsymbol{Q}_0, \boldsymbol{V}_0)$ be described by the random variables $\xi_j^{(\rho)} \in \mathbb{R}_{>0}$, $\varsigma_j^{(\rho)} \in \Sigma$ and $\boldsymbol{V}_j^{(\rho)} \in \mathrm{S}_1^{d-1}$, where $\xi_j^{(\rho)}$ is the length of the jth path segment, $\varsigma_j^{(\rho)}$ is the mark of the scatterer involved in the jth collision and $\boldsymbol{V}_j^{(\rho)} \in \mathrm{S}_1^{d-1}$ is the velocity after the jth collision, then as $\rho \to 0$, the random process

$$\big(\langle \xi_j^{(\rho)}, \varsigma_j^{(\rho)}, \boldsymbol{V}_j^{(\rho)} \rangle\big)_{j=1,2,\ldots}$$

converges in distribution to the second-order Markov process

$$\big((\xi_j, \varsigma_j, \boldsymbol{V}_j)\big)_{j=1,2,\ldots}, \tag{1.8}$$

where for any Borel set $A \subset \mathbb{R}_{\geq 0} \times \Sigma \times \mathrm{S}_1^{d-1}$,

$$\mathbb{P}\Big((\xi_1, \varsigma_1, \boldsymbol{V}_1) \in A \,\Big|\, (\boldsymbol{Q}_0, \boldsymbol{V}_0)\Big) = \int_A p(\boldsymbol{V}_0; \xi, \varsigma, \boldsymbol{V})\, d\xi\, d\mathsf{m}(\varsigma)\, d\boldsymbol{V}, \tag{1.9}$$

and for $j \geq 2$,

$$\mathbb{P}\Big((\xi_j, \varsigma_j, \boldsymbol{V}_j) \in A \,\Big|\, (\boldsymbol{Q}_0, \boldsymbol{V}_0),\ \langle (\xi_i, \varsigma_i, \boldsymbol{V}_i)\rangle_{i=1}^{j-1}\Big) = \int_A p_{\mathbf{0}}(\boldsymbol{V}_{j-2}, \varsigma_{j-1}, \boldsymbol{V}_{j-1}; \xi, \varsigma, \boldsymbol{V})\, d\xi\, d\mathsf{m}(\varsigma)\, d\boldsymbol{V} \tag{1.10}$$

[20, Theorems 1.2 and 4.6]. Here the *collision kernels* $p, p_{\mathbf{0}}$ are functions which we define in the next paragraph; they depend on $\mathcal{P}$ but are independent of Λ, and for any fixed $\boldsymbol{V}_0, \varsigma, \boldsymbol{V}$ both $p(\boldsymbol{V}_0\,;\,\cdot\,)$ and $p_{\mathbf{0}}(\boldsymbol{V}_0, \varsigma, \boldsymbol{V}\,;\,\cdot\,)$ are probability densities on $\mathbb{R}_{>0} \times \Sigma \times \mathrm{S}_1^{d-1}$. Note that the fact that the finite subset Σ' satisfies $\mathsf{m}(\Sigma') = 1$ (as mentioned below Theorem 1.1) implies that the limiting Markov process in (1.8) can in fact be taken to have state space $\mathbb{R}_{>0} \times \Sigma' \times \mathrm{S}_1^{d-1}$.

The collision kernels $p, p_{\mathbf{0}}$ are defined as follows:

$$p\big(\boldsymbol{V}; \xi, \varsigma_+, \boldsymbol{V}_+\big) = \frac{\sigma(\boldsymbol{V}, \boldsymbol{V}_+)}{v_{d-1}}\, k^{\mathrm{g}}\big(\xi, (\boldsymbol{w}, \varsigma_+)\big), \tag{1.11}$$

$$p_{\mathbf{0}}\big(\boldsymbol{V}_0, \varsigma, \boldsymbol{V}; \xi, \varsigma_+, \boldsymbol{V}_+\big) = \frac{\sigma(\boldsymbol{V}, \boldsymbol{V}_+)}{v_{d-1}} k\big((\boldsymbol{w}', \varsigma), \xi, (\boldsymbol{w}, \varsigma_+)\big), \tag{1.12}$$

where $\sigma(\boldsymbol{V}, \boldsymbol{V}_+)$ is the differential cross section of the scattering map through which the particles interacts with the scatterers (cf. [20, Sec. 3.4]; in particular in the case of elastic reflection, $\sigma(\boldsymbol{V}, \boldsymbol{V}_+) = \frac{1}{4}\|\boldsymbol{V} - \boldsymbol{V}_+\|^{3-d}$), and where k^{g} and k are *transition* kernels, defined below, that quantify the probability of hitting the next scatterer at distance ξ with impact parameter $\boldsymbol{w} \in \mathcal{B}_1^{d-1}$, which is a function of $\boldsymbol{V}$ and $\boldsymbol{V}_+$. The kernel k^{g} corresponds to the case of generic initial data, and $k((\boldsymbol{w}', \varsigma), \,\cdot\,)$ to the case of an initial condition relative to previous scattering event with marking ς and exit parameter $\boldsymbol{w}' \in \mathcal{B}_1^{d-1}$. The exit parameter $\boldsymbol{w}'$ can be viewed as the time-reversed impact parameter and thus is a function of $\boldsymbol{V}_0$ and $\boldsymbol{V}$.

Precise formulas for the transitions kernels k^{g} and k are as follows [20, Props. 3.20 and 3.21][4]: For any $\xi > 0$ and $(\boldsymbol{w}', \varsigma)$ and $(\boldsymbol{w}, \varsigma_+)$ in $\mathcal{B}_1^{d-1} \times \Sigma'$,

$$k^{\mathrm{g}}(\xi, (\boldsymbol{w}, \varsigma_+)) = v_{d-1} c_{\mathcal{P}}\, \mu_{\varsigma_+}\big(\{Y \in N_s(\mathcal{X}) \,:\, Y \cap (\mathfrak{Z}_\xi + (-\xi, \boldsymbol{w})) = \emptyset\}\big), \tag{1.13}$$

and

$$k((\boldsymbol{w}', \varsigma), \xi, (\boldsymbol{w}, \varsigma_+)) = v_{d-1} c_{\mathcal{P}}\, \nu_\varsigma\big((\xi, \boldsymbol{w}' - \boldsymbol{w}, \varsigma_+);\ \{Y \in N_s(\mathcal{X}) \,:\, Y \cap (\mathfrak{Z}_\xi + (0, \boldsymbol{w}')) = \emptyset\}\big), \tag{1.14}$$

where v_{d-1} is the volume of the unit ball in $\mathbb{R}^{d-1}$, $\mathfrak{Z}_\xi$ is the open cylinder $(0, \xi) \times \mathcal{B}_1^{d-1}$ in $\mathbb{R}^d$, and ν_ς is a version of the Palm distributions of a point process with distribution μ_ς.[5]

[4] The fact that [20, Prop. 3.20] indeed applies follows from the last part of Lemma 5.8 below.

[5] That is, ν_ς is a function $\mathcal{X} \times \mathcal{N} \to [0, 1]$, where $\mathcal{N}$ is the Borel σ-algebra of $N_s(\mathcal{X})$, such that $\nu_\varsigma(\boldsymbol{x}; A)$ is Borel measurable in $\boldsymbol{x} \in \mathcal{X}$ for each $A \in \mathcal{N}$, is a probability measure in $A \in \mathcal{N}$ for each $\boldsymbol{x} \in \mathcal{X}$, and for any Borel sets $B \subset \mathcal{X}$ and $A \in \mathcal{N}$ one has $\int_A \#(Y \cap B)\, d\mu_\varsigma(Y) = c_{\mathcal{P}} \int_B \nu_\varsigma(\boldsymbol{y}; A)\, d\mu_{\mathcal{X}}(\boldsymbol{y})$. Cf. [11, Ch. 10].

1.3. **The generalized linear Boltzmann equation.** The limiting random flight process $\Xi(t)$ discussed in Section 1.2 is closely related to the dynamics of a particle cloud in the Boltzmann-Grad limit of the Lorentz gas. Indeed, the time evolution of an initial particle density $f_0 \in \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d))$ in the Lorentz gas with fixed scatterer radius ρ is given by $f_t = L_\rho^t f_0$ where L_ρ^t is the Liouville operator defined by

$$[L_\rho^t f_0](\boldsymbol{Q},\boldsymbol{V}) := f_0(F_{-t,\rho}(\boldsymbol{Q},\boldsymbol{V})).$$

The existence of the limiting stochastic process $\Xi(t)$ implies that for every $t>0$ there exists a linear operator $L_t : \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d)) \to \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d))$ such that for every $f_0 \in \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d))$ and every set $A \subset \mathrm{T}^1(\mathbb{R}^d)$ with boundary of Liouville measure zero,

$$\lim_{\rho\to 0}\int_A [L_t^{(\rho)} f_0](\boldsymbol{Q},\boldsymbol{V})\, d\boldsymbol{Q}\, d\sigma(\boldsymbol{V}) = \int_A L_t f_0(\boldsymbol{Q},\boldsymbol{V})\, d\boldsymbol{Q}\, d\sigma(\boldsymbol{V})$$

[20, Cor. 1.4]. Under suitable continuity assumptions, we can in fact express $L_t f_0$ as

$$[L_t f_0](\boldsymbol{Q},\boldsymbol{V}) = \int_{\mathbb{R}_{>0}\times\Sigma'\times\mathrm{S}_1^{d-1}} f(t,\boldsymbol{Q},\boldsymbol{V},\xi,\varsigma,\boldsymbol{V}_+)\, d\xi\, d\mathsf{m}(\varsigma)\, d\sigma(\boldsymbol{V}_+)$$

where f is the unique solution of the differential equation

$$\begin{aligned}&\big(\partial_t + \boldsymbol{V}\cdot\nabla_{\boldsymbol{Q}} - \partial_\xi\big) f(t,\boldsymbol{Q},\boldsymbol{V},\xi,\varsigma,\boldsymbol{V}_+)\\ &\quad = \int_{\Sigma'\times\mathrm{S}_1^{d-1}} f\big(t,\boldsymbol{Q},\boldsymbol{V}_0,0,\varsigma',\boldsymbol{V}\big) p_{\mathbf{0}}(\boldsymbol{V}_0,\varsigma',\boldsymbol{V};\xi,\varsigma,\boldsymbol{V}_+)\, d\mathsf{m}(\varsigma')\, d\sigma(\boldsymbol{V}_0),\end{aligned} \tag{1.15}$$

subject to the initial condition $f(0,\boldsymbol{Q},\boldsymbol{V},\xi,\varsigma_+,\boldsymbol{V}_+) = f_0(\boldsymbol{Q},\boldsymbol{V})\, p(\boldsymbol{V};\xi,\varsigma_+,\boldsymbol{V}_+)$ [20, Sections 1.4 and 4.6].

Equation (1.15) may be viewed as a generalization of the linear Boltzmann equation; it is the forward Kolmogorov equation (or Fokker-Planck-Kolmogorov equation) of a natural *extension* of Ξ to a Markov flight process on the space $\mathrm{T}^1(\mathbb{R}^d)\times\mathbb{R}_{>0}\times\Sigma\times\mathrm{S}_1^{d-1}$ (see [20, (1.24)]).

Remark 1.3. We verify in Section 5.3 below that in the special case when the grids $\mathcal{L}_i$ in (1.2) are pairwise incommensurable, the limiting random flight process described in Section 1.2, as well as the generalized linear Boltzmann equation in (1.15), agree with the corresponding limits conjectured in [19].

1.4. **Outline of the paper.** In Section 2 we introduce some basic set-up and notation: Given an arbitrary union of grids $\mathcal{P}$ as in (1.2), we split the grids into so called *commensurability* classes, and also introduce some conventions on how the grids are arranged within each commensurability class. The number of commensurability classes of grids appearing in our presentation of $\mathcal{P}$ is denoted by N, and we write r_j for the number of grids in the jth commensurability class; thus the grids in our presentation of $\mathcal{P}$ are naturally indexed by the following set:

$$\Psi = \{\psi = (j,i) \,:\, j\in\{1,\ldots,N\},\ i\in\{1,\ldots,r_j\}\}.$$

We also introduce a certain product of homogeneous spaces $\mathbb{X} = \mathbb{X}_1\times\cdots\times\mathbb{X}_N$ which naturally parametrizes the families of unions of grids which appear when studying the condition [P2]. Here each space $\mathbb{X}_j$ is a torus fiber bundle over the space $\mathrm{SL}_d(\mathbb{Z})\backslash\mathrm{SL}_d(\mathbb{R})$ of d-dimensional lattices of covolume one in $\mathbb{R}^d$; the fiber space is an $r_j d$-dimensional torus which we will denote by $\mathbb{T}_j^d$; it is equipped with a natural action by $\mathrm{SL}_d(\mathbb{Z})$ from the right.

In Section 3 we state an equidistribution result in the space $\mathbb{X}$, Theorem 3.2, which can be viewed as a precise description of the limit considered in the spherical equidistribution condition [P2]. In fact Theorem 3.2 applies for an arbitrary translation vector $\boldsymbol{q}$ in $\mathbb{R}^d$; however the theorem includes no statement about uniformity with respect to $\boldsymbol{q}$, as is required in [P2]. The corresponding limit probability measure on $\mathbb{X}$ is called $\mu^{(\boldsymbol{q})}$. In the remainder of Section 3 we give an alternative way to represent these measures $\mu^{(\boldsymbol{q})}$ via certain measures

$\omega_1^{(\boldsymbol{q})}, \ldots, \omega_N^{(\boldsymbol{q})}$, where for each j, $\omega_j^{(\boldsymbol{q})}$ is an $\mathrm{SL}_d(\mathbb{Z})$-invariant probability measures on $\mathbb{T}_j^d$. We will denote by $P(\mathbb{T}_j^d)'$ the set of all $\mathrm{SL}_d(\mathbb{Z})$-invariant probability measures on $\mathbb{T}_j^d$; this is a compact subset of $P(\mathbb{T}_j^d)$.

In Section 4 we study the behaviour of each measure $\omega_j^{(\boldsymbol{q})}$ as $\boldsymbol{q}$ tends to infinity within $\mathcal{P}$, and we will explicitly identify the limit measures obtained. In order for these limit results to have a uniqueness property which is required for our later treatment to work, we need to assume that our fixed presentation of $\mathcal{P}$ as a union of grids has a certain "admissibility" property; the fact that there exists such a presentation of $\mathcal{P}$ is non-trivial, and is proved in Section 4.3.

In Section 5, building on the results from the previous sections, we are finally able to give the precise definition of the space of marks Σ and the marking $\varsigma : \mathcal{P} \to \Sigma$, as well as the map $\varsigma \mapsto \mu_\varsigma$ and measure m which we will use in the proof of our main Theorem 1.1. In fact we will take Σ to be a certain compact subset of the product space $\Psi \times \Omega$ where $\Omega := \prod_{j=1}^N P(\mathbb{T}_j^d)'$. In the remainder of Section 5 we prove that the conditions [Q1]–[Q3] and [P1] and [P3] hold, and we reduce the remaining condition, [P2], to a certain equidistribution result taking place in the homogeneous space $\mathbb{X}$, Theorem 6.6. Thus, after Section 5, in order to complete the proof of our main result Theorem 1.1, it only remains to prove Theorem 6.6.

In Section 7 we prove a key result on equidistribution of certain expanding unipotent orbits in a slightly generalized version of the homogeneous space $\mathbb{X} = \Gamma\backslash G$. This result is then used in the final Section 8 in order to complete the proof of Theorem 6.6, and thus also of Theorem 1.1.

2. Set-up and notation

2.1. Representing the point set $\mathcal{P}$.

As mentioned in the introduction, we will always view vectors in $\mathbb{R}^d$ as *row* vectors.[6]

Recall that we are assuming that the scatterer configuration $\mathcal{P}$ is of the form

$$\mathcal{P} = \bigcup_{i=1}^{M} \mathcal{L}_i \tag{2.1}$$

where each $\mathcal{L}_i$ is a grid (viz., a translate of a lattice) in $\mathbb{R}^d$.

Recall that two grids $\mathcal{L}$ and $\mathcal{L}'$ in $\mathbb{R}^d$ are said to be *commensurable* if there exist $c > 0$ and $\boldsymbol{v} \in \mathbb{R}^d$ such that $\mathcal{L} \cap (c\mathcal{L}' + \boldsymbol{v})$ is a grid. This is an equivalence relation on the family of grids in $\mathbb{R}^d$. By collecting the grids $\mathcal{L}_1, \ldots, \mathcal{L}_N$ in (2.1) into commensurability classes, and then applying Lemma 2.1 below to each class, one sees that the set $\mathcal{P}$ can be represented as follows:

$$\mathcal{P} = \bigcup_{j=1}^{N} \bigcup_{i=1}^{r_j} c_{j,i}(\mathcal{L}_j + \boldsymbol{v}_{j,i}), \tag{2.2}$$

where $\mathcal{L}_1, \ldots, \mathcal{L}_N$ are *pairwise incommensurable* lattices in $\mathbb{R}^d$ of *covolume one*, $r_1, \ldots, r_N$ are positive integers, $c_{j,i}$ are positive real numbers and $\boldsymbol{v}_{j,i}$ are vectors in $\mathbb{R}^d$, and furthermore:

(2.3)
$$\forall j \in \{1, \ldots, N\} : \ \forall i \neq i' \in \{1, \ldots, r_j\} : \ c_{j,i}/c_{j,i'} \in \mathbb{Q} \Rightarrow \big[c_{j,i} = c_{j,i'} \text{ and } \boldsymbol{v}_{j,i} - \boldsymbol{v}_{j,i'} \notin \mathcal{L}_j\big].$$

In order to arrive at the above representation, we made use of the following lemma.

Lemma 2.1. *Let $\mathcal{L}_1, \ldots, \mathcal{L}_m$ be commensurable grids in $\mathbb{R}^d$. Then there exists a lattice $\mathcal{L}$ in $\mathbb{R}^d$ of covolume one, a positive integer r, positive real numbers $c_1, \ldots, c_r$ and vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r \in$*

[6]But note that for other spaces $\mathbb{R}^r$ appearing below, it will be natural to instead consider the vectors to be *column* vectors. The first instance of this is on p. 12, for the space $\mathbb{R}^{r_j}$ in the definition of the torus $\mathbb{T}_j = \mathbb{R}^{r_j}/\mathbb{Z}^{r_j}$.

$\mathbb{R}^d$ such that $\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_m = \bigcup_{i=1}^r c_i(\mathcal{L}+\boldsymbol{v}_i)$. In this representation, we may furthermore assume that for any $i \neq i'$ in $\{1,\ldots,r\}$, if $c_i/c_{i'} \in \mathbb{Q}$ then $c_i = c_{i'}$ and $\boldsymbol{v}_i - \boldsymbol{v}_{i'} \notin \mathcal{L}$.

Proof. Choose $\boldsymbol{w}_1 \in \mathbb{R}^d$ such that $\mathcal{L}' := \mathcal{L}_1 + \boldsymbol{w}_1$ is a lattice. Now for each $j \in \{2,\ldots,m\}$, the fact that $\mathcal{L}_1$ and $\mathcal{L}_j$ are commensurable implies that we can choose $\delta_j > 0$ and $\boldsymbol{w}_j \in \mathbb{R}^d$ such that $\mathcal{L}' \cap (\delta_j\mathcal{L}_j + \boldsymbol{w}_j)$ is a lattice. (In particular then $\mathbf{0} \in \delta_j\mathcal{L}_j + \boldsymbol{w}_j$, meaning that $\delta_j\mathcal{L}_j + \boldsymbol{w}_j$ is a lattice.) It follows that also $\cap_{j=2}^m(\delta_j\mathcal{L}_j + \boldsymbol{w}_j)$ intersects $\mathcal{L}'$ in a lattice, and we can express this intersection as $\delta\mathcal{L}$, where $\mathcal{L}$ is a lattice of covolume one and δ is a positive real number. The fact that $\delta\mathcal{L}$ is a finite index subgroup of $\mathcal{L}'$ implies that $\mathcal{L}'$ is a union of a finite number of translates of $\delta\mathcal{L}$; hence so is $\mathcal{L}_1 = \mathcal{L}' - \boldsymbol{w}_1$. Similarly for each $j \in \{2,\ldots,m\}$, since $\delta\mathcal{L}$ is a finite index subgroup of $\delta_j\mathcal{L}_j + \boldsymbol{w}_j$, the grid $\mathcal{L}_j$ is a union of a finite number of translates of $(\delta/\delta_j)\mathcal{L}$. The first statement of the lemma follows by collecting all these unions into a single union.

To prove the second statement, we start from a union $\bigcup_{i=1}^r c_i(\mathcal{L}+\boldsymbol{v}_i)$ with arbitrary numbers $c_1,\ldots,c_r \in \mathbb{R}_{>0}$ and vectors $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r \in \mathbb{R}^d$. Then partition the index set $\{1,\ldots,r\}$ by the equivalence relation $i \sim i' \overset{\text{def}}{\Leftrightarrow} c_i/c_{i'} \in \mathbb{Q}$. For each equivalence class $J \subset \{1,\ldots,r\}$, let c_J be the least common multiple of the numbers $\{c_i \,:\, i \in J\}$ (this exists since $c_i/c_{i'} \in \mathbb{Q}$ for all $i,i' \in J$). Then for each $i \in J$ we have $n_i := c_J/c_i \in \mathbb{Z}^+$, and now $\bigcup_{i\in J} c_i(\mathcal{L}+\boldsymbol{v}_i) = \bigcup_{i\in J}\bigcup_{\boldsymbol{w}\in R_i} c_J(\mathcal{L}+\boldsymbol{w}+n_i^{-1}\boldsymbol{v}_i)$, where $R_i \subset n_i^{-1}\mathcal{L}$ is any fixed set of representatives for the quotient $n_i^{-1}\mathcal{L}/\mathcal{L}$. Removing any duplicate translate of $c_J\mathcal{L}$ from the last union, we arrive at an expression of the form $\bigcup_{i\in J} c_i(\mathcal{L}+\boldsymbol{v}_i) = \bigcup_{i=1}^{r_J} c_J(\mathcal{L}+\boldsymbol{v}'_i)$ for some vectors $\boldsymbol{v}'_1,\ldots,\boldsymbol{v}'_{r_J} \in \mathbb{R}^d$ which are pairwise incongruent modulo $\mathcal{L}$. Applying this rewriting procedure to each equivalence class $J \subset \{1,\ldots,r\}$, we arrive at a representation of $\mathcal{L}_1 \cup \cdots \cup \mathcal{L}_m$ with all the required properties. □

Remark 2.1. Note that for a given $\mathcal{P}$, the representation (2.2) is *far* from unique, even when we require (2.3).

Let us rewrite (2.2) slightly: Choose $M_1,\ldots,M_N \in \mathrm{SL}_d(\mathbb{R})$ so that $\mathcal{L}_j = \mathbb{Z}^dM_j$ (this is possible since $\mathcal{L}_j$ has covolume one). Set

$$\Psi = \{(j,i) \,:\, j \in \{1,\ldots,N\},\, i \in \{1,\ldots,r_j\}\}. \tag{2.4}$$

Also for each $\psi = (j,i) \in \Psi$, we set $c_\psi = c_{j,i}$, $\boldsymbol{w}_\psi = \boldsymbol{w}_{j,i} = \boldsymbol{v}_{j,i}M_j^{-1} \in \mathbb{R}^d$, and

$$\mathcal{L}_\psi := c_{j,i}(\mathcal{L}_j + \boldsymbol{v}_{j,i}) = c_\psi(\mathbb{Z}^d + \boldsymbol{w}_\psi)M_j. \tag{2.5}$$

Then:

$$\mathcal{P} = \bigcup_{\psi\in\Psi} \mathcal{L}_\psi. \tag{2.6}$$

Given $\psi \in \Psi$, we will often denote by j_ψ and i_ψ the indices such that $\psi = (j_\psi, i_\psi)$. The condition (2.3) now takes the following form:

$$\forall\psi \neq \psi' \in \Psi: \ \big[j_\psi = j_{\psi'} \text{ and } c_\psi/c_{\psi'} \in \mathbb{Q}\big] \Rightarrow \big[c_\psi = c_{\psi'} \text{ and } \boldsymbol{w}_\psi - \boldsymbol{w}_{\psi'} \notin \mathbb{Z}^d\big]. \tag{2.7}$$

Note also that the fact that $\mathcal{L}_1,\ldots,\mathcal{L}_N$ are pairwise incommensurable implies that

$$M_jM_{j'}^{-1} \notin \mathcal{S}, \qquad \forall j \neq j' \in \{1,\ldots,N\}, \tag{2.8}$$

where $\mathcal{S}$ is the commensurator of $\mathrm{SL}_d(\mathbb{Z})$ in $\mathrm{SL}_d(\mathbb{R})$, i.e.

$$\mathcal{S} = \{(\det T)^{-1/d}T \,:\, T \in \mathrm{GL}(d,\mathbb{Q}),\ \det T > 0\}. \tag{2.9}$$

The presentation of $\mathcal{P}$ in (2.6) is the one which we will work with throughout the paper, and the conditions (2.7) and (2.8) will always be assumed to hold.

Lemma 2.2. *In the above representation of $\mathcal{P}$, for any two $\psi \neq \psi' \in \Psi$, the intersection $\mathcal{L}_\psi \cap \mathcal{L}_{\psi'}$ is contained in an affine hyperplane of $\mathbb{R}^d$.*

Proof. If $j_\psi \neq j_{\psi'}$, then $\mathcal{L}_\psi$ and $\mathcal{L}_{\psi'}$ are incommensurable, and the statement follows. [Details: Assume that $\mathcal{L}_\psi \cap \mathcal{L}_{\psi'}$ is *not* contained in an affine hyperplane. Then there exist points $\boldsymbol{q}_0, \ldots, \boldsymbol{q}_d \in \mathcal{L}_\psi \cap \mathcal{L}_{\psi'}$ such that $\boldsymbol{q}_j - \boldsymbol{q}_0$ for $j = 1, \ldots, d$ are linearly independent and thus form an linear basis of $\mathbb{R}^d$. But now both $\mathcal{L}_\psi - \boldsymbol{q}_0$ and $\mathcal{L}_{\psi'} - \boldsymbol{q}_0$ are lattices containing the vectors $\boldsymbol{q}_j - \boldsymbol{q}_0$ for $j = 1, \ldots, d$; hence these lattices also contain the lattice $\mathbb{Z}(\boldsymbol{q}_1 - \boldsymbol{q}_0) + \cdots + \mathbb{Z}(\boldsymbol{q}_d - \boldsymbol{q}_0)$, and so $\mathcal{L}_\psi \cap \mathcal{L}_{\psi'}$ contains the grid $\boldsymbol{q}_0 + \mathbb{Z}(\boldsymbol{q}_1 - \boldsymbol{q}_0) + \cdots + \mathbb{Z}(\boldsymbol{q}_d - \boldsymbol{q}_0)$, contradicting the fact that $\mathcal{L}_\psi$ and $\mathcal{L}_{\psi'}$ are incommensurable.]

Next assume $j_\psi = j_{\psi'}$. If $c_\psi / c_{\psi'} \notin \mathbb{Q}$ then the intersection $c_\psi(\mathbb{Z}^d + \boldsymbol{w}_\psi) \cap c_{\psi'}(\mathbb{Z}^d + \boldsymbol{w}_{\psi'})$ contains at most one point; hence so does $\mathcal{L}_\psi \cap \mathcal{L}_{\psi'}$. Finally, if $c_\psi / c_{\psi'} \in \mathbb{Q}$ then by (2.7) we have $c_\psi = c_{\psi'}$ and $\boldsymbol{w}_\psi - \boldsymbol{w}_{\psi'} \notin \mathbb{Z}^d$; hence $\mathcal{L}_\psi \cap \mathcal{L}_{\psi'}$ is empty. □

Given $\mathcal{P}$ as in (2.6) we now also introduce a "crude marking". This is a fixed choice of a function

$$\psi : \mathcal{P} \to \Psi, \tag{2.10}$$

such that $\boldsymbol{q} \in \mathcal{L}_{\psi(\boldsymbol{q})}$ every $\boldsymbol{q} \in \mathcal{P}$.

Remark 2.2. Note that this makes $\psi(\boldsymbol{q})$ uniquely defined for every $\boldsymbol{q} \in \mathcal{P}$ which does not lie in more than one of the sets $\mathcal{L}_\psi$, i.e. for every $\boldsymbol{q} \in \mathcal{P}$ outside the union $\bigcup_{\psi \neq \psi' \in \Psi} (\mathcal{L}_\psi \cap \mathcal{L}_{\psi'})$. By Lemma 2.2, this union is contained in a finite union of affine hyperplanes. In particular $\psi(\boldsymbol{q})$ is uniquely defined for every $\boldsymbol{q} \in \mathcal{P}$ away from a subset of density zero.

Finally, for each $\psi = (j, i) \in \Psi$ we set

$$\overline{n}_\psi = \overline{n}_{j,i} := c_\psi^{-d}; \tag{2.11}$$

this is the asymptotic density of the grid $\mathcal{L}_\psi$. We also set

$$c_{\mathcal{P}} := \sum_{\psi \in \Psi} \overline{n}_\psi. \tag{2.12}$$

This is the asymptotic density of the point set $\mathcal{P}$.

2.2. **Lie groups and homogeneous spaces.** We now introduce appropriate homogeneous spaces to keep track of the unions of grids which will appear in our discussion. (This is very standard; cf. [20, Sections 5.2 & 5.3.3], [5, Sec. 7], but there certainly exist many other references as well.)

For any positive integer r, we let $\mathrm{M}_{r \times d}(\mathbb{R})$ be the space of real $r \times d$ matrices, and let $\mathrm{S}_r(\mathbb{R})$ be the semidirect product

$$\mathrm{S}_r(\mathbb{R}) := \mathrm{SL}_d(\mathbb{R}) \ltimes \mathrm{M}_{r \times d}(\mathbb{R}) \tag{2.13}$$

with multiplication law

$$(M, U)(M', U') = (MM', UM' + U'). \tag{2.14}$$

We also set

$$\mathrm{S}_r(\mathbb{Z}) := \mathrm{SL}_d(\mathbb{Z}) \ltimes \mathrm{M}_{r \times d}(\mathbb{Z}).$$

In the special case $r = 1$, we have $\mathrm{S}_1(\mathbb{R}) = \mathrm{ASL}_d(\mathbb{R})$, the affine special linear group. This group acts on $\mathbb{R}^d$ from the right through

$$\boldsymbol{w}(M, \boldsymbol{u}) := \boldsymbol{w}M + \boldsymbol{u} \qquad (\boldsymbol{w} \in \mathbb{R}^d,\ (M, \boldsymbol{u}) \in \mathrm{ASL}_d(\mathbb{R})).$$

It is natural to identify $\mathrm{S}_r(\mathbb{R})$ with a subgroup of $\mathrm{SL}_{d+r}(\mathbb{R})$ through $(M, U) \leftrightarrow \begin{pmatrix} M & 0 \\ U & I \end{pmatrix}$, since the group law (2.14) then corresponds to multiplication of matrices; however we will not make use of this identification in the present paper.

We will always consider $\mathrm{SL}_d(\mathbb{R})$ to be embedded in $\mathrm{S}_r(\mathbb{R})$ through the isomorphism $M \mapsto (M, 0)$; in other words any $M \in \mathrm{SL}_d(\mathbb{R})$ is understood to also denote the element $(M, 0)$ in $\mathrm{S}_r(\mathbb{R})$. In the opposite direction, we denote by ι the projection homomorphism

$$\iota : \mathrm{S}_r(\mathbb{R}) \to \mathrm{SL}_d(\mathbb{R}); \qquad \iota(M, U) := M.$$

For any $U \in \mathrm{M}_{r\times d}(\mathbb{R})$ we let I_U be the element

$$\mathrm{I}_U := (\mathrm{I}, U). \tag{2.15}$$

in $\mathrm{S}_r(\mathbb{R})$. Note that the map $U \mapsto \mathrm{I}_U$ is a homomorphism from $\mathrm{M}_{r\times d}(\mathbb{R})$ to $\mathrm{S}_r(\mathbb{R})$. Furthermore, for each $i \in \{1, \ldots, r\}$, we let $\mathsf{r}_i : \mathrm{M}_{r\times d}(\mathbb{R}) \to \mathbb{R}^d$ be the projection map which takes any matrix to its ith row, and we also define the following Lie group homomorphism:

$$\mathsf{a}_i : \mathrm{S}_r(\mathbb{R}) \to \mathrm{ASL}_d(\mathbb{R}); \qquad \mathsf{a}_i(M, U) := (M, \mathsf{r}_i(U)).$$

From now on we assume a point set $\mathcal{P}$ with presentation as in (2.4)–(2.6) to be *fixed.* In particular this means that the numbers $N, r_1, \ldots, r_N$ are now fixed; and so are the numbers c_ψ and vectors $\boldsymbol{w}_\psi$ $(\psi \in \Psi)$. We then set:

$$G = \mathrm{S}_{r_1}(\mathbb{R}) \times \cdots \times \mathrm{S}_{r_N}(\mathbb{R}); \qquad \Gamma = \mathrm{S}_{r_1}(\mathbb{Z}) \times \cdots \times \mathrm{S}_{r_N}(\mathbb{Z});$$

and

$$\mathbb{X} = \Gamma\backslash G.$$

We also write $G_j = \mathrm{S}_{r_j}(\mathbb{R})$, $\Gamma_j = \mathrm{S}_{r_j}(\mathbb{Z})$ and $\mathbb{X}_j = \Gamma_j\backslash G_j$, so that $G = G_1 \times \cdots \times G_N$ and $\mathbb{X} = \mathbb{X}_1 \times \cdots \times \mathbb{X}_N$; and we write p_j and $\widetilde{p}_j$ for the projection maps onto the jth factor:

$$p_j : G \to \mathrm{S}_{r_j}(\mathbb{R}) \qquad \text{and} \qquad \widetilde{p}_j : \mathbb{X} \to \mathbb{X}_j \qquad (j \in \{1, \ldots, N\}). \tag{2.16}$$

Also for each $\psi = (j, i) \in \Psi$, we introduce the Lie group homomorphism p_ψ through

$$p_\psi := \mathsf{a}_i \circ p_j : G \to \mathrm{ASL}_d(\mathbb{R}). \tag{2.17}$$

We will use the space $\mathbb{X}$ to parametrize the set of all point sets that can be obtained from $\mathcal{P} = \cup_{\psi\in\Psi}\mathcal{L}_\psi$ by applying an arbitrary affine linear map of determinant one to each individual grid $\mathcal{L}_\psi$. Specifically, we take Γg in $\mathbb{X}$ to parametrize the point set $J_0(\Gamma g)$, where

$$J_0 : \mathbb{X} \to N_s(\mathbb{R}^d); \qquad J_0(\Gamma g) := \bigcup_{\psi\in\Psi} c_\psi\big(\mathbb{Z}^d\, p_\psi(g)\big) \qquad (g \in G). \tag{2.18}$$

To see that this map is well-defined, note that for each ψ we have $p_\psi(\Gamma) = \mathrm{ASL}_d(\mathbb{Z})$; hence if $\Gamma g = \Gamma g'$ then $p_\psi(g') = \gamma\, p_\psi(g)$ for some $\gamma \in \mathrm{ASL}_d(\mathbb{Z})$, implying that $\mathbb{Z}^d p_\psi(g') = \mathbb{Z}^d p_\psi(g)$. However it should be noted that the map J_0 is discontinuous at any point $\Gamma g \in \mathbb{X}$ for which the grids $c_\psi\big(\mathbb{Z}^d\, p_\psi(g)\big)$ $(\psi \in \Psi)$ are not pairwise disjoint.

We next introduce some notation for expressing the parametrization in $\mathbb{X}$ of an arbitrary translate of the fixed point set $\mathcal{P}$. For any $\boldsymbol{q} \in \mathbb{R}^d$ and $j \in \{1, \ldots, N\}$ we define the matrix $U_j^{(\boldsymbol{q})}$ by specifying its row vectors:

$$U_j^{(\boldsymbol{q})} \in \mathrm{M}_{r_j\times d}(\mathbb{R}); \qquad \mathsf{r}_i(U_j^{(\boldsymbol{q})}) = \boldsymbol{w}_{j,i} - c_{j,i}^{-1}\boldsymbol{q}M_j^{-1} \qquad (i = 1, \ldots, r_j). \tag{2.19}$$

We also write

$$U^{(\boldsymbol{q})} := (U_1^{(\boldsymbol{q})}, \cdots, U_N^{(\boldsymbol{q})}) \in \prod_{j=1}^{N} \mathrm{M}_{r_j\times d}(\mathbb{R}), \tag{2.20}$$

and we introduce the notation (cf. (2.15))

$$\mathrm{I}_V := \big(\mathrm{I}_{V_1}, \ldots, \mathrm{I}_{V_N}\big) \in G \qquad \text{for any } \; V = (V_1, \ldots, V_N) \in \prod_{j=1}^{N} \mathrm{M}_{r_j\times d}(\mathbb{R}). \tag{2.21}$$

We also set

$$\widetilde{M} := (M_1, \ldots, M_N) \in G. \tag{2.22}$$

Finally we define:

$$g_0^{(\boldsymbol{q})} := \mathrm{I}_{U^{(\boldsymbol{q})}}\ \widetilde{M} \in G \qquad (\boldsymbol{q} \in \mathbb{R}^d). \tag{2.23}$$

The point of these definitions is that we now have, for any $\psi = (j,i) \in \Psi$:

$$c_\psi\big(\mathbb{Z}^d p_\psi(g_0^{(\boldsymbol{q})})\big) = c_\psi\big(\mathbb{Z}^d \mathsf{a}_i(\mathrm{I}_{U_j^{(\boldsymbol{q})}} M_j)\big) = c_\psi\left(\mathbb{Z}^d + \boldsymbol{w}_\psi - c_\psi^{-1}\boldsymbol{q}M_j^{-1}\right) M_j = \mathcal{L}_\psi - \boldsymbol{q} \tag{2.24}$$

(cf. (2.5)), and hence

$$J_0(\Gamma\, g_0^{(\boldsymbol{q})}) = \mathcal{P} - \boldsymbol{q} \qquad (\forall \boldsymbol{q} \in \mathbb{R}^d). \tag{2.25}$$

Finally we introduce some further notation relating to the structure of the homogeneous spaces $\mathbb{X}_j$ and $\mathbb{X}$. Recall that for each $j \in \{1,\ldots,N\}$ we have a projection map $\iota : \mathrm{S}_{r_j}(\mathbb{R}) \to \mathrm{SL}_d(\mathbb{R})$; this map takes Γ_j to $\mathrm{SL}_d(\mathbb{Z})$, and hence induces a projection map from $\mathbb{X}_j$ to $\mathrm{SL}_d(\mathbb{Z})\backslash \mathrm{SL}_d(\mathbb{R})$, which we will denote by $\widetilde{\iota}$:

$$\widetilde{\iota} : \mathbb{X}_j \to \mathrm{SL}_d(\mathbb{Z})\backslash \mathrm{SL}_d(\mathbb{R}), \qquad \widetilde{\iota}(\Gamma_j g) = \mathrm{SL}_d(\mathbb{Z})\, \iota(g) \quad (g \in G_j). \tag{2.26}$$

We also set

$$\mathbb{T}_j := \mathbb{R}^{r_j}/\mathbb{Z}^{r_j} \qquad \text{and} \qquad \mathbb{T}_j^d := \underbrace{\mathbb{T}_j \times \cdots \times \mathbb{T}_j}_{d \text{ copies}}.$$

Note that $\mathbb{T}_j^d$ is an $r_j d$-dimensional torus. We will use "π" to denote both the corresponding projection maps;

$$\pi : \mathbb{R}^{r_j} \to \mathbb{T}_j \qquad \text{and} \qquad \pi : (\mathbb{R}^{r_j})^d \to \mathbb{T}_j^d. \tag{2.27}$$

We will identify $\mathrm{M}_{r_j\times d}(\mathbb{R})$ with $(\mathbb{R}^{r_j})^d$, by identifying any matrix in $\mathrm{M}_{r_j\times d}(\mathbb{R})$ with the sequence of its d column vectors (in order). This also induces an identification

$$\mathbb{T}_j^d = \mathrm{M}_{r_j\times d}(\mathbb{R}/\mathbb{Z}).$$

We write $\widetilde{\mathsf{r}}_i : \mathbb{T}_j^d \to (\mathbb{R}/\mathbb{Z})^d$ for the projection induced by the map r_i. For each j we also introduce the embedding

$$x : \mathbb{T}_j^d \to \mathbb{X}_j; \qquad x(\pi(U)) = \Gamma_j\, \mathrm{I}_U, \quad \forall U \in \mathrm{M}_{r_j\times d}(\mathbb{R}). \tag{2.28}$$

3. Spherical equidistribution (without uniformity)

In this section we state a result in homogeneous dynamics, Theorem 3.2 below, which gives a precise description of the limit considered in the spherical equidistribution condition [P2] in Section 1.1. Theorem 3.2 expresses the answer in terms of the parametrizing space $\mathbb{X}$, and in fact applies for an arbitrary point $\boldsymbol{q}$ in $\mathbb{R}^d$; however the theorem includes no statement about uniformity with respect to the translation vector $\boldsymbol{q}$, as is required in [P2].

This Theorem 3.2 is a special case of more general equidistribution results which we will prove in Sections 7 and 8 (see in particular Theorem 8.1), and the main reason for stating this special case already here is to help motivating the introduction of the measure $\mu^{(\boldsymbol{q})}$ (see (3.2) below), which will be a crucial object of study in the remainder of Section 3 as well as in Section 4. Note that we will not make use of the statement of Theorem 3.2 before Section 8.

For a given $\boldsymbol{q} \in \mathbb{R}^d$ and each $j \in \{1,\ldots,N\}$, we let $U_{j,1}^{(\boldsymbol{q})},\ldots,U_{j,d}^{(\boldsymbol{q})} \in \mathbb{R}^{r_j}$ be the column vectors of $U_j^{(\boldsymbol{q})}$, and then define $L_j^{(\boldsymbol{q})}$ to be the identity component of the smallest closed subgroup of $\mathbb{R}^{r_j}$ containing both $\mathbb{Z}^{r_j}$ and $U_{j,1}^{(\boldsymbol{q})},\ldots,U_{j,d}^{(\boldsymbol{q})}$. This $L_j^{(\boldsymbol{q})}$ is a *rational* (linear) subspace of $\mathbb{R}^{r_j}$, i.e., $L_j^{(\boldsymbol{q})} \cap \mathbb{Z}^{r_j}$ is a lattice in $L_j^{(\boldsymbol{q})}$. Next, given any linear subspace L or $\mathbb{R}^{r_j}$, we let $\mathrm{S}_L(\mathbb{R})$ be the closed connected subgroup of $\mathrm{S}_{r_j}(\mathbb{R})$ given by

$$\mathrm{S}_L(\mathbb{R}) := \mathrm{SL}_d(\mathbb{R}) \ltimes L^d = \big\{(M,U) \in \mathrm{S}_{r_j}(\mathbb{R})\ :\ U \in L^d\big\}. \tag{3.1}$$

Here "L^d" should be understood via our identification $\mathrm{M}_{r_j\times d}(\mathbb{R}) = (\mathbb{R}^{r_j})^d$, viz., L^d is the set of matrices in $\mathrm{M}_{r_j\times d}(\mathbb{R})$ all of whose column vectors lie in L.

Lemma 3.1. *For any $\boldsymbol{q} \in \mathbb{R}^d$, the $\mathrm{S}_{L_j^{(\boldsymbol{q})}}(\mathbb{R})$-orbit of the point $\Gamma_j \, \mathrm{I}_{U_j^{(\boldsymbol{q})}}$ in $\mathbb{X}_j = \Gamma_j \backslash G_j$ is a closed embedded submanifold of $\mathbb{X}_j$ which carries a unique $\mathrm{S}_{L_j^{(\boldsymbol{q})}}(\mathbb{R})$-invariant probability measure.*

Proof. Set $U := U_j^{(\boldsymbol{q})}$ and $L := L_j^{(\boldsymbol{q})}$. It follows from the construction of L that there exists a matrix $X \in \mathrm{M}_{r_j \times d}(\mathbb{Q})$ such that $U - X \in L^d$ (indeed, cf. Lemma 3.3 below). This implies $\mathrm{I}_U \, \mathrm{I}_X^{-1} \in \mathrm{S}_L(\mathbb{R})$, and hence the $\mathrm{S}_L(\mathbb{R})$-orbit in the statement of the lemma can be expressed as $\Gamma_j \backslash \Gamma_j \, \mathrm{I}_U \, \mathrm{S}_L(\mathbb{R}) = \Gamma_j \backslash \Gamma_j \, \mathrm{I}_X \, \mathrm{S}_L(\mathbb{R})$. Hence by [23, Theorem 1.13], it suffices to verify that Γ_j intersects $\mathrm{I}_X \, \mathrm{S}_L(\mathbb{R}) \, \mathrm{I}_X^{-1}$ in a lattice. To show this, let n be a denominator of X, i.e. a positive integer such that $X \in n^{-1} \, \mathrm{M}_{r_j \times d}(\mathbb{Z})$, and then set

$$\Lambda := \{(M, V) \, : \, M \in \Gamma(n), \, V \in XM - X + (L \cap \mathbb{Z}^{r_j})^d\},$$

where $\Gamma(n)$ is the principal congruence subgroup of $\mathrm{SL}_d(\mathbb{Z})$ of level n. One verifies that Λ is a subgroup of $\Gamma_j \cap \mathrm{I}_X \, \mathrm{S}_L(\mathbb{R}) \, \mathrm{I}_X^{-1}$, and a lattice in $\mathrm{I}_X \, \mathrm{S}_L(\mathbb{R}) \, \mathrm{I}_X^{-1}$. Hence the lemma is proved. □

In view of Lemma 3.1, it makes sense to define $\mu_j^{(\boldsymbol{q})} \in P(\mathbb{X}_j)$ to be the unique $\mathrm{S}_{L_j^{(\boldsymbol{q})}}(\mathbb{R})$-invariant probability measure on the orbit $\Gamma_j \backslash \Gamma_j \, \mathrm{I}_{U_j^{(\boldsymbol{q})}} \, \mathrm{S}_{L_j^{(\boldsymbol{q})}}(\mathbb{R})$ in $\mathbb{X}_j$. Finally we set

$$\mu^{(\boldsymbol{q})} := \mu_1^{(\boldsymbol{q})} \otimes \cdots \otimes \mu_N^{(\boldsymbol{q})} \in P(\mathbb{X}). \tag{3.2}$$

Let $\varphi : \mathrm{SL}_d(\mathbb{R}) \to G$ be the diagonal embedding.

Theorem 3.2. *Given any $\boldsymbol{q} \in \mathbb{R}^d$, $f \in \mathrm{C}_b(\mathbb{X})$ and $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$, we have*

$$\int_{\mathrm{S}_1^{d-1}} f(\Gamma g_0^{(\boldsymbol{q})} \varphi(R(\boldsymbol{v}) D_\rho)) \, d\lambda(\boldsymbol{v}) \to \int_{\mathbb{X}} f \, d\mu^{(\boldsymbol{q})} \tag{3.3}$$

as $\rho \to 0$.

We will prove Theorem 3.2 at the end of Section 8.1, as a corollary of more general and difficult results which we prove in Section 7. We remark that it is possible to give a considerably simpler deduction of Theorem 3.2 by combining the ideas in the proof of [19, Theorem 10] and the proof of [20, Lemma 5.22]. In any case, the crucial ingredient in either of these two proofs is the deep measure classification theorem of Ratner [24].

3.1. **Re-expressing the limit measures $\mu_j^{(\boldsymbol{q})}$ as $\overline{\omega_j^{(\boldsymbol{q})}}$.** The task of finding the appropriate space of marks Σ and marking $\varsigma : \mathcal{P} \to \Sigma$ to satisfy the conditions in Section 1.1 is closely related to the task of understanding the limit measures $\mu^{(\boldsymbol{q})} = \mu_1^{(\boldsymbol{q})} \otimes \cdots \otimes \mu_N^{(\boldsymbol{q})}$ appearing in Theorem 3.2, and in particular how these vary as $\boldsymbol{q}$ varies within the set $\mathcal{P}$. As a first step towards this goal, in this section we will introduce an alternative way to define the measures $\mu_j^{(\boldsymbol{q})}$. Recall that each space $\mathbb{X}_j$ is a torus bundle over the space $\mathrm{SL}_d(\mathbb{Z}) \backslash \mathrm{SL}_d(\mathbb{R})$ (cf. (2.26)), and the point here is that each each measure $\mu_j^{(\boldsymbol{q})}$ can be expressed as a *product measure*, of the normalized Haar measure on $\mathrm{SL}_d(\mathbb{Z}) \backslash \mathrm{SL}_d(\mathbb{R})$ and some fixed measure on the fiber $\mathbb{T}_j^d$. We make this precise in Lemma 3.4 and Proposition 3.7 below.

We keep $j \in \{1, \ldots, N\}$ throughout this section. Our first step will be to introduce the relevant measures which can appear on the fiber $\mathbb{T}_j^d$.

Recall that π denotes (among other things) the projection map $\mathbb{R}^{r_j} \to \mathbb{T}^{r_j} := \mathbb{R}^{r_j} / \mathbb{Z}^{r_j}$. For any subset S of $\mathbb{R}^{r_j}$ (resp. $\mathbb{T}^{r_j}$), we write $\langle S \rangle$ for the subgroup of $\mathbb{R}^{r_j}$ (resp. $\mathbb{T}^{r_j}$) generated by S, and $\overline{\langle S \rangle}$ for its closure. For any topological group G we write G° for its identity component. Now for any non-empty subset S of $\mathbb{T}^{r_j}$, we introduce the notation:

$$\mathfrak{L}(S) := \overline{\langle \pi^{-1}(S) \rangle}^{\,\circ}. \tag{3.4}$$

This is a *rational* subspace of $\mathbb{R}^{r_j}$, i.e. a linear subspace of $\mathbb{R}^{r_j}$ which is spanned by its vectors in $\mathbb{Z}^{r_j}$. If S is a non-empty subset *of* $\mathbb{R}^{r_j}$ then we also write $\mathfrak{L}(S) := \mathfrak{L}(\pi(S)) = \overline{\langle S + \mathbb{Z}^{r_j} \rangle}^{\,\circ}$, and if $V_1,\ldots,V_m$ is any finite sequence of points in $\mathbb{T}^{r_j}$ or $\mathbb{R}^{r_j}$, then we also write $\mathfrak{L}(V_1,\ldots,V_m) := \mathfrak{L}(\{V_1,\ldots,V_m\})$.

Lemma 3.3. *For any non-empty subset $S \subset \mathbb{R}^{r_j}$, $\mathfrak{L}(S)$ is the unique smallest rational subspace $L \subset \mathbb{R}^{r_j}$ with the property that there is some $n \in \mathbb{Z}^+$ such that $\langle S\rangle \subset n^{-1}\mathbb{Z}^{r_j} + L$.*

Proof. Set $G = \overline{\langle \pi(S)\rangle}$; this is a closed subgroup of $\mathbb{T}^{r_j}$; hence its group of components, G/G°, is finite, i.e. there exists a positive integer n such that $n\boldsymbol{y} \in G^\circ$ for all $\boldsymbol{y} \in G$. We also have $G^\circ = \pi(\mathfrak{L}(S))$. It follows that $n \cdot \pi(S) \subset \pi(\mathfrak{L}(S))$, i.e. $n \cdot S \subset \mathbb{Z}^{r_j} + \mathfrak{L}(S)$. This is equivalent with $S \subset n^{-1}\mathbb{Z}^{r_j} + \mathfrak{L}(S)$, and also with $\langle S\rangle \subset n^{-1}\mathbb{Z}^{r_j} + \mathfrak{L}(S)$.

Next assume that L is *any* rational subspace of $\mathbb{R}^{r_j}$ satisfying $\langle S\rangle \subset n^{-1}\mathbb{Z}^{r_j} + L$ for some $n \in \mathbb{Z}^+$. Now $n^{-1}\mathbb{Z}^{r_j} + L$ is a closed subgroup of $\mathbb{R}^{r_j}$ which contains $S + \mathbb{Z}^{r_j}$; hence $\overline{\langle S + \mathbb{Z}^{r_j}\rangle} \subset n^{-1}\mathbb{Z}^{r_j} + L$, and so $\mathfrak{L}(S) := \overline{\langle S + \mathbb{Z}^{r_j}\rangle}^{\,\circ} \subset (n^{-1}\mathbb{Z}^{r_j} + L)^\circ = L$. □

Next, for any $V = (V_1,\ldots,V_d) \in \mathbb{T}_j^d$, we introduce the notation

$$\mathbb{S}_j^{(V)} := \overline{\langle V_1,\ldots,V_d\rangle} \tag{3.5}$$

and

$$L_j^{(V)} := \mathfrak{L}(V_1,\ldots,V_d) = \big(\pi^{-1}(\mathbb{S}_j^{(V)})\big)^\circ. \tag{3.6}$$

Thus $\mathbb{S}_j^{(V)}$ is a closed subgroup of $\mathbb{T}_j$ and $L_j^{(V)}$ is a rational subspace of $\mathbb{R}^{r_j}$.

Recall that we have identified $\mathrm{M}_{r_j\times d}(\mathbb{R})$ with $(\mathbb{R}^{r_j})^d$; this means that for any linear subspace $L \subset \mathbb{R}^{r_j}$, $L^d = L \times \cdots \times L$ is a linear subspace of $\mathrm{M}_{r_j\times d}(\mathbb{R})$. Similarly for any subgroup $\mathbb{S} \subset \mathbb{T}_j$, $\mathbb{S}^d$ is a subgroup of $\mathbb{T}_j^d = \mathrm{M}_{r_j\times d}(\mathbb{R})/\mathrm{M}_{r_j\times d}(\mathbb{Z})$. Note that $\mathrm{SL}_d(\mathbb{R})$ acts from the right on $\mathrm{M}_{r_j\times d}(\mathbb{R})$ by matrix multiplication, and this action preserves the subspace L^d for any $L \subset \mathbb{R}^{r_j}$. Furthermore, the subgroup $\mathrm{SL}_d(\mathbb{Z})$ preserves $\mathrm{M}_{r_j\times d}(\mathbb{Z})$, and hence we obtain an induced right action of $\mathrm{SL}_d(\mathbb{Z})$ on $\mathbb{T}_j^d = \mathrm{M}_{r_j\times d}(\mathbb{R}/\mathbb{Z})$. This action preserves the subgroup $\mathbb{S}^d$, for any subgroup $\mathbb{S} \subset \mathbb{T}_j$.

Now for any $V \in \mathbb{T}_j^d$, we define:

$$\mathcal{O}_j^{(V)} = \bigcup_{\gamma \in \mathrm{SL}_d(\mathbb{Z})} \big(V\gamma + \big(\mathbb{S}_j^{(V)}\big)^{\circ\, d}\big). \tag{3.7}$$

Here it should be noted that $V \in \big(\mathbb{S}_j^{(V)}\big)^d$; hence $\mathcal{O}_j^{(V)}$ is a union of some of the connected components of the Lie group $\big(\mathbb{S}_j^{(V)}\big)^d$. In particular $\mathcal{O}_j^{(V)}$ is open and closed in $\big(\mathbb{S}_j^{(V)}\big)^d$. Note also that since $\big(\mathbb{S}_j^{(V)}\big)^d$ is compact, its total number of components is finite.

We also define $\omega_j^{(V)} \in P(\mathbb{T}_j^d)$ to be the restriction to $\mathcal{O}_j^{(V)}$ of the Haar measure on $\big(\mathbb{S}_j^{(V)}\big)^d$. normalized so that $\omega_j^{(V)}(\mathcal{O}_j^{(V)}) = 1$.

Note that for each $V \in \mathbb{T}_j^d$, the measure $\omega_j^{(V)}$ is $\mathrm{SL}_d(\mathbb{Z})$-invariant by construction. We denote by $P(\mathbb{T}_j^d)'$ the subset of all $\mathrm{SL}_d(\mathbb{Z})$-invariant measures $\omega \in P(\mathbb{T}_j^d)$. This is a closed, and hence compact, subset of $P(\mathbb{T}_j^d)$.

Next, recall that for any $\boldsymbol{q} \in \mathbb{R}^d$ we have defined a matrix $U_j^{(\boldsymbol{q})} \in \mathrm{M}_{r_j\times d}(\mathbb{R})$, cf. (2.19); and via our identification $\mathbb{T}_j^d = \mathrm{M}_{r_j\times d}(\mathbb{R})/\mathrm{M}_{r_j\times d}(\mathbb{Z})$ we have $\pi(U_j^{(\boldsymbol{q})}) \in \mathbb{T}_j^d$. Note that the rational subspace $L_j^{(\boldsymbol{q})} \subset \mathbb{R}^{r_j}$ defined in Section 3 equals $L_j^{(\pi(U_j^{(\boldsymbol{q})}))}$ in our present notation. From now on we will also write $\mathbb{S}_j^{(\boldsymbol{q})}$, $\mathcal{O}_j^{(\boldsymbol{q})}$ and $\omega_j^{(\boldsymbol{q})}$ to denote $\mathbb{S}_j^{(\pi(U_j^{(\boldsymbol{q})}))}$, $\mathcal{O}_j^{(\pi(U_j^{(\boldsymbol{q})}))}$ and $\omega_j^{(\pi(U_j^{(\boldsymbol{q})}))}$, respectively. Thus in particular, $\mathcal{O}_j^{(\boldsymbol{q})}$ is a union of some of the components of the closed subgroup $\big(\mathbb{S}_j^{(\boldsymbol{q})}\big)^d \subset \mathbb{T}_j^d$, and $\omega_j^{(\boldsymbol{q})}$ is a measure on $P(\mathbb{T}_j^d)'$ supported on $\mathcal{O}_j^{(\boldsymbol{q})}$. We will see that

$\omega_j^{(\boldsymbol{q})}$ is the measure on the fiber $\mathbb{T}_j^d$ which gives back the measure $\mu_j^{(\boldsymbol{q})}$ on $\mathbb{X}_j$ via the product construction which we will presently describe.

We now introduce a map

$$P(\mathbb{T}_j^d)' \to P(\mathbb{X}_j), \qquad \omega \mapsto \overline{\omega}, \tag{3.8}$$

as follows. For any $\omega \in P(\mathbb{T}_j^d)$, let $\widetilde{\omega}$ be corresponding $\mathrm{M}_{r_j\times d}(\mathbb{Z})$-invariant Borel measure on $\mathrm{M}_{r_j\times d}(\mathbb{R})$, and define the Borel measure $\widetilde{\overline{\omega}}$ on G_j through

$$d\widetilde{\overline{\omega}}(g) = d\widetilde{\omega}(U)\,d\nu(A) \qquad \text{when } \ g = \mathrm{I}_U\,A \in G_j \quad (U \in \mathrm{M}_{r_j\times d}(\mathbb{R}),\ A \in \mathrm{SL}_d(\mathbb{R})), \tag{3.9}$$

where ν denotes the Haar measure on $\mathrm{SL}_d(\mathbb{R})$, normalized by $\nu(\mathrm{SL}_d(\mathbb{Z})\backslash \mathrm{SL}_d(\mathbb{R})) = 1$. By the following lemma, if $\omega \in P(\mathbb{T}_j^d)'$ then the measure $\widetilde{\overline{\omega}}$ is left Γ_j-invariant, and we finally define $\overline{\omega}$ to be the corresponding Borel measure on $\mathbb{X}_j$.

Lemma 3.4. *For any $\omega \in P(\mathbb{T}_j^d)'$, the measure $\widetilde{\overline{\omega}}$ is left Γ_j-invariant, and the corresponding measure $\overline{\omega}$ on $\mathbb{X}_j$ satisfies, for any Borel set $E \subset \mathbb{X}_j$,*

$$\overline{\omega}(E) = \int_{F_d}\int_{\mathbb{T}_j^d} I\big(x(U)A \in E\big)\,d\omega(U)\,d\nu(A), \tag{3.10}$$

where F_d is any fixed Borel set in $\mathrm{SL}_d(\mathbb{R})$ which is a fundamental domain for $\mathrm{SL}_d(\mathbb{Z})\backslash \mathrm{SL}_d(\mathbb{R})$.

Note that (3.10) in particular shows that $\overline{\omega}$ is a probability measure, i.e. $\overline{\omega} \in P(\mathbb{X}_j)$, and so we indeed have a map as in (3.8).

Proof. Let $\omega \in P(\mathbb{T}_j^d)'$. In order to verify that $\widetilde{\overline{\omega}}$ is left Γ_j-invariant, it suffices to verify that for any Borel set $E \subset G_j$ we have $\widetilde{\overline{\omega}}(\mathrm{I}_M\,E) = \widetilde{\overline{\omega}}(E)$ for all $M \in \mathrm{M}_{r_j\times d}(\mathbb{Z})$ and $\widetilde{\overline{\omega}}(\gamma\,E) = \widetilde{\overline{\omega}}(E)$ for all $\gamma \in \mathrm{SL}_d(\mathbb{Z})$. The first of these two relations is immediate from (3.9) and the fact that $\widetilde{\omega}$ is $\mathrm{M}_{r_j\times d}(\mathbb{Z})$-invariant. The second relation follows by noticing that, for any $U \in \mathrm{M}_{r_j\times d}(\mathbb{R})$ and $A \in \mathrm{SL}_d(\mathbb{R})$, $\mathrm{I}_U\,A \in \gamma E$ holds if and only if $\gamma^{-1}\,\mathrm{I}_U\,A \in E$, viz., $\mathrm{I}_{U\gamma}\,\gamma^{-1}A \in E$, and then using the invariance of the Haar measure ν, and the fact that $\widetilde{\omega}$ is $\mathrm{SL}_d(\mathbb{Z})$-invariant, since $\omega \in P(\mathbb{T}_j^d)'$.

Next, in order to verify (3.10), note that if C_d is the set of matrices in $\mathrm{M}_{r_j\times d}(\mathbb{R})$ all of whose entries lie in $[0,1)$, then the set

$$F_d' := \{\mathrm{I}_U\,A \ : \ U \in C_d,\ A \in F_d\} \quad \subset G_j$$

is a fundamental domain for $\Gamma_j\backslash G_j$. Hence for any Borel set $E \subset \mathbb{X}_j$ we have

$$\overline{\omega}(E) = \widetilde{\overline{\omega}}(\pi^{-1}(E) \cap F_d') = \int_{F_d}\int_{C_d} I\big(\mathrm{I}_U\,A \in \pi^{-1}(E)\big)\,d\widetilde{\omega}(U)\,d\nu(A),$$

which is the same as (3.10). □

For later use we record a few simple properties of the map $\omega \mapsto \overline{\omega}$ just defined.

Lemma 3.5. *For any $\omega \in P(\mathbb{T}_j^d)'$, the measure $\overline{\omega}$ is $\mathrm{SL}_d(\mathbb{R})$ invariant, and $\iota_*\overline{\omega} = \nu$.*

(Here, by abuse of notation, we write ν also for the measure on $\mathrm{SL}_d(\mathbb{Z})\backslash \mathrm{SL}_d(\mathbb{R})$ induced by the Haar measure ν.)

Proof. It is obvious from (3.9) that $\widetilde{\overline{\omega}}$ is right $\mathrm{SL}_d(\mathbb{R})$ invariant; hence also $\overline{\omega}$ is right $\mathrm{SL}_d(\mathbb{R})$ invariant. The fact that $\iota_*\overline{\omega} = \nu$ is immediate from (3.10). □

Lemma 3.6. *The map in (3.8) is continuous. (Here $P(\mathbb{T}_j^d)'$ is equipped with the subspace topology from $P(\mathbb{T}_j^d)$.)*

Proof. Recall that $P(S)$ is metrizable for any separable and metrizable topological space S [1, pp. 72–73]; in particular $P(\mathbb{T}_j^d)$ and $P(\mathbb{X}_j)$ are metrizable. Hence it suffices to prove that if (ω_k) is any sequence in $P(\mathbb{T}_j^d)$ converging to $\omega \in P(\mathbb{T}_j^d)$, we have $\overline{\omega}_k \to \overline{\omega}$ in $P(\mathbb{X}_j)$. To prove this, we have to prove that $\overline{\omega}_k(\varphi) \to \overline{\omega}(\varphi)$ for any $\varphi \in \mathrm{C}_b(\mathbb{X}_j)$; but it is a well-known fact that it suffices to prove this for $\varphi \in \mathrm{C}_c(\mathbb{X}_j)$. Thus for a fixed $\varphi \in \mathrm{C}_c(\mathbb{X}_j)$, our task is to prove that

$$\int_{\mathbb{T}_j^d} \int_{F_d} \varphi(x(U)A)\, d\nu(A)\, d\omega_k(U)$$

converges to the corresponding integral with ω, as $k \to \infty$. The fact that φ has compact support in $\mathbb{X}_j$ implies that $\alpha(U) := \int_{F_d} \varphi(x(U)A)\, d\nu(A)$ is a continuous function on $\mathbb{T}_j^d$.

Now the task is to prove that $\int_{\mathbb{T}_j^d} \alpha\, d\omega_k \to \int_{\mathbb{T}_j^d} \alpha\, d\omega$, and this holds by definition since $\omega_k \to \omega$ in $P(\mathbb{T}_j^d)$. □

Finally, for any $V \in \mathbb{T}_j^d$ we will now identify the measure $\overline{\omega^{(V)}} \in P(\mathbb{X}_j)$ as an invariant measure on a certain homogeneous submanifold of $\mathbb{X}_j$. Given $V \in \mathbb{T}_j^d$, it follows from Lemma 3.3 that there exists some $X \in \mathrm{M}_{r_j \times d}(\mathbb{Q})$ such that $V - \pi(X) \in \big(\mathbb{S}_j^{(V)}\big)^{\circ\, d}$, and this means that we can choose some $\widetilde{V} \in X + \big(L_j^{(V)}\big)^d \subset \mathrm{M}_{r_j \times d}(\mathbb{R})$ with $\pi(\widetilde{V}) = V$. Let $L = L_j^{(V)}$ and recall the definition of $\mathrm{S}_L(\mathbb{R})$ in (3.1). It follows that the $\mathrm{S}_L(\mathbb{R})$-orbit of the point $x(V)$ in $\mathbb{X}_j$ can be expressed as:

$$x(V) \cdot \mathrm{S}_L(\mathbb{R}) = \Gamma_j \backslash \Gamma_j \, \mathrm{I}_{\widetilde{V}} \, \mathrm{S}_L(\mathbb{R}) = \Gamma_j \backslash \Gamma_j \, \mathrm{I}_X \, \mathrm{S}_L(\mathbb{R}). \tag{3.11}$$

By the proof of Lemma 3.1 (applied to $\widetilde{V}$ in place of $U_j^{(\boldsymbol{q})}$), the orbit in (3.11) is a closed embedded submanifold of $\mathbb{X}_j$ which carries a unique $\mathrm{S}_L(\mathbb{R})$-invariant probability measure.

Proposition 3.7. *For any $V \in \mathbb{T}_j^d$, the unique $\mathrm{S}_{L_j^{(V)}}(\mathbb{R})$-invariant probability measure on the orbit $x(V) \cdot \mathrm{S}_{L_j^{(V)}}(\mathbb{R})$ equals $\overline{\omega_j^{(V)}}$. In particular, for any $\boldsymbol{q} \in \mathbb{R}^d$ we have $\mu_j^{(\boldsymbol{q})} = \overline{\omega_j^{(\boldsymbol{q})}}$.*

Proof. Note that the second part of the proposition is an immediate consequence of the first part, since, by the definition in Section 3, $\mu_j^{(\boldsymbol{q})}$ is the $\mathrm{S}_{L_j^{(\boldsymbol{q})}}$-invariant probability measure on the orbit $x(\pi(U_j^{(\boldsymbol{q})})) \cdot \mathrm{S}_{L_j^{(\boldsymbol{q})}}(\mathbb{R})$ in $\mathbb{X}_j$, and furthermore we have $L_j^{(\boldsymbol{q})} = L_j^{(\pi(U_j^{(\boldsymbol{q})}))}$ and $\omega_j^{(\boldsymbol{q})} = \omega_j^{(\pi(U_j^{(\boldsymbol{q})}))}$.

To prove the first part of the proposition, set $\omega := \omega_j^{(V)}$, and let X, L and $\widetilde{V}$ be as in (3.11). Since $\overline{\omega} = \overline{\omega_j^{(V)}}$ by definition is the probability measure on $\mathbb{X}_j$ which corresponds to the Γ_j-invariant measure $\widetilde{\overline{\omega}}$ on G_j given by (3.9), it suffices to verify that $\widetilde{\overline{\omega}}$ is supported on $\Gamma_j \, \mathrm{I}_{\widetilde{V}} \, \mathrm{S}_L(\mathbb{R})$ and that $\widetilde{\overline{\omega}}$ is right $\mathrm{S}_L(\mathbb{R})$ invariant. However, it follows from (3.9) that

$$\mathrm{supp}(\widetilde{\overline{\omega}}) = \{\mathrm{I}_U\, A \,:\, U \in \pi^{-1}(\mathcal{O}_j(V)),\ A \in \mathrm{SL}_d(\mathbb{R})\}, \tag{3.12}$$

and here we have

$$\pi^{-1}(\mathcal{O}_j(V)) = \{(\widetilde{V} + W)\gamma \,:\, \gamma \in \mathrm{SL}_d(\mathbb{Z}),\ W \in L^d + \mathrm{M}_{r_j \times d}(\mathbb{Z})\} \tag{3.13}$$

(cf. (3.7)). Note here that both L^d and $\mathrm{M}_{r_j \times d}(\mathbb{Z})$ are $\mathrm{SL}_d(\mathbb{Z})$-invariant subsets of $\mathrm{M}_{r_j \times d}(\mathbb{R})$. Using (3.12), (3.13) and $\mathrm{I}_{(\widetilde{V}+W)\gamma}\, A = \gamma^{-1}\, \mathrm{I}_{\widetilde{V}+W}\, \gamma A$ it follows that $\mathrm{supp}(\widetilde{\overline{\omega}}) = \Gamma_j \, \mathrm{I}_{\widetilde{V}} \, \mathrm{S}_L(\mathbb{R})$, as desired.

Finally we verify that $\widetilde{\overline{\omega}}$ is right $\mathrm{S}_L(\mathbb{R})$-invariant. We have noted that $\widetilde{\overline{\omega}}$ is right $\mathrm{SL}_d(\mathbb{R})$ invariant (cf. Lemma 3.5); hence it suffices to verify that $\widetilde{\overline{\omega}}$ is right I_W-invariant for every $W \in L^d$. However this also follows from (3.9), by noticing that $\mathrm{I}_U\, A\, \mathrm{I}_W = \mathrm{I}_{U+WA^{-1}}\, A$ for all $U \in \mathrm{M}_{r_j \times d}(\mathbb{R})$ and $A \in \mathrm{SL}_d(\mathbb{R})$, and using $WA^{-1} \in L^d$ and the fact that $\widetilde{\omega}$ is invariant under L^d-translations (since $\omega = \omega_j^{(V)}$ is invariant under $(\mathbb{S}_j(V)^\circ)^d$-translations). □

3.2. **A Siegel integration formula for the measure $\overline{\omega}$.** The classical Siegel integration formula [26] states that for any measurable function $f : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ (or for any $f \in \mathrm{C}_c(\mathbb{R}^d)$), defining the *Siegel transform* $\widehat{f}$ on $\mathrm{SL}_d(\mathbb{Z})\backslash \mathrm{SL}_d(\mathbb{R})$ by

$$\widehat{f}(\mathrm{SL}_d(\mathbb{Z})g) = \sum_{\boldsymbol{m}\in\mathbb{Z}^d} f(\boldsymbol{m}g) \qquad (g \in \mathrm{SL}_d(\mathbb{R})),$$

then

$$\int_{\mathrm{SL}_d(\mathbb{Z})\backslash \mathrm{SL}_d(\mathbb{R})} \widehat{f}\, d\nu = \int_{\mathbb{R}^d} f(\boldsymbol{x})\, d\boldsymbol{x} + f(\boldsymbol{0}).$$

The following proposition gives an analogous formula involving the measure $\overline{\omega}$, for any given $\omega \in P(\mathbb{T}_j^d)'$. This formula will be used later in our proof of [P2] (uniform spherical equidistribution); see the proof of Lemma 6.7 below.

Proposition 3.8. *Let $\psi = (j, i) \in \Psi$. For any Borel measurable function $f \in \mathbb{R}^d \to \mathbb{R}_{\geq 0}$, we define its "$\psi$-Siegel transform" $\widehat{f}_\psi : \mathbb{X}_j \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ by*

$$\widehat{f}_\psi(\Gamma_j g) = \sum_{\boldsymbol{m}\in\mathbb{Z}^d} f\big(c_\psi \cdot (\boldsymbol{m}\,\mathsf{a}_i(g))\big) \qquad (g \in G_j). \tag{3.14}$$

Then for any $\omega \in P(\mathbb{T}_j^d)'$ we have

$$\int_{\mathbb{X}_j} \widehat{f}_\psi\, d\overline{\omega} = c_\psi^{-d} \int_{\mathbb{R}^d} f(\boldsymbol{x})\, d\boldsymbol{x} + \omega(\widetilde{\mathsf{r}}_i^{-1}(\{\boldsymbol{0}\}))f(\boldsymbol{0}), \tag{3.15}$$

as an equality of extended real numbers in $\mathbb{R}_{\geq 0} \cup \{+\infty\}$.

Remark 3.1. The reason for the formula (3.14) is that this corresponds to the "ψ-part" of the union in (2.18). Note that $\widehat{f_\psi}$ is well-defined, in the sense that the right hand side of (3.14) remains the same if g is replaced by γg for any $\gamma \in \Gamma_j$.

Proof. Let $\omega \in P(\mathbb{T}_j^d)'$ be given. Define the Borel measure μ on $\mathbb{R}^d$ by $\mu(B) := \int_{\mathbb{X}_j} \widehat{(\chi_B)_\psi}\, d\overline{\omega}$ for any Borel set $B \subset \mathbb{R}^d$. Let us prove that μ is finite on compact sets, so that μ is a Radon measure. For this, it suffices to prove that $\mu(B) < \infty$ for any ball $B = \mathcal{B}_R^d$. In this case we have, for all $g \in G_j$:

$$\widehat{(\chi_B)_\psi}(\Gamma_j g) = \#\big(B \cap c_\psi(\mathbb{Z}^d\mathsf{a}_i(g))\big) \ll \#\big(B \cap c_\psi \mathbb{Z}^d \iota(g)\big), \tag{3.16}$$

by [9, Proposition 5] (the implied constant in the last bound depends only on d). Furthermore,

$$\int_{\mathbb{X}_j} \#\big(B \cap c_\psi \mathbb{Z}^d \iota(g)\big)\, d\overline{\omega}(g) = \int_{\mathrm{SL}_d(\mathbb{Z})\backslash \mathrm{SL}_d(\mathbb{R})} \#\big(B \cap c_\psi \mathbb{Z}^d g\big)\, d\nu(g) = 1 + \mathrm{vol}(c_\psi^{-1} B) < \infty,$$

where we applied Siegel's original integration formula [26].

Next, one verifies that for any Borel set $B \subset \mathbb{R}^d$ and any $T \in \mathrm{SL}_d(\mathbb{R})$ and $g \in G_j$, we have $\widehat{(\chi_{BT})_\psi}(\Gamma_j g) = \widehat{(\chi_B)_\psi}(\Gamma_j g T^{-1})$. Using this identity, and the fact that $\overline{\omega}$ is $\mathrm{SL}_d(\mathbb{R})$ invariant (cf. Lemma 3.5), it follows that $\mu(BT) = \mu(B)$. Hence μ is $\mathrm{SL}_d(\mathbb{R})$ invariant. By the well-known characterization of $\mathrm{SL}_d(\mathbb{R})$ invariant Radon measures on $\mathbb{R}^d$, it follows that

$$\mu = c_1\delta_{\boldsymbol{0}} + c_2\,\mathrm{vol} \tag{3.17}$$

for some constants $c_1, c_2 \in \mathbb{R}_{\geq 0}$, where vol is the Lebesgue measure on $\mathbb{R}^d$. Here

$$c_1 = \mu(\{\boldsymbol{0}\}) = \int_{F_d} \int_{\mathbb{T}_j^d} \widehat{(\chi_{\{\boldsymbol{0}\}})_\psi}\big(x(U)A\big)\, d\omega(U)\, d\nu(A) = \omega(\widetilde{\mathsf{r}}_i^{-1}(\{\boldsymbol{0}\})),$$

where the second equality holds by (3.10), and the last equality holds since $\widehat{(\chi_{\{\mathbf{0}\}})_\psi}\big(x(U)A\big) = I(\widetilde{\mathsf{r}}_i(U) = \mathbf{0})$ for all $U \in \mathbb{T}_j^d$ and all $A \in \mathrm{SL}_d(\mathbb{R})$. Furthermore, it follows from (3.17) that

$$c_2 = \lim_{R\to\infty} \frac{\mu(\mathcal{B}_R^d)}{\mathrm{vol}(\mathcal{B}_R^d)} = \lim_{R\to\infty} \int_{\mathbb{X}_j} \frac{\widehat{(\chi_{\mathcal{B}_R^d})_\psi}}{\mathrm{vol}(\mathcal{B}_R^d)}\, d\overline{\omega},$$

and here for each fixed point in $\Gamma_j g \in \mathbb{X}_j$, the value of the integrand tends to c_ψ^{-d} as $R \to \infty$, by (3.14) and since $c_\psi \cdot (\mathbb{Z}^d \mathsf{a}_i(g))$ is an affine lattice of covolume c_ψ^d in $\mathbb{R}^d$. Hence by Lebesgue's dominated convergence theorem, the application of which is justified by the bound

$$\frac{\widehat{(\chi_{\mathcal{B}_R^d})_\psi}(\Gamma_j g)}{\mathrm{vol}(\mathcal{B}_R^d)} \ll \frac{\#(\mathcal{B}_R^d \cap c_\psi \mathbb{Z}^d \iota(g))}{\mathrm{vol}(\mathcal{B}_R^d)} \ll \frac{\#(\mathcal{B}_1^d \cap c_\psi \mathbb{Z}^d \iota(g))}{\mathrm{vol}(\mathcal{B}_1^d)}$$

(where we first used (3.16) and then [9, Proposition 4]) and Siegel's integration formula, [26], we conclude that $c_2 = c_\psi^{-d}$.

We have thus proved that (3.15) holds whenever f is the characteristic function of a Borel set $B \subset \mathbb{R}^d$. By taking finite linear combinations, it follows that (3.15) holds whenever f is a simple function, and finally by a standard approximation argument using the monotone convergence theorem one proves (3.15) in the general case. □

4. Limit behaviour of $\omega_j^{(\boldsymbol{q})}$

As we have mentioned, it will be important for us to understand how $\omega_j^{(\boldsymbol{q})}$ varies as $\boldsymbol{q}$ varies in $\mathcal{P}$; in particular we are interested in the behaviour of $\omega_j^{(\boldsymbol{q})}$ as $\boldsymbol{q}$ tends to infinity within $\mathcal{P}$. The main result of the present section states that, under a certain admissibility assumption on the presentation of the given point set $\mathcal{P}$ as a union of grids, it holds for each $\psi \in \Psi$ that the measure $\omega_j^{(\boldsymbol{q})}$ tends to a unique limit measure $\omega_j^\psi \in P(\mathbb{T}_j^d)'$ as $\boldsymbol{q}$ tends to infinity within a full density subset of $\mathcal{L}_\psi$.

4.1. The limit spaces L_j^ψ and L_j.

As a first step, we will study the limiting behaviour of the linear spaces $L_j^{(\boldsymbol{q})}$ as $\boldsymbol{q}$ varies through a fixed grid $\mathcal{L}_\psi$. For each $\psi \in \Psi$ and each $j \in \{1, \ldots, N\}$ we will define a certain rational space $L_j^\psi \subset \mathbb{R}^{r_j}$, and will then prove that this space is, in a certain sense, the limit of the spaces $L_j^{(\boldsymbol{q})}$ as $\boldsymbol{q}$ tends to infinity within a full density subset of $\mathcal{L}_\psi$.

Let $\psi \in \Psi$ and $j \in \{1, \ldots, N\}$. Recall that $L_j^{(\boldsymbol{q})} = L_j^{(\pi(U_j^{(\boldsymbol{q})}))}$. We next introduce some auxiliary notation for keeping track of $U_j^{(\boldsymbol{q})}$ for $\boldsymbol{q}$ varying in $\mathcal{L}_\psi$. Recall that we have expressed $\mathcal{L}_\psi$ as $\mathcal{L}_\psi = c_\psi(\mathbb{Z}^d + \boldsymbol{w}_\psi)M_{j_\psi}$; cf. (2.5). We set

$$\boldsymbol{c}_j^\psi = c_\psi \begin{pmatrix} c_{j,1}^{-1} \\ \vdots \\ c_{j,r_j}^{-1} \end{pmatrix} \in \mathbb{R}^{r_j}, \tag{4.1}$$

and let $c_{j,i}^\psi = c_\psi c_{j,i}^{-1}$ be the ith coordinate of $\boldsymbol{c}_j^\psi$ $(i = 1, \ldots, r_j)$. We also set

$$T_j^\psi := M_{j_\psi} M_j^{-1} \in \mathrm{SL}_d(\mathbb{R}),$$

and define the matrix W_j^ψ by specifying its row vectors:

$$W_j^\psi \in \mathrm{M}_{r_j \times d}(\mathbb{R}); \qquad \mathsf{r}_i(W_j^\psi) = \boldsymbol{w}_{j,i} - c_{j,i}^\psi \boldsymbol{w}_\psi T_j^\psi \qquad (i = 1, \ldots, r_j). \tag{4.2}$$

The point of this notation is that for and arbitrary point $\boldsymbol{q}$ in $\mathcal{L}_\psi = c_\psi(\mathbb{Z}^d + \boldsymbol{w}_\psi)M_{j_\psi}$, we can now write:

$$U_j^{(\boldsymbol{q})} = W_j^\psi - \boldsymbol{c}_j^\psi \boldsymbol{m} T_j^\psi \qquad \text{when } \boldsymbol{q} = c_\psi(\boldsymbol{m} + \boldsymbol{w}_\psi)M_{j_\psi} \ (\forall \boldsymbol{m} \in \mathbb{Z}^d). \tag{4.3}$$

This is verified by comparing the two matrices row by row, using (2.19) and (4.2).

Let L_j^ψ be the rational subspace of $\mathbb{R}^{r_j}$ given by

$$L_j^\psi := \begin{cases} \mathfrak{L}\big(\{\boldsymbol{c}_j^\psi\} \cup \{W_{j,\ell}^\psi \,:\, \ell \in \{1,\ldots,d\}\}\big) & \text{if } j = j_\psi \\ \mathfrak{L}\big(\mathbb{R}\boldsymbol{c}_j^\psi \cup \{W_{j,\ell}^\psi \,:\, \ell \in \{1,\ldots,d\}\}\big) & \text{if } j \neq j_\psi, \end{cases} \tag{4.4}$$

where $W_{j,1}^\psi, \ldots, W_{j,d}^\psi \in \mathbb{R}^{r_j}$ are the column vectors of W_j^ψ.

Lemma 4.1. *Let $\psi \in \Psi$ and $j \in \{1,\ldots,N\}$. Let $\mathcal{L}'$ be any subgrid of $\mathcal{L}_\psi$. Then*

$$L_j^\psi = \mathfrak{L}\big(\{U_{j,\ell}^{(\boldsymbol{q})} \,:\, \boldsymbol{q} \in \mathcal{L}',\ \ell \in \{1,\ldots,d\}\}\big),$$

where $U_{j,1}^{(\boldsymbol{q})}, \ldots, U_{j,d}^{(\boldsymbol{q})} \in \mathbb{R}^{r_j}$ are the column vectors of $U_j^{(\boldsymbol{q})}$.

Proof. Set $S_0 = \{W_{j,\ell}^\psi \,:\, \ell \in \{1,\ldots,d\}\}$ and $S = \{\boldsymbol{c}_j^\psi\} \cup S_0$ if $j = j_\psi$; otherwise $S = \mathbb{R}\boldsymbol{c}_j^\psi \cup S_0$. Also set $S' := \{U_{j,\ell}^{(\boldsymbol{q})} \,:\, \boldsymbol{q} \in \mathcal{L}',\ \ell \in \{1,\ldots,d\}\}$. Then the task is to prove that $\mathfrak{L}(S) = \mathfrak{L}(S')$. Let $T_{j,1}^\psi, \ldots, T_{j,d}^\psi \in \mathbb{R}^{r_j}$ be the column vectors of T_j^ψ. Then by (4.3) we have

$$U_{j,\ell}^{(\boldsymbol{q})} = W_{j,\ell}^\psi - \boldsymbol{c}_j^\psi \boldsymbol{m} T_{j,\ell}^\psi \qquad \text{when } \boldsymbol{q} = c_\psi(\boldsymbol{m} + \boldsymbol{w}_\psi)M_{j_\psi} \ (\forall \boldsymbol{m} \in \mathbb{Z}^d,\ \ell \in \{1,\ldots,d\}). \tag{4.5}$$

It is immediate from (4.5) that $S' \subset \langle S \rangle$, and so $\mathfrak{L}(S') \subset \mathfrak{L}(S)$. It remains to prove that $\mathfrak{L}(S) \subset \mathfrak{L}(S')$.

Let $\mathcal{L}''$ be the inverse image of $\mathcal{L}'$ under the bijection $\boldsymbol{m} \mapsto c_\psi(\boldsymbol{m} + \boldsymbol{w}_\psi)M_{j_\psi}$ from $\mathbb{Z}^d$ onto $\mathcal{L}_\psi$. Then $\mathcal{L}''$ is a coset of a full rank sublattice of $\mathbb{Z}^d$; hence there is some $n \in \mathbb{Z}^+$ and some $\boldsymbol{m}_0 \in \mathbb{Z}^d$ such that $\mathcal{L}''$ contains $\boldsymbol{m}_0 + n\mathbb{Z}^d$, and so, by (4.5):

$$W_{j,\ell}^\psi - \boldsymbol{c}_j^\psi \boldsymbol{m} T_{j,\ell}^\psi \in S', \qquad \forall \boldsymbol{m} \in \boldsymbol{m}_0 + n\mathbb{Z}^d,\ \ell \in \{1,\ldots,d\}. \tag{4.6}$$

By Lemma 3.3 there is some $m \in \mathbb{Z}^+$ such that $S' \subset m^{-1}\mathbb{Z}^{r_j} + \mathfrak{L}(S')$. Hence, since every point in $\mathbb{Z}^d$ can be written as an affine linear combination of points in $\boldsymbol{m}_0 + n\mathbb{Z}^d$, with all weights in $n^{-1}\mathbb{Z}$, it follows that

$$W_{j,\ell}^\psi - \boldsymbol{c}_j^\psi \boldsymbol{m} T_{j,\ell}^\psi \in \mathcal{A}, \qquad \forall \boldsymbol{m} \in \mathbb{Z}^d,\ \ell \in \{1,\ldots,d\}, \tag{4.7}$$

where $\mathcal{A} := (nm)^{-1}\mathbb{Z}^{r_j} + \mathfrak{L}(S')$ (this is a closed subgroup of $\mathbb{R}^{r_j}$). We will prove that $S \subset \mathcal{A}$; by Lemma 3.3 this implies $\mathfrak{L}(S) \subset \mathfrak{L}(S')$, and so the proof of Lemma 4.1 will be complete. By taking $\boldsymbol{m} = \boldsymbol{0}$ in (4.7) it follows that $S_0 \subset \mathcal{A}$; hence it suffices to prove that $\boldsymbol{c}_j^\psi \in \mathcal{A}$ if $j = j_\psi$, and $\mathbb{R}\boldsymbol{c}_j^\psi \subset \mathcal{A}$ if $j \neq j_\psi$.

We denote by $\boldsymbol{e}_k$ the kth standard unit vector in $\mathbb{R}^d$. If $j = j_\psi$ then T_j^ψ is the identity matrix, and applying (4.7) with $\ell = 1$ and $\boldsymbol{m} \in \{\boldsymbol{e}_1, \boldsymbol{0}\}$ it follows that $\mathcal{A}$ contains both $W_{j,\ell}^\psi - \boldsymbol{c}_j^\psi$ and $W_{j,\ell}^\psi$; hence also $\boldsymbol{c}_j^\psi \in \mathcal{A}$.

Next assume $j \neq j_\psi$. Then $T_j^\psi \notin \mathcal{S}$ (cf. (2.8)), and so there exist two (non-zero) entries of the matrix T_j^ψ which have an irrational ratio; say entries k,ℓ and k',ℓ', respectively. Writing $T_j^\psi = (t_{r,s})$ we thus have $t_{k,\ell}/t_{k',\ell'} \notin \mathbb{Q}$, which implies that the set $\{a t_{k,\ell} + b t_{k',\ell'} \,:\, a,b \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. By considering the difference of two arbitrary vectors as in (4.7) we have $\boldsymbol{c}_j^\psi \boldsymbol{m} T_{j,\ell}^\psi \in \mathcal{A}$ for all $\boldsymbol{m} \in \mathbb{Z}^d$. Taking here $\boldsymbol{m} = \boldsymbol{e}_k$ gives $t_{k,\ell}\boldsymbol{c}_j^\psi \in \mathcal{A}$. Similarly we also have $t_{k',\ell'}\boldsymbol{c}_j^\psi \in \mathcal{A}$, and hence $(a t_{k,\ell} + b t_{k',\ell'})\boldsymbol{c}_j^\psi \in \mathcal{A}$ for all $a,b \in \mathbb{Z}$. Hence since $\mathcal{A}$ is closed, we have $\mathbb{R}\boldsymbol{c}_j^\psi \subset \mathcal{A}$. □

Lemma 4.2. *Let $\psi \in \Psi$ and $j \in \{1,\ldots,N\}$. For every $\boldsymbol{q} \in \mathcal{L}_\psi$ we have $L_j^{(\boldsymbol{q})} \subset L_j^\psi$.*

Proof. By definition, $L_j^{(\boldsymbol{q})} = \mathfrak{L}(\{U_{j,\ell}^{(\boldsymbol{q})} : \ell \in \{1,\ldots,d\}\})$; hence the statement follows from Lemma 4.1, and the fact that $\mathfrak{L}(S)$ is increasing in S. □

Our goal is now to prove Lemma 4.4 below, which says that, in an appropriate sense, the space $L_j^{(\boldsymbol{q})}$ approaches L_j^{ψ} as $\boldsymbol{q}$ tends to infinity within a full density subset of $\mathcal{L}_\psi$. (Cf. also Remark 4.3 below.) We will need the following auxiliary lemma. We denote by " · " the standard scalar product in the space $\mathbb{R}^{r_j}$.

Lemma 4.3. *Let $\psi \in \Psi$ and $j \in \{1,\ldots,N\}$. For any non-empty subset $S \subset \mathbb{R}^{r_j}$ and any vector $\boldsymbol{a} \in \mathbb{Q}^{r_j}$, we have $\boldsymbol{a} \perp \mathfrak{L}(S)$ if and only if $\{\boldsymbol{a}\cdot\boldsymbol{v} : \boldsymbol{v} \in S\} \subset n^{-1}\mathbb{Z}$ for some $n \in \mathbb{Z}^+$.*

Proof. The case $\boldsymbol{a} = \boldsymbol{0}$ is trivial; hence from now on we assume $\boldsymbol{a} \neq \boldsymbol{0}$. The statement of the lemma is invariant under rescaling of $\boldsymbol{a}$ by a non-zero rational number; hence we may assume that $\boldsymbol{a} \in \mathbb{Z}^{r_j}$ and $\gcd(a_1,\ldots,a_{r_j}) = 1$. It follows that $\{\boldsymbol{a}\cdot\boldsymbol{m} : \boldsymbol{m} \in \mathbb{Z}^{r_j}\} = \mathbb{Z}$, and hence $\{\boldsymbol{a}\cdot\boldsymbol{m} : \boldsymbol{m} \in n^{-1}\mathbb{Z}^{r_j}\} = n^{-1}\mathbb{Z}$ for any $n \in \mathbb{Z}^+$. Therefore, if $\{\boldsymbol{a}\cdot\boldsymbol{v} : \boldsymbol{v} \in S\} \subset n^{-1}\mathbb{Z}$ then $S \subset n^{-1}\mathbb{Z} + \boldsymbol{a}^\perp$, and so by Lemma 3.3, $\mathfrak{L}(S) \subset \boldsymbol{a}^\perp$, i.e. $\boldsymbol{a} \perp \mathfrak{L}(S)$. Conversely, assume $\boldsymbol{a} \perp \mathfrak{L}(S)$. By Lemma 3.3 there is some $n \in \mathbb{Z}^+$ such that $S \subset n^{-1}\mathbb{Z}^{r_j} + \mathfrak{L}(S) \subset n^{-1}\mathbb{Z}^{r_j} + \boldsymbol{a}^\perp$, and this implies $\{\boldsymbol{a}\cdot\boldsymbol{v} : \boldsymbol{v} \in S\} \subset n^{-1}\mathbb{Z}$. □

Remark 4.1. If S is a *finite* subset of $\mathbb{R}^{r_j}$, then Lemma 4.3 implies that a vector $\boldsymbol{a} \in \mathbb{Q}^r$ is orthogonal to $\mathfrak{L}(S)$ if and only if $\{\boldsymbol{a}\cdot\boldsymbol{v} : \boldsymbol{v} \in S\} \subset \mathbb{Q}$. Furthermore, if $S = \mathbb{R}\boldsymbol{c} \cup S'$ for some vector $\boldsymbol{c} \in \mathbb{R}^r$ and a finite subset $S' \subset \mathbb{R}^r$, then Lemma 4.3 implies that a vector $\boldsymbol{a} \in \mathbb{Q}^r$ is orthogonal to $\mathfrak{L}(S)$ if and only if $\boldsymbol{a} \perp \boldsymbol{c}$ and $\{\boldsymbol{a}\cdot\boldsymbol{v} : \boldsymbol{v} \in S'\} \subset \mathbb{Q}$.

Lemma 4.4. *Let $\psi \in \Psi$ and $j \in \{1,\ldots,N\}$. For every $\boldsymbol{a} \in \mathbb{Z}^{r_j}$ with $\boldsymbol{a} \not\perp L_j^\psi$, we have*

$$\#\{\boldsymbol{q} \in \mathcal{L}_\psi \cap \mathcal{B}_R^d : \boldsymbol{a} \perp L_j^{(\boldsymbol{q})}\} \ll R^{d-1} \qquad \text{as } R \to \infty. \tag{4.8}$$

Proof. As before, let us parametrize the points in $\mathcal{L}_\psi = c_\psi(\mathbb{Z}^d + \boldsymbol{w}_\psi)M_{j_\psi}$ as $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{m}) = c_\psi(\boldsymbol{m} + \boldsymbol{w}_\psi)M_{j_\psi}$, with $\boldsymbol{m}$ running through $\mathbb{Z}^d$. We will prove the following bound, which clearly implies (4.8):

$$\#\{\boldsymbol{m} \in \mathbb{Z}^d \cap [-R,R]^d : \boldsymbol{a} \perp L_j^{(\boldsymbol{q}(\boldsymbol{m}))}\} \leq (2R+1)^{d-1} \qquad (\forall R > 0). \tag{4.9}$$

Using $L_j^{(\boldsymbol{q})} = \mathfrak{L}(\{U_{j,\ell}^{(\boldsymbol{q})} : \ell \in \{1,\ldots,d\}\})$ together with (4.5) and Lemma 4.3, the bound (4.9) can equivalently be stated as:

$$\#\big\{\boldsymbol{m} \in \mathbb{Z}^d \cap [-R,R]^d : \boldsymbol{a}\cdot\big(W_{j,\ell}^\psi - \boldsymbol{c}_j^\psi \boldsymbol{m} T_{j,\ell}^\psi\big) \in \mathbb{Q} \ \forall \ell \in \{1,\ldots,d\}\big\} \leq (2R+1)^{d-1}. \tag{4.10}$$

We are assuming that $\boldsymbol{a}$ is not orthogonal to L_j^ψ. By (4.4) and Lemma 4.3 (cf. also Remark 4.1), this implies that

$$\begin{cases} \text{If } j = j_\psi: & \boldsymbol{a}\cdot\boldsymbol{c}_j^\psi \notin \mathbb{Q} \ \text{ or } \ \boldsymbol{a}\cdot W_{j,\ell}^\psi \notin \mathbb{Q} \text{ for some } \ell \in \{1,\ldots,d\}, \\ \text{If } j \neq j_\psi: & \boldsymbol{a}\cdot\boldsymbol{c}_j^\psi \neq 0 \ \text{ or } \ \boldsymbol{a}\cdot W_{j,\ell}^\psi \notin \mathbb{Q} \text{ for some } \ell \in \{1,\ldots,d\}. \end{cases}$$

Let us first assume $j \neq j_\psi$. If $\boldsymbol{a}\cdot\boldsymbol{c}_j^\psi = 0$ then $\boldsymbol{a}\cdot W_{j,\ell}^\psi \notin \mathbb{Q}$ for some ℓ, and it follows that the set in (4.10) is *empty*, for all R. Hence we may assume that $\boldsymbol{a}\cdot\boldsymbol{c}_j^\psi \neq 0$. Now $j \neq j_\psi$ implies that $T_j^\psi \notin \mathcal{S}$, which means that there exist two (non-zero) entries of the matrix T_j^ψ which have an irrational ratio. Hence there exist $k, \ell \in \{1,\ldots,d\}$ such that $(\boldsymbol{a}\cdot\boldsymbol{c}_j^\psi)(\boldsymbol{e}_k T_{j,\ell}^\psi) \notin \mathbb{Q}$. This implies that, for any R, the set in (4.10) contains at most one point $\boldsymbol{m}$ along any line parallel to $\boldsymbol{e}_k$. Hence the bound in (4.10) holds.

Next assume $j = j_\psi$. Then $T_j^\psi = \mathrm{I}$. If $\boldsymbol{a}\cdot\boldsymbol{c}_j^\psi \in \mathbb{Q}$ then $\boldsymbol{a}\cdot W_{j,\ell}^\psi \notin \mathbb{Q}$ for some $\ell \in \{1,\ldots,d\}$, and it follows that the set in (4.10) is *empty*, for all R. Hence we may assume that $\boldsymbol{a}\cdot\boldsymbol{c}_j^\psi \notin \mathbb{Q}$. Then for each $\ell \in \{1,\ldots,d\}$ there is at most one integer m such that $\boldsymbol{a}\cdot(W_{j,\ell}^\psi - m\boldsymbol{c}_j^\psi) \in \mathbb{Q}$, and so the set in (4.10) contains at most one point, and the bound in (4.10) holds. □

We end this section by introducing a certain rational space $L_j \subset \mathbb{R}^{r_j}$ which equals $L_j^{(\boldsymbol{q})}$ for $\boldsymbol{q}$ *generic within* $\mathbb{R}^d$ (cf. Lemma 4.5 below). These spaces are closely analogous to (but in general larger than) the spaces L_j^ψ introduced above, and they appear in the explicit description of the macroscopic limit measure μ^{g} which we discuss in Section 6.4 below.

Given $j \in \{1, \ldots, N\}$, we set

$$\widetilde{\boldsymbol{c}}_j = \begin{pmatrix} c_{j,1}^{-1} \\ \vdots \\ c_{j,r_j}^{-1} \end{pmatrix} \in \mathbb{R}^{r_j}. \tag{4.11}$$

(Thus $\widetilde{\boldsymbol{c}}_j = c_\psi^{-1} \boldsymbol{c}_j^\psi$ for every $\psi \in \Psi$.) We also define the matrix W_j through

$$W_j \in \mathrm{M}_{r_j \times d}(\mathbb{R}); \qquad \mathsf{r}_i(W_j) = \boldsymbol{w}_{j,i} \qquad (i = 1, \ldots, r_j). \tag{4.12}$$

The point of this notation is that we now have (cf. (2.19))

$$U_j^{(\boldsymbol{q})} = W_j - \widetilde{\boldsymbol{c}}_j \boldsymbol{q} M_j^{-1}, \qquad \forall \boldsymbol{q} \in \mathbb{R}^d. \tag{4.13}$$

Let $W_{j,1}, \ldots, W_{j,d} \in \mathbb{R}^{r_j}$ be the column vectors of W_j. Finally we define:

$$L_j := \mathfrak{L}\big(\mathbb{R}\widetilde{\boldsymbol{c}}_j \cup \{W_{j,\ell} \,:\, \ell \in \{1, \ldots, d\}\}\big). \tag{4.14}$$

Lemma 4.5. *Let $j \in \{1, \ldots, N\}$. For every $\boldsymbol{q} \in \mathbb{R}^d$ we have $L_j^{(\boldsymbol{q})} \subset L_j$, and for all except countably many $\boldsymbol{q} \in \mathbb{R}^d$ we even have $L_j^{(\boldsymbol{q})} = L_j$.*

Proof. Let $S_0 = \{W_{j,\ell} \,:\, \ell \in \{1, \ldots, d\}$ and $S = \mathbb{R}\widetilde{\boldsymbol{c}}_j \cup S_0$, so that $L_j = \mathfrak{L}(S)$. Recall that for every $\boldsymbol{q} \in \mathbb{R}^d$ we have $L_j^{(\boldsymbol{q})} = \mathfrak{L}(\{U_{j,\ell}^{(\boldsymbol{q})} \,:\, \ell \in \{1, \ldots, d\}\})$; and by (4.13), $U_{j,\ell}^{(\boldsymbol{q})} = W_{j,\ell} - \widetilde{\boldsymbol{c}}_j \boldsymbol{q} M'_{j,\ell}$, where $M'_{j,\ell} \in \mathbb{R}^{r_j}$ is the ℓth column vector of M_j^{-1}. Here $\widetilde{\boldsymbol{c}}_j \boldsymbol{q} M'_{j,\ell} \in \mathbb{R}\widetilde{\boldsymbol{c}}_j$, and thus $U_{j,\ell}^{(\boldsymbol{q})} \in \langle S \rangle$ for every ℓ. Therefore $L_j^{(\boldsymbol{q})} \subset L_j$.

Next, since both $L_j^{(\boldsymbol{q})}$ and L_j are rational subspaces of $\mathbb{R}^{r_j}$, if $L_j^{(\boldsymbol{q})} \subsetneq L_j$ then there exists some $\boldsymbol{a} \in \mathbb{Z}^{r_j}$ satisfying $\boldsymbol{a} \perp L_j^{(\boldsymbol{q})}$ but $\boldsymbol{a} \not\perp L_j$. By Lemma 4.3 (cf. also Remark 4.1) this means that

$$\boldsymbol{a} \cdot (W_{j,\ell} - \widetilde{\boldsymbol{c}}_j \boldsymbol{q} M'_{j,\ell}) \in \mathbb{Q}, \qquad \forall \ell \in \{1, \ldots, d\}, \tag{4.15}$$

but either $\boldsymbol{a} \not\perp \widetilde{\boldsymbol{c}}_j$ or $\boldsymbol{a} \cdot W_{j,\ell} \notin \mathbb{Q}$ for some ℓ. Clearly this is not possible if $\boldsymbol{a} \perp \widetilde{\boldsymbol{c}}_j$; hence we must have $\boldsymbol{a} \not\perp \widetilde{\boldsymbol{c}}_j$. Noticing also that the condition (4.15) is equivalent to[7] $\boldsymbol{a}^{\mathrm{T}}(W_j - \widetilde{\boldsymbol{c}}_j \boldsymbol{q} M_j^{-1}) \in \mathbb{Q}^d$, we conclude that:

$$\{\boldsymbol{q} \in \mathbb{R}^d \,:\, L_j^{(\boldsymbol{q})} \subsetneq L_j\} = \bigcup_{\substack{\boldsymbol{a} \in \mathbb{Z}^{r_j} \\ (\boldsymbol{a} \not\perp \widetilde{\boldsymbol{c}}_j)}} \bigcup_{\boldsymbol{b} \in \mathbb{Q}^d} \big\{\boldsymbol{q} \in \mathbb{R}^d \,:\, \boldsymbol{a}^{\mathrm{T}}(W_j - \widetilde{\boldsymbol{c}}_j \boldsymbol{q} M_j^{-1}) = \boldsymbol{b}\big\}. \tag{4.16}$$

But for every $\boldsymbol{a} \in \mathbb{Z}^{r_j}$ with $\boldsymbol{a} \not\perp \widetilde{\boldsymbol{c}}_j$ and every $\boldsymbol{b} \in \mathbb{Q}^d$, the set $\big\{\boldsymbol{q} \in \mathbb{R}^d : \boldsymbol{a}^{\mathrm{T}}(W_j - \widetilde{\boldsymbol{c}}_j \boldsymbol{q} M_j^{-1}) = \boldsymbol{b}\big\}$ consists of exactly one point, namely $\boldsymbol{q} = (\boldsymbol{a}^{\mathrm{T}} \widetilde{\boldsymbol{c}}_j)^{-1}(\boldsymbol{a}^{\mathrm{T}} W_j - \boldsymbol{b}) M_j$. Hence the set in (4.16) is countable, and the lemma is proved. □

Lemma 4.6. *For any $\psi \in \Psi$ and $j \in \{1, \ldots, N\}$ we have $L_j^\psi \subset L_j$, and if $j \neq j_\psi$ then even $L_j^\psi = L_j$.*

Proof. Recall that $\boldsymbol{c}_j^\psi = c_\psi \widetilde{\boldsymbol{c}}_j$; hence $\mathbb{R}\boldsymbol{c}_j^\psi = \mathbb{R}\widetilde{\boldsymbol{c}}_j$. Furthermore, by comparing the definitions (4.2) and (4.12) we note that $W_{j,\ell}^\psi = W_{j,\ell} - \boldsymbol{c}_j^\psi \boldsymbol{w}_\psi T_{j,\ell}^\psi \in W_{j,\ell} + \mathbb{R}\boldsymbol{c}_j^\psi = W_{j,\ell} + \mathbb{R}\widetilde{\boldsymbol{c}}_j$. Hence the two sets $\mathbb{R}\boldsymbol{c}_j^\psi \cup \{W_{j,\ell}^\psi \,:\, \ell \in \{1, \ldots, d\}\}$ and $\mathbb{R}\widetilde{\boldsymbol{c}}_j \cup \{W_{j,\ell} \,:\, \ell \in \{1, \ldots, d\}\}$ generate the same subgroups of $\mathbb{R}^{r_j}$. In view of this fact, the lemma now follows by inspecting the definitions of L_j^ψ and L_j, (4.4) and (4.14). □

[7]Note: We view $\boldsymbol{a}$ as a column vector, or equivalently as an $r_j \times 1$ matrix; hence $\boldsymbol{a}^{\mathrm{T}}$ is a $1 \times r_j$ matrix.

4.2. **The limit measures ω_j^ψ and ω_j.** The following condition will be of crucial importance for us. It involves the vectors $\boldsymbol{c}_j^\psi \in \mathbb{R}^{r_j}$ and the subspaces L_j^ψ which were defined in (4.1) and (4.4) in the previous section.

Definition 4.1. We say that a presentation of $\mathcal{P}$ as in (2.6) (and subject to (2.7) and (2.8)) is *admissible* if $\boldsymbol{c}_j^\psi \in L_j^\psi + \mathbb{Z}^{r_j}$ for all $\psi = (j,i) \in \Psi$.

It should be carefully noted that the condition in Definition 4.1 only involves vectors $\boldsymbol{c}_j^\psi$ and spaces L_j^ψ for pairs of ψ and j with $j = j_\psi$.

As we will see below, the admissibility of the presentation of $\mathcal{P}$ is a necessary (as well as sufficient) condition to ensure that all the measures $\omega_j^{(\boldsymbol{q})}$ have a unique generic limit as $\boldsymbol{q}$ tends to infinity within any of the grids $\mathcal{L}_\psi$. Luckily, it turns out that any point set $\mathcal{P}$ which is a finite union of grids possesses an admissible presentation; we will prove this in Section 4.3 below.

From now on we assume that the given presentation (2.6) *of $\mathcal{P}$ is admissible.*

We will start by defining, for any $\psi \in \Psi$ and $j \in \{1, \ldots, N\}$, a measure $\omega_j^\psi \in P(\mathbb{T}_j^d)'$. We will then prove that ω_j^ψ is in fact the generic limit of the measures $\omega_j^{(\boldsymbol{q})}$ as $\boldsymbol{q}$ tends to infinity within the grid $\mathcal{L}_\psi$. As we will see, the assumption about admissibility is needed already for the definition of ω_j^ψ to make sense. (Cf. the proof of Lemma 4.7 below.)

Given $\psi \in \Psi$ and $j \in \{1, \ldots, N\}$ we set $\mathbb{S}_j^\psi := \pi(L_j^\psi)$. This is a closed subtorus of $\mathbb{T}_j$, since L_j^ψ is a rational subspace of $\mathbb{R}^{r_j}$. Furthermore, we pick an arbitrary point $\boldsymbol{q} \in \mathcal{L}_\psi$, and define:

$$\widetilde{\mathbb{S}}_j^\psi := \mathbb{S}_j^{(\boldsymbol{q})} + \mathbb{S}_j^\psi \subset \mathbb{T}_j; \qquad \text{and} \qquad \mathcal{O}_j^\psi := \bigcup_{\gamma \in \mathrm{SL}_d(\mathbb{Z})} (\pi(U_j^{(\boldsymbol{q})})\gamma + (\mathbb{S}_j^\psi)^d) \subset \mathbb{T}_j^d. \tag{4.17}$$

Lemma 4.7. *Both $\widetilde{\mathbb{S}}_j^\psi$ and $\mathcal{O}_j^\psi$ are well-defined, i.e. the expressions in* (4.17) *are independent of the choice of $\boldsymbol{q}$. We have $\mathcal{O}_j^{(\boldsymbol{q})} \subset \mathcal{O}_j^\psi$ for all $\boldsymbol{q} \in \mathcal{L}_\psi$. Furthermore, $\widetilde{\mathbb{S}}_j^\psi$ is a closed subgroup of $\mathbb{T}_j$ whose connected component subgroup equals $\mathbb{S}_j^\psi$, and $\mathcal{O}_j^\psi$ is a union of some of the connected components of $(\widetilde{\mathbb{S}}_j^\psi)^d$.*

(It should be noted that since $(\widetilde{\mathbb{S}}_j^\psi)^d$ is compact, its total number of connected components is finite.)

In view of Lemma 4.7, we may now define $\omega_j^\psi \in P(\mathbb{T}_j^d)$ to be the restriction to $\mathcal{O}_j^\psi$ of the Haar measure on $(\widetilde{\mathbb{S}}_j^\psi)^d$, normalized so that $\omega_j^\psi(\mathcal{O}_j^\psi) = 1$. Note that ω_j^ψ is $\mathrm{SL}_d(\mathbb{Z})$-invariant by construction, i.e. we actually have $\omega_j^\psi \in P(\mathbb{T}_j^d)'$.

Proof of Lemma 4.7. Let us first prove that

$$U_{j,\ell}^{(\boldsymbol{q})} - U_{j,\ell}^{(\boldsymbol{q}')} \in L_j^\psi + \mathbb{Z}^{r_j}, \qquad \forall \boldsymbol{q}, \boldsymbol{q}' \in \mathcal{L}_\psi,\ \forall \ell \in \{1, \ldots, d\}. \tag{4.18}$$

Indeed, pick $\boldsymbol{m}, \boldsymbol{m}' \in \mathbb{Z}^d$ so that $\boldsymbol{q} = c_\psi(\boldsymbol{m} + \boldsymbol{w}_\psi)M_{j_\psi}$ and $\boldsymbol{q}' = c_\psi(\boldsymbol{m}' + \boldsymbol{w}_\psi)M_{j_\psi}$; then by (4.5) we have $U_{j,\ell}^{(\boldsymbol{q})} - U_{j,\ell}^{(\boldsymbol{q}')} = \boldsymbol{c}_j^\psi(\boldsymbol{m}' - \boldsymbol{m})T_{j,\ell}^\psi$. If $j \neq j_\psi$ then $\mathbb{R}\boldsymbol{c}_j^\psi \subset L_j^\psi$ by (4.4) and hence $U_{j,\ell}^{(\boldsymbol{q})} - U_{j,\ell}^{(\boldsymbol{q}')} \in L_j^\psi$ for each ℓ. On the other hand if $j = j_\psi$ then $T_j^\psi = I$ and so $U_{j,\ell}^{(\boldsymbol{q})} - U_{j,\ell}^{(\boldsymbol{q}')} \in \mathbb{Z}\,\boldsymbol{c}_j^\psi$, and using the assumption that $\mathcal{P}$ *is admissible* (cf. Def. 4.1), this implies that (4.18) holds.

In order to prove that $\widetilde{\mathbb{S}}_j^\psi$ is well-defined it suffices to verify that for any two $\boldsymbol{q}, \boldsymbol{q}' \in \mathcal{L}_\psi$ we have $\mathbb{S}_j^{(\boldsymbol{q}')} \subset \mathbb{S}_j^{(\boldsymbol{q})} + \mathbb{S}_j^\psi$. However this follows from the definition of $\mathbb{S}_j^{(\boldsymbol{q}')}$ and the fact that, by (4.18), $\pi(U_{j,\ell}^{(\boldsymbol{q}')}) \in \pi(U_{j,\ell}^{(\boldsymbol{q})}) + \mathbb{S}_j^\psi \subset \mathbb{S}_j^{(\boldsymbol{q})} + \mathbb{S}_j^\psi$. Note also that (4.18) implies that $\pi(U_j^{(\boldsymbol{q})}) - \pi(U_j^{(\boldsymbol{q}')}) \in (\mathbb{S}_j^\psi)^d$; and this implies that $\mathcal{O}_j^\psi$ is well-defined.

Now consider an arbitrary point $\boldsymbol{q} \in \mathcal{L}_\psi$. Recall that $L_j^{(\boldsymbol{q})} \subset L_j^\psi$, by Lemma 4.2; hence $(\mathbb{S}_j^{(\boldsymbol{q})})^\circ \subset \mathbb{S}_j^\psi$, and by inspection in (3.1) and (4.17) this implies that $\mathcal{O}_j^{(\boldsymbol{q})} \subset \mathcal{O}_j^\psi$. Recall also that $\mathbb{S}_j^{(\boldsymbol{q})}$ is a finite union of $(\mathbb{S}_j^{(\boldsymbol{q})})^\circ$-cosets in $\mathbb{T}_j$; therefore $\widetilde{\mathbb{S}}_j^\psi = \mathbb{S}_j^{(\boldsymbol{q})} + \mathbb{S}_j^\psi$ is a finite union of $\mathbb{S}_j^\psi$-cosets. Hence $\widetilde{\mathbb{S}}_j^\psi$ is indeed a closed subgroup of $\mathbb{T}_j$ whose connected component subgroup equals $\mathbb{S}_j^\psi$. Finally, from the definition of $\mathcal{O}_j^\psi$ and the fact that $\pi(U_j^{(\boldsymbol{q})}) \in (\mathbb{S}_j^{(\boldsymbol{q})})^d$, it follows that $\mathcal{O}_j^\psi \subset (\widetilde{\mathbb{S}}_j^\psi)^d$; and $\mathcal{O}_j^\psi$ is by definition a union of $(\mathbb{S}_j^\psi)^d$-cosets, i.e. a union of some of the connected components of $(\widetilde{\mathbb{S}}_j^\psi)^d$. □

This is a convenient point to interject the definition of a measure $\omega_j^{\mathsf{g}} \in P(\mathbb{T}_j^d)'$, which is closely analogous to ω_j^ψ, and which we will need later for the explicit description of the macroscopic limit measure μ^{g} which we discuss in Section 6.4 below.

Given $j \in \{1, \ldots, N\}$ we set $\mathbb{S}_j := \pi(L_j)$. This is a closed subtorus of $\mathbb{T}_j$. Picking an arbitrary $\boldsymbol{q} \in \mathbb{R}^d$, we define:

$$\widetilde{\mathbb{S}}_j := \mathbb{S}_j^{(\boldsymbol{q})} + \mathbb{S}_j \qquad \text{and} \qquad \mathcal{O}_j := \bigcup_{\gamma \in \mathrm{SL}_d(\mathbb{Z})} (\pi(U_j^{(\boldsymbol{q})})\gamma + \mathbb{S}_j^d) \subset \mathbb{T}_j^d. \tag{4.19}$$

Lemma 4.8. *Both $\widetilde{\mathbb{S}}_j$ and $\mathcal{O}_j$ are well-defined, i.e. the expressions in (4.19) are independent of the choice of $\boldsymbol{q}$. We have $\mathcal{O}_j^{(\boldsymbol{q})} \subset \mathcal{O}_j$ for all $\boldsymbol{q} \in \mathbb{R}^d$. Furthermore, $\widetilde{\mathbb{S}}_j$ is a closed subgroup of $\mathbb{T}_j$ whose connected component subgroup equals $\mathbb{S}_j$, and $\mathcal{O}_j$ is a union of some of the connected components of $\widetilde{\mathbb{S}}_j^d$.*

In view of Lemma 4.8 we may now define $\omega_j^{\mathsf{g}} \in P(\mathbb{T}_j^d)'$ to be the restriction to $\mathcal{O}_j$ of Haar measure on $\widetilde{\mathbb{S}}_j^d$, normalized so that $\omega_j^{\mathsf{g}}(\mathcal{O}_j) = 1$.

Proof. It follows from (4.13) that for all $\boldsymbol{q}, \boldsymbol{q}' \in \mathbb{R}^d$ we have $U_{j,\ell}^{(\boldsymbol{q})} - U_{j,\ell}^{(\boldsymbol{q}')} \in \overline{\mathbb{R}\boldsymbol{c}_j \subset L_j}$, and thus $\pi(U_{j,\ell}^{(\boldsymbol{q})}) - \pi(U_{j,\ell}^{(\boldsymbol{q}')}) \in \mathbb{S}_j$, for every $\ell \in \{1, \ldots, d\}$. Recall that $\mathbb{S}_j^{(\boldsymbol{q})} = \langle \pi(U_{j,1}^{(\boldsymbol{q})}), \ldots, \pi(U_{j,d}^{(\boldsymbol{q})}) \rangle$. It follows that $\mathbb{S}_j^{(\boldsymbol{q}')} \subset \mathbb{S}_j^{(\boldsymbol{q})} + \mathbb{S}_j$ for any $\boldsymbol{q}, \boldsymbol{q}' \in \mathbb{R}^d$. Hence we get that $\widetilde{\mathbb{S}}_j$ is well-defined. It also follows that $\pi(U_j^{(\boldsymbol{q})}) - \pi(U_j^{(\boldsymbol{q}')}) \in \mathbb{S}_j^d$ for any $\boldsymbol{q}, \boldsymbol{q}' \in \mathbb{R}^d$; hence $\mathcal{O}_j$ is well-defined. The proof of the remaining assertions is essentially the same as in Lemma 4.7. □

Lemma 4.9. *For all except at most countably many $\boldsymbol{q} \in \mathbb{R}^d$ we have $(\mathbb{S}_j^{(\boldsymbol{q})})^\circ = \mathbb{S}_j$, $\mathbb{S}_j^{(\boldsymbol{q})} = \widetilde{\mathbb{S}}_j$, $\mathcal{O}_j^{(\boldsymbol{q})} = \mathcal{O}_j$, and $\omega_j^{(\boldsymbol{q})} = \omega_j^{\mathsf{g}}$.*

Proof. We will prove that the stated equalities hold whenever $L_j^{(\boldsymbol{q})} = L_j$; by Lemma 4.5 this gives the statement of the present lemma. Thus assume that $L_j^{(\boldsymbol{q})} = L_j$. Then $\big(\mathbb{S}_j^{(\boldsymbol{q})}\big)^\circ = \pi(L_j^{(\boldsymbol{q})}) = \pi(L_j) = \mathbb{S}_j$. In particular $\mathbb{S}_j \subset \mathbb{S}_j^{(\boldsymbol{q})}$, and so and $\mathbb{S}_j^{(\boldsymbol{q})} = \mathbb{S}_j^{(\boldsymbol{q})} + \mathbb{S}_j = \widetilde{\mathbb{S}}_j$. By comparing (4.19) with (3.7) (applied with $V = \pi(U_j^{(\boldsymbol{q})})$), we have $\mathcal{O}_j^{(\boldsymbol{q})} = \mathcal{O}_j$. Finally now also $\omega_j^{(\boldsymbol{q})} = \omega_j^{\mathsf{g}}$ is immediate from the definitions. □

Lemma 4.10. *For any $\psi \in \Psi$ and $j \in \{1, \ldots, N\}$, if $j \neq j_\psi$ then $\mathbb{S}_j^\psi = \mathbb{S}_j$, $\widetilde{\mathbb{S}}_j^\psi = \widetilde{\mathbb{S}}_j$, $\mathcal{O}_j^\psi = \mathcal{O}_j$ and $\omega_j^\psi = \omega_j^{\mathsf{g}}$.*

Proof. If $j \neq j_\psi$. then $L_j^\psi = L_j$ by Lemma 4.6, and the stated equalities are immediate from this fact, by inspection in the relevant definitions. □

We now return to the discussion on the measures ω_j^ψ. In Proposition 4.12 and Remark 4.2 below, we will prove that ω_j^ψ is the generic limit of the measures $\omega_j^{(\boldsymbol{q})}$ as $\boldsymbol{q}$ tends to infinity within the grid $\mathcal{L}_\psi$. We will make use of the following auxiliary lemma.

Lemma 4.11. *Let $\psi \in \Psi$, $j \in \{1, \ldots, N\}$ and $\boldsymbol{q} \in \mathcal{L}_\psi$. Let $m = m(\boldsymbol{q})$ be the number of $(\mathbb{S}_j^{(\boldsymbol{q})})^{\circ\, d}$-cosets contained in $\mathcal{O}_j(\boldsymbol{q})$, and let n be the number of $(\mathbb{S}_j^{\psi})^d$-cosets contained in $\mathcal{O}_j^\psi$. Then n divides m, and every $(\mathbb{S}_j^{\psi})^d$-coset contained in $\mathcal{O}_j^\psi$ contains exactly m/n distinct $(\mathbb{S}_j^{(\boldsymbol{q})})^{\circ\, d}$-cosets.*

Proof. Let $U := \pi(U_j^{(\boldsymbol{q})}) \in (\mathbb{S}_j^{(\boldsymbol{q})})^d$, and let Λ be the stabilizer of the coset $U + (\mathbb{S}_j^{(\boldsymbol{q})})^{\circ\, d}$, for the action of $\mathrm{SL}_d(\mathbb{Z})$ on the finite group $(\mathbb{S}_j^{(\boldsymbol{q})})^d / (\mathbb{S}_j^{(\boldsymbol{q})})^{\circ\, d}$. It then follows from the definition in (3.1) that $\mathcal{O}_j^{(\boldsymbol{q})}$ equals the disjoint union of the cosets $U\gamma + (\mathbb{S}_j^{(\boldsymbol{q})})^{\circ\, d}$, when γ runs through any set of representatives for $\Lambda \backslash \mathrm{SL}_d(\mathbb{Z})$. In particular we have $m = \#(\Lambda \backslash \mathrm{SL}_d(\mathbb{Z}))$. Similarly let Λ' be the stabilizer of the coset $U + (\mathbb{S}_j^\psi)^d$ in $(\widetilde{\mathbb{S}}_j^\psi)^d / (\mathbb{S}_j^\psi)^d$; then by (4.17), $\mathcal{O}_j^\psi$ equals the disjoint union of the cosets $U\gamma + (\mathbb{S}_j^\psi)^d$, when γ runs through any set of representatives for $\Lambda' \backslash \mathrm{SL}_d(\mathbb{Z})$; and in particular we have $n = \#(\Lambda' \backslash \mathrm{SL}_d(\mathbb{Z}))$. Note that $\Lambda \subset \Lambda'$, since $(\mathbb{S}_j^{(\boldsymbol{q})})^\circ \subset \mathbb{S}_j^\psi$. Therefore n divides m, and the last statement of the lemma follows from the fact that each right Λ'-coset in $\mathrm{SL}_d(\mathbb{Z})$ contains exactly m/n right Λ-cosets. $\square$

Proposition 4.12. *Let $\psi \in \Psi$ and $j \in \{1, \ldots, N\}$, and let U be any open neighbourhood of ω_j^ψ in $P(\mathbb{T}_j^d)$. Then the set $\{\boldsymbol{q} \in \mathcal{L}_\psi \,:\, \omega_j^{(\boldsymbol{q})} \notin U\}$ has density zero.*

Proof. To each matrix $A \in \mathrm{M}_{r_j \times d}(\mathbb{Z})$ corresponds a character χ_A on $\mathbb{T}_j^d = \mathrm{M}_{r_j \times d}(\mathbb{R}) / \mathrm{M}_{r_j \times d}(\mathbb{Z})$ given by $\chi_A(X) = e^{2\pi i \, \mathrm{Tr}(AX^t)}$, and every character on $\mathbb{T}_j^d$ can be so expressed. Using the fact that the set of finite linear combinations of characters on $\mathbb{T}_j^d$ is dense in $C(\mathbb{T}_j^d)$, it follows that there exists a finite subset $S \subset \mathrm{M}_{r_j \times d}(\mathbb{Z})$ and some $\varepsilon > 0$ such that U contains the set

$$\{\mu \in P(\mathbb{T}_j^d) \,:\, |\mu(\chi_A) - \omega_j^\psi(\chi_A)| < \varepsilon \quad \forall A \in S\}.$$

Hence, using also the fact that a finite union of density zero sets again has density zero, it suffices to prove that for any fixed $A \in \mathrm{M}_{r_j \times d}(\mathbb{Z})$, the set

$$\{\boldsymbol{q} \in \mathcal{L}_\psi \,:\, |\omega_j^{(\boldsymbol{q})}(\chi_A) - \omega_j^\psi(\chi_A)| \geq \varepsilon\} \tag{4.20}$$

has density zero.

For each $\boldsymbol{q} \in \mathcal{L}_\psi$, let $\nu_j^{(\boldsymbol{q})} \in P(\mathbb{T}_j^d)$ be the normalized Haar measure on $(\mathbb{S}_j^{(\boldsymbol{q})})^{\circ\, d}$, and let $R(\boldsymbol{q})$ be a set of representatives containing one point in each $(\mathbb{S}_j^{(\boldsymbol{q})})^{\circ\, d}$-coset contained in $\mathcal{O}_j^{(\boldsymbol{q})}$ (recall that $\mathcal{O}_j^{(\boldsymbol{q})}$ is a finite union of $(\mathbb{S}_j^{(\boldsymbol{q})})^{\circ\, d}$-cosets). Then by the definition of $\omega_j^{(\boldsymbol{q})}$ around (3.1), we have

$$\omega_j^{(\boldsymbol{q})} = \frac{1}{\# R(\boldsymbol{q})} \sum_{X \in R(\boldsymbol{q})} \nu_j^{(\boldsymbol{q})} \circ \tau_X^{-1},$$

where $\tau_X : \mathbb{T}_j^d \to \mathbb{T}_j^d$ denotes translation by X. Also let $\nu_j^\psi \in P(\mathbb{T}_j^d)$ be the normalized Haar measure on $(\mathbb{S}_j^\psi)^d$. Comparing (3.1) with (4.17) and the definition of ω_j^ψ on p. 22, and using Lemmas 4.7 and 4.11, we then have:

$$\omega_j^\psi = \frac{1}{\# R(\boldsymbol{q})} \sum_{X \in R(\boldsymbol{q})} \nu_j^{\psi_0} \circ \tau_X^{-1}. \tag{4.21}$$

It follows that

$$\big|\omega_j^{(\boldsymbol{q})}(\chi_A) - \omega_j^\psi(\chi_A)\big| \leq \frac{1}{\# R(\boldsymbol{q})} \sum_{X \in R(\boldsymbol{q})} \big|\nu_j^{(\boldsymbol{q})}(\chi_A \circ \tau_X) - \nu_j^\psi(\chi_A \circ \tau_X)\big| = \big|\nu_j^{(\boldsymbol{q})}(\chi_A) - \nu_j^\psi(\chi_A)\big|, \tag{4.22}$$

where the last equality holds since $\chi_A \circ \tau_X = \chi_A(X) \cdot \chi_A$ for any $X \in \mathbb{T}_j^d$.

Recall that $\big(\mathbb{S}_j^{(\boldsymbol{q})}\big)^{\circ\, d} = \pi\big(\big(L_j^{(\boldsymbol{q})}\big)^d\big)$; hence $\nu_j^{(\boldsymbol{q})}(\chi_A) = 1$ if A is orthogonal to $\big(L_j^{(\boldsymbol{q})}\big)^d$, otherwise $\nu_j^{(\boldsymbol{q})}(\chi_A) = 0$. Hence, letting $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_d \in \mathbb{Z}^{r_j}$ be the column vectors of A, we have $\nu_j^{(\boldsymbol{q})}(\chi_A) = 1$ if each $\boldsymbol{a}_\ell$ is orthogonal to $L_j^{(\boldsymbol{q})}$, otherwise $\nu_j^{(\boldsymbol{q})}(\chi_A) = 0$. Similarly $\nu_j^{\psi}(\chi_A) = 1$ if each $\boldsymbol{a}_\ell$ is orthogonal to L_j^{ψ}, otherwise $\nu_j^{(\boldsymbol{q})}(\chi_A) = 0$. Recall also that $L_j^{(\boldsymbol{q})} \subset L_j^{\psi}$ for all $\boldsymbol{q} \in \mathcal{L}_\psi$ (cf. Lemma 4.2). These facts, together with (4.22), imply that the set in (4.20) is contained in the following finite union:

$$\bigcup_{\substack{\ell \in \{1,\ldots,d\} \\ (\boldsymbol{a}_\ell \not\perp L_j^{\psi})}} \big\{\boldsymbol{q} \in \mathcal{L}_\psi \;:\; \boldsymbol{a}_\ell \perp L_j^{(\boldsymbol{q})}\big\}.$$

This set has density zero by Lemma 4.4. □

Remark 4.2. We have stated Proposition 4.12 in a way which will be convenient for later applications. However let us note that it can be reformulated as follows: *We have $\omega_j^{(\boldsymbol{q})} \to \omega_j^{\psi}$ in $P(\mathbb{T}_j^d)$ as $\boldsymbol{q}$ tends to infinity within a full density subset of $\mathcal{L}_\psi$.* Indeed, applying Proposition 4.12 to a sequence of shrinking open sets U containing no other element than ω_j^{ψ} in their intersection, and using Lemma 4.13 below, we conclude that there exists a subset $\mathcal{Z} \subset \mathcal{L}_\psi$ of density zero such that $\lim_{k\to\infty} \omega_j^{(\boldsymbol{q}_k)} = \omega_j^{\psi}$ for any sequence of points $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots$ in $\mathcal{L}_\psi \setminus \mathcal{Z}$ with $\|\boldsymbol{q}_k\| \to \infty$.

Lemma 4.13. *Let $\mathcal{L}$ be a grid in $\mathbb{R}^d$, and let $\mathcal{Z}_1, \mathcal{Z}_2, \ldots$ be subsets of $\mathcal{L}$ which each have density zero. Then there exists a subset $\mathcal{Z}$ of $\mathcal{L}$ of density zero which has the property that for every k there exists some $R > 0$ such that $\mathcal{Z}_k \setminus \mathcal{B}_R^d \subset \mathcal{Z}$.*

Proof. Set $z_j(R) = R^{-d}\#(\mathcal{Z}_j \cap \mathcal{B}_R^d)$. Using the fact that $z_j(R) \to 0$ for each j, we can pick $0 < R_1 < R_2 < \cdots$ such that for each $k \in \mathbb{Z}^+$ and each $R \geq R_k$ we have $\sum_{j=1}^k z_j(R) < k^{-1}$. Now set

$$\mathcal{Z} = \bigcup_{j=1}^{\infty} (\mathcal{Z}_j \setminus \mathcal{B}_{R_j}^d).$$

This set $\mathcal{Z}$ has density zero, since for each $k \in \mathbb{Z}^+$ and each $R \in [R_k, R_{k+1})$ we have $R^{-d}\#(\mathcal{Z} \cap \mathcal{B}_R^d) = R^{-d}\# \cup_{j=1}^k (\mathcal{Z}_j \cap \mathcal{B}_R^d \setminus \mathcal{B}_{R_j^d}) \leq \sum_{j=1}^k z_j(R) < k^{-1}$. It is also clear that $\mathcal{Z}$ has the last property stated in the lemma. □

Remark 4.3. By analogy with Remark 4.2 we also note that Lemma 4.4 implies that there exists a subset $\mathcal{Z} \subset \mathcal{L}_\psi$ of density zero such that for every $\boldsymbol{a} \in \mathbb{Z}^{r_j}$ with $\boldsymbol{a} \not\perp L_j^{\psi}$ there is some $R > 0$ such that $\boldsymbol{a} \not\perp L_j^{(\boldsymbol{q})}$ holds for all $\boldsymbol{q} \in \mathcal{L}_\psi \setminus \mathcal{Z}$ with $\|\boldsymbol{q}\| > R$. As in the proof of Proposition 4.12, this implies that the normalized Haar measure on $\big(\mathbb{S}_j^{(\boldsymbol{q})}\big)^{\circ} = \pi(L_j^{(\boldsymbol{q})}) \subset \mathbb{T}_j$ tends to the normalized Haar measure ν_j^{ψ} on $\mathbb{S}_j^{\psi} = \pi(L_j^{\psi})$ as $\boldsymbol{q}$ tends to infinity within $\mathcal{L}_\psi \setminus \mathcal{Z}$. In this sense, we may say that "$L_j^{(\boldsymbol{q})}$ approaches L_j^{ψ}" as $\boldsymbol{q}$ tends to infinity within a full density subset of $\mathcal{L}_\psi$.

4.3. **Admissible presentations of $\mathcal{P}$.**

In this section we will prove that an admissible presentation of $\mathcal{P}$ can always be obtained:

Proposition 4.14. *Let $\mathcal{P}$ be a finite union of grids. Then $\mathcal{P}$ possesses an admissible presentation, i.e. there exist $N \in \mathbb{Z}^+$, $r_1, \ldots, r_N \in \mathbb{Z}^+$, $M_1, \ldots, M_N \in \mathrm{SL}_d(\mathbb{R})$, and numbers $c_\psi \in \mathbb{R}$ and vectors $\boldsymbol{w}_\psi \in \mathbb{R}^d$ for $\psi \in \Psi$ (with Ψ as in* (2.4)*), such that $\mathcal{P}$ is given by* (2.6), (2.5), *and this presentation of $\mathcal{P}$ satisfies the admissibility condition in Definition 4.1.*

We will prove Proposition 4.14 by showing that there exists a presentation of $\mathcal{P}$ as in (2.6), (2.5), such that (2.7) and (2.8) hold and also

$$\boldsymbol{c}_j^{\psi} \in \mathfrak{L}\big(\{\boldsymbol{c}_j^{\psi}\}\big) + \mathbb{Z}^{r_j}, \qquad \forall \psi = (j,i) \in \Psi. \tag{4.23}$$

This implies that the presentation is admissible, since we always have $\mathfrak{L}\big(\{\boldsymbol{c}_j^{\psi}\}\big) \subset L_j^{\psi}$ for all $\psi = (j,i) \in \Psi$, because of (4.4) and the fact that $\mathfrak{L}(S)$ is increasing in S.

For any $r \geq 1$ and any vector $\boldsymbol{u} = (u_1, \ldots, u_r)^t \in \mathbb{R}^r_{>0}$, let us write $\widetilde{\boldsymbol{u}} := (u_1^{-1}, \ldots, u_r^{-1})^t \in \mathbb{R}^r_{>0}$. We call the vector $\boldsymbol{u} \in \mathbb{R}^r_{>0}$ *admissible* if $u_i \widetilde{\boldsymbol{u}} \in \mathfrak{L}(u_i \widetilde{\boldsymbol{u}}) + \mathbb{Z}^r$ for every $i \in \{1, \ldots, r\}$. In view of (4.1), we then have that (4.23) is equivalent the condition that $(c_{j,1}, \ldots, c_{j,r_j})^t$ is admissible, for every $j \in \{1, \ldots, N\}$.

To prove Proposition 4.14, we will start from an arbitrary presentation of $\mathcal{P}$ as obtained in Section 2.1, i.e. we assume that $\mathcal{P}$ is expressed as in (2.6), (2.5), and that the conditions (2.7) and (2.8) hold. If this presentation is not already admissible, then we will modify it by making use of the simple fact that for any positive integer q, we can express $\mathbb{Z}^d$ as the (disjoint) union of the grids $q\mathbb{Z}^d + \boldsymbol{\alpha}$, with $\boldsymbol{\alpha}$ running through the set $\{1, \ldots, q\}^d$; hence for any $\boldsymbol{w} \in \mathbb{R}^d$ and $M \in \mathrm{SL}_d(\mathbb{R})$, the grid $(\mathbb{Z}^d + \boldsymbol{w})M$ equals the union of the grids $(q\mathbb{Z}^d + \boldsymbol{\alpha} + \boldsymbol{w})M$. It follows that for any choice of positive integers q_ψ $(\psi \in \Psi)$, we have

$$\mathcal{P} = \bigcup_{\psi \in \Psi} \mathcal{L}_\psi = \bigcup_{\psi \in \Psi} c_\psi(\mathbb{Z}^d + \boldsymbol{w}_\psi) M_{j_\psi} = \bigcup_{\psi \in \Psi} \bigcup_{\boldsymbol{\alpha} \in \{1,\ldots,q_\psi\}^d} q_\psi c_\psi \big(\mathbb{Z}^d + q_\psi^{-1}(\boldsymbol{\alpha} + \boldsymbol{w}_\psi)\big) M_{j_\psi}. \tag{4.24}$$

In other words, we have obtained a new presentation of $\mathcal{P}$, analogous to the original one:

$$\mathcal{P} = \bigcup_{\vartheta \in \Theta} c'_\vartheta (\mathbb{Z}^d + \boldsymbol{w}'_\vartheta) M_{j_\vartheta}, \tag{4.25}$$

where

$$\Theta = \{(j,i) \ : \ j \in \{1, \ldots, N\}, \ i \in \{1, \ldots, r'_j\}\}$$

with

$$r'_j = \sum_{i=1}^{r_j} q_{j,i}^d \qquad (j = 1, \ldots, N),$$

and where

$$\boldsymbol{w}'_\vartheta = q_{\psi(\vartheta)}^{-1} (\boldsymbol{\alpha}_\vartheta + \boldsymbol{w}_{\psi(\vartheta)}); \qquad c'_\vartheta = q_{\psi(\vartheta)} c_{\psi(\vartheta)},$$

with $\psi(\vartheta) \in \Psi$ and $\boldsymbol{\alpha}_\vartheta \in \mathbb{Z}^d$ chosen in such a way that $j_{\psi(\vartheta)} = j_\vartheta$ for all $\vartheta \in \Theta$, and the map $\vartheta \mapsto \langle \psi(\vartheta), \boldsymbol{\alpha}_\vartheta \rangle$ is a bijection from Θ onto $\{\langle \psi, \boldsymbol{\alpha} \rangle \ : \ \psi \in \Psi, \ \boldsymbol{\alpha} \in \{1, \ldots, q_\psi\}^d\}$. It is obvious that the new presentation again satisfies the condition (2.8), since the matrices $M_1, \ldots, M_N$ are unchanged. One also verifies that the new presentation satisfies the analogue of the condition (2.7), so long as the integers q_ψ are chosen so that

$$\forall \psi, \psi' \in \Psi : \ \big[j_\psi = j_{\psi'} \text{ and } c_\psi = c_{\psi'} \big] \Rightarrow q_\psi = q_{\psi'}. \tag{4.26}$$

Hence, recalling our initial discussion, it remains to prove that we can choose the positive integers q_ψ subject to (4.26) in such a way that the analogue of (4.23) holds for the new presentation, i.e. so that $\big(c'_{j,1}, \ldots, c'_{j,r'_j}\big)^{\mathrm{T}}$ is admissible, for every $j \in \{1, \ldots, N\}$.

The proof of this fact is essentially completed by the following three lemmas.

Lemma 4.15. *Let $1 \leq r \leq r'$ and let $T \in \mathrm{M}_{r,r'}(\mathbb{Q})$. Then for any non-empty subset $S \subset \mathbb{R}^r$ we have $\mathfrak{L}(TS) = T\,\mathfrak{L}(S)$.*

(Here for any subset $A \subset \mathbb{R}^r$ we write $TA := \{T\boldsymbol{v} \ : \ \boldsymbol{v} \in A\}$. Recall also that we view vectors in $\mathbb{R}^r$ as column matrices; hence $T\boldsymbol{v} \in \mathbb{R}^{r'}$ for every $\boldsymbol{v} \in \mathbb{R}^r$.)

Proof. Both $\mathfrak{L}(TS)$ and $T\,\mathfrak{L}(S)$ are rational subspaces of $\mathbb{R}^{r'}$; hence it suffices to prove that the equivalence $[\boldsymbol{a} \perp \mathfrak{L}(TS) \Leftrightarrow \boldsymbol{a} \perp T\,\mathfrak{L}(S)]$ holds for all $\boldsymbol{a} \in \mathbb{Q}^{r'}$. However, $\boldsymbol{a} \perp T\,\mathfrak{L}(S)$ holds if and only if $\boldsymbol{a} \cdot T\boldsymbol{v} = 0$ for all $\boldsymbol{v} \in \mathfrak{L}(S)$, or equivalently $(T^{\mathrm{T}}\boldsymbol{a}) \cdot \boldsymbol{v} = 0$ for all $\boldsymbol{v} \in \mathfrak{L}(S)$. By Lemma 4.3, this holds if and only if there is some $n \in \mathbb{Z}^+$ such that $T^{\mathrm{T}}\boldsymbol{a} \cdot \boldsymbol{v} \in n^{-1}\mathbb{Z}$ for all $\boldsymbol{v} \in S$. On the other hand, Lemma 4.3 also gives that $\boldsymbol{a} \perp \mathfrak{L}(TS)$ holds if and only if there is some $n \in \mathbb{Z}^+$ such that $\boldsymbol{a} \cdot T\boldsymbol{v} \in n^{-1}\mathbb{Z}$ for all $\boldsymbol{v} \in S$, or equivalently $T^{\mathrm{T}}\boldsymbol{a} \cdot \boldsymbol{v} \in n^{-1}\mathbb{Z}$ for all $\boldsymbol{v} \in S$. Hence the equivalence is established. □

Lemma 4.16. *If the vectors* $\boldsymbol{u} = (u_1, \ldots, u_r)^{\mathrm{T}} \in \mathbb{R}^r_{>0}$ *and* $\boldsymbol{u}' = (u'_1, \ldots, u'_{r'})^{\mathrm{T}} \in \mathbb{R}^{r'}_{>0}$ *have the same set of coordinates, i.e.* $\{u_1, \ldots, u_r\} = \{u'_1, \ldots, u'_{r'}\}$*, then* $\boldsymbol{u}$ *is admissible if and only if* $\boldsymbol{u}'$ *is admissible.*

Proof. Assume that $\{u_1, \ldots, u_r\} = \{u'_1, \ldots, u'_{r'}\}$. This means that there exist uniquely determined matrices $T \in \mathrm{M}_{r,r'}(\mathbb{Z})$ and $T' \in \mathrm{M}_{r',r}(\mathbb{Z})$ with exactly one entry of 1 in each row and 0s elsewhere, such that $\boldsymbol{u}' = T\boldsymbol{u}$ and $\boldsymbol{u} = T'\boldsymbol{u}'$; thus also $\widetilde{\boldsymbol{u}'} = T\widetilde{\boldsymbol{u}}$ and $\widetilde{\boldsymbol{u}} = T'\widetilde{\boldsymbol{u}'}$. Assume that $\boldsymbol{u}$ is admissible. Then for every $i \in \{1, \ldots, r\}$ we have $u_i\widetilde{\boldsymbol{u}} \in \mathfrak{L}(u_i\widetilde{\boldsymbol{u}}) + \mathbb{Z}^r$. Multiplying this relation by T from the left, and using Lemma 4.15 and $T\mathbb{Z}^r \subset \mathbb{Z}^{r'}$, we obtain $u_i\widetilde{\boldsymbol{u}'} \in \mathfrak{L}(u_i\widetilde{\boldsymbol{u}'}) + \mathbb{Z}^{r'}$. This holds for all $i \in \{1, \ldots, r\}$, and for each $i' \in \{1, \ldots, r'\}$ there exists some $i \in \{1, \ldots, r\}$ such that $u'_{i'} = u_i$. Hence $\boldsymbol{u}'$ is admissible. The opposite implication is proved analogously, using T'. □

Lemma 4.17. *For any* $\boldsymbol{u} = (u_1, \ldots, u_r)^{\mathrm{T}} \in \mathbb{R}^r_{>0}$*, there exist positive integers* $q_1, \ldots, q_r$ *such that the vector* $(q_1u_1, \ldots, q_ru_r)^{\mathrm{T}}$ *is admissible.*

Proof. Given $\boldsymbol{q} = (q_1, \ldots, q_r)^{\mathrm{T}} \in \mathbb{Z}^r_{>0}$ we write $D_{\boldsymbol{q}} = \mathrm{diag}(q_1, \ldots, q_r)$, Then the task is to prove that we there exists some $\boldsymbol{q} \in \mathbb{Z}^r_{>0}$ such that $q_iu_iD_{\boldsymbol{q}}^{-1}\widetilde{\boldsymbol{u}} \in \mathfrak{L}(q_iu_iD_{\boldsymbol{q}}^{-1}\widetilde{\boldsymbol{u}}) + \mathbb{Z}^r$ for all $i \in \{1, \ldots, r\}$. By Lemma 4.15 applied for the matrix $q_iD_{\boldsymbol{q}}^{-1} \in \mathrm{M}_r(\mathbb{Q})$, this is equivalent to:

$$q_iu_i\widetilde{\boldsymbol{u}} \in \mathfrak{L}(u_i\widetilde{\boldsymbol{u}}) + D_{\boldsymbol{q}}\,\mathbb{Z}^r \qquad \text{for all } i \in \{1, \ldots, r\}. \tag{4.27}$$

For each i, $\mathbb{R}\widetilde{\boldsymbol{u}} + \mathfrak{L}(u_i\widetilde{\boldsymbol{u}})$ is a rational subspace of $\mathbb{R}^r$, and $u_i\widetilde{\boldsymbol{u}} \in \mathbb{Q}^r + \mathfrak{L}(u_i\widetilde{\boldsymbol{u}})$ by Lemma 3.3, and $\mathbb{R}\widetilde{\boldsymbol{u}} = \mathbb{R}u_i\widetilde{\boldsymbol{u}}$. But $\mathfrak{L}(\mathbb{R}\widetilde{\boldsymbol{u}})$ is the unique smallest rational subspace of $\mathbb{R}^r$ containing $\mathbb{R}\widetilde{\boldsymbol{u}}$; hence

$$\mathfrak{L}(\mathbb{R}\widetilde{\boldsymbol{u}}) \subset \mathbb{R}\widetilde{\boldsymbol{u}} + \mathfrak{L}(u_i\widetilde{\boldsymbol{u}}). \tag{4.28}$$

We will now describe a choice of $q_1, \ldots, q_r$ which makes (4.27) hold. Let $p_i : \mathbb{R}^r \to \mathbb{R}$ denote projection onto the ith coordinate. Note that $p_i(\mathbb{Z}^r \cap \mathfrak{L}(\mathbb{R}\widetilde{\boldsymbol{u}}))$ is a subgroup of $\mathbb{Z}$. By considering the expansion of $\widetilde{\boldsymbol{u}}$ with respect to a basis of $\mathfrak{L}(\mathbb{R}\widetilde{\boldsymbol{u}})$ consisting of vectors in $\mathbb{Z}^r$, it follows that $p_i(\mathbb{Z}^r \cap \mathfrak{L}(\mathbb{R}\widetilde{\boldsymbol{u}})) \neq \{0\}$ for each i, and hence there exist unique positive integers q_i such that

$$p_i(\mathbb{Z}^r \cap \mathfrak{L}(\mathbb{R}\widetilde{\boldsymbol{u}})) = q_i\mathbb{Z} \qquad \text{for all } i \in \{1, \ldots, r\}. \tag{4.29}$$

Choose vectors $\boldsymbol{h}^{(i)} \in \mathbb{Z}^r \cap \mathfrak{L}(\mathbb{R}\widetilde{\boldsymbol{u}})$ with $p_i(\boldsymbol{h}^{(i)}) = q_i$. We now claim that

$$q_iu_i\widetilde{\boldsymbol{u}} \in \mathfrak{L}(u_i\widetilde{\boldsymbol{u}}) + \boldsymbol{h}^{(i)} \qquad \text{for all } i \in \{1, \ldots, r\}. \tag{4.30}$$

This implies that (4.27) holds, since $\boldsymbol{h}^{(i)} \in \mathbb{Z}^r \cap \mathfrak{L}(\mathbb{R}\widetilde{\boldsymbol{u}}) \subset D_{\boldsymbol{q}}\mathbb{Z}^r$, where the last inclusion follows from (4.29).

In order to prove (4.30), let i be given, and set $\boldsymbol{w} := q_iu_i\widetilde{\boldsymbol{u}} - \boldsymbol{h}^{(i)}$. We have $\boldsymbol{w} \in \mathfrak{L}(\mathbb{R}\widetilde{\boldsymbol{u}})$, since $\widetilde{\boldsymbol{u}}$ and $\boldsymbol{h}^{(i)}$ lie in $\mathfrak{L}(\mathbb{R}\widetilde{\boldsymbol{u}})$; hence by (4.28) there exists some $t \in \mathbb{R}$ such that $\boldsymbol{w} - t\widetilde{\boldsymbol{u}} \in \mathfrak{L}(u_i\widetilde{\boldsymbol{u}})$. However, both $\boldsymbol{w}$ and $\mathfrak{L}(u_i\widetilde{\boldsymbol{u}})$ lie in the orthogonal complement of $\boldsymbol{e}_i$, the ith standard unit vector in $\mathbb{R}^r$; for $\boldsymbol{w}$ this holds since $\boldsymbol{e}_i \cdot \widetilde{\boldsymbol{u}} = u_i$ and $\boldsymbol{e}_i \cdot \boldsymbol{h}^{(i)} = p_i(\boldsymbol{h}^{(i)}) = q_i$, and for $\mathfrak{L}(u_i\widetilde{\boldsymbol{u}})$ it holds by Lemma 4.3, since $\boldsymbol{e}_i \cdot u_i\widetilde{\boldsymbol{u}} = 1$. On the other hand, $\widetilde{\boldsymbol{u}}$ is not orthogonal to $\boldsymbol{e}_i$. Hence we must have $t = 0$, i.e. $\boldsymbol{w} \in \mathfrak{L}(u_i\widetilde{\boldsymbol{u}})$, and so (4.30) holds. □

Now to finally prove Proposition 4.14 we proceed as follows, for each fixed $j \in \{1, \ldots, N\}$: Choose $0 < u_1 < \cdots < u_r$ so that $\{c_{j,1}, \ldots, c_{j,r_j}\} = \{u_1, \ldots, u_r\}$, and let $\tau : \{1, \ldots, r_j\} \to \{1, \ldots, r\}$ be the map so that $c_{j,i} = u_{\tau(i)}$ for all $i \in \{1, \ldots, r_j\}$; then by Lemma 4.17 there exist positive integers $q_1, \ldots, q_r$ such that the vector $(q_1 u_1, \ldots, q_r u_r)^{\mathrm{T}}$ is admissible. Setting now $q_{j,i} = q_{\tau(i)}$, Lemma 4.16 implies that the vector $\big(q_{j,1} c_{j,1}, \ldots, q_{j,r_j} c_{j,r_j}\big)^{\mathrm{T}}$ is admissible; and since $c'_{j,i} = q_{\psi(j,i)} c_{\psi(j,i)}$ for all $i = 1, \ldots, r'_j$, another application of Lemma 4.16 gives that also the vector $\big(c'_{j,1}, \ldots, c'_{j,r'_j}\big)^{\mathrm{T}}$ is admissible, which was exactly the desired condition. It also follows from our construction that $q_{j,i} = q_{j,i'}$ whenever $c_{j,i} = c_{j,i'}$; hence the condition (4.26) is fulfilled. This completes the proof of Proposition 4.14. □

5. Precise statement of the main result

In this section we will make our main result, Theorem 1.1, more explicit, by making the precise choice of the data $\varsigma, \Sigma, \mathsf{m}$ and $\varsigma \mapsto \mu_\varsigma$ for which we will later prove that the six conditions [P1]–[P3] and [Q1]–[Q3] in Section 1.1 hold.

5.1. The subspace $\mathbb{X}^\psi \subset \mathbb{X}$.

We first need to make a preparation of a technical nature. For any $\psi \in \Psi$ we let $\mathbb{X}^\psi$ be the closed set consisting of those $\Gamma g \in \mathbb{X}$ for which $\mathbb{Z}^d \, p_\psi(g)$ is a *lattice,* i.e.

$$\mathbb{X}^\psi := \{\Gamma g \in \mathbb{X} \,:\, g \in G,\ \mathbf{0} \in \mathbb{Z}^d \, p_\psi(g)\}. \tag{5.1}$$

(Recall that the grid $\mathbb{Z}^d \, p_\psi(g)$ is independent of the choice of representative g, i.e. if $\Gamma g = \Gamma g'$ then $\mathbb{Z}^d \, p_\psi(g) = \mathbb{Z}^d \, p_\psi(g')$.) Clearly, for $\psi = (j, i)$,

$$\mathbb{X}^\psi = p_j^{-1}(\mathbb{X}_j^{(i)}), \qquad \text{with } \mathbb{X}_j^{(i)} = \{\Gamma_j g \,:\, g \in G_j,\ \mathbf{0} \in \mathbb{Z}^d \mathsf{a}_i(g)\}. \tag{5.2}$$

Lemma 5.1. *For any $\psi \in \Psi$ and $\boldsymbol{q} \in \mathcal{L}_\psi$, $\Gamma g_0^{(\boldsymbol{q})} \in \mathbb{X}^\psi$.*

Proof. This is immediate from (2.24). □

We will write $\delta_{\mathbf{0}}$ for the Dirac measure on $(\mathbb{R}/\mathbb{Z})^d$ at the point $\mathbf{0} \in (\mathbb{R}/\mathbb{Z})^d$. Recall that $\widetilde{\mathsf{r}}_i : \mathbb{T}_j^d \to (\mathbb{R}/\mathbb{Z})^d$ is the projection induced by the ith row map $\mathsf{r}_i : \mathrm{M}_{r_j \times d}(\mathbb{R}) \to \mathbb{R}^d$.

Lemma 5.2. *Assume that $\psi = (j, i) \in \Psi$, $\omega \in P(\mathbb{T}_j^d)'$ and $\widetilde{\mathsf{r}}_{i\,*}\,\omega = \delta_{\mathbf{0}}$. Then $\overline{\omega}(\mathbb{X}_j^{(i)}) = 1$.*

Proof. In view of the definition of $\overline{\omega}$ in (3.9), it suffices to verify that $\Gamma_j \, \mathrm{I}_U \, A \in \mathbb{X}_j^{(i)}$ for all $U \in \mathsf{r}_i^{-1}(\mathbb{Z}^d) \subset \mathrm{M}_{r_j \times d}(\mathbb{R})$ and all $A \in \mathrm{SL}_d(\mathbb{R})$. The verification of this fact is immediate. □

Lemma 5.3. *Let $\psi = (j, i) \in \Psi$ and $V \in \mathbb{T}_j^d$. If $\widetilde{\mathsf{r}}_i(V) = \mathbf{0}$ in $(\mathbb{R}/\mathbb{Z})^d$ then $\widetilde{\mathsf{r}}_{i\,*}\,\omega_j^{(V)} = \delta_{\mathbf{0}}$, while if $\widetilde{\mathsf{r}}_i(V) \neq \mathbf{0}$ then $\big(\widetilde{\mathsf{r}}_{i\,*}\,\omega_j^{(V)}\big)(\{\mathbf{0}\}) = 0$.*

Proof. Write $V = (V_1, \ldots, V_d)$. If $\widetilde{\mathsf{r}}_i(V) = \mathbf{0}$, then the ith coordinate of each V_ℓ vanishes; hence all points in $\mathbb{S}_j^{(V)}$ have vanishing ith coordinate, and so $\mathsf{r}_i\big(\big(\mathbb{S}_j^{(V)}\big)^d\big) = \{\mathbf{0}\}$ in $(\mathbb{R}/\mathbb{Z})^d$, which forces $\omega_j^{(V)}(\widetilde{\mathsf{r}}_i^{-1}(\{\mathbf{0}\})) = 1$, i.e., $\widetilde{\mathsf{r}}_{i\,*}\,\omega_j^{(V)} = \delta_{\mathbf{0}}$.

Next assume $\widetilde{\mathsf{r}}_i(V) \neq \mathbf{0}$. If $L_j^{(V)} \not\perp \boldsymbol{e}_i$, then $\widetilde{\mathsf{r}}_i\big(\big(\mathbb{S}_j^{(V)}\big)^{\circ\, d}\big) = (\mathbb{R}/\mathbb{Z})^d$ and so by (3.7) and the definition of $\omega^{(V)}$, we have $\omega^{(V)}(\widetilde{\mathsf{r}}_i^{-1}(\{\mathbf{0}\})) = 0$. In the remaining case, when $L_j^{(V)} \perp \boldsymbol{e}_i$ and therefore $\widetilde{\mathsf{r}}_i\big(\big(\mathbb{S}_j^{(V)}\big)^{\circ\, d}\big) = \{\mathbf{0}\}$, we again have $\omega^{(V)}(\widetilde{\mathsf{r}}_i^{-1}(\{\mathbf{0}\})) = 0$, since $\widetilde{\mathsf{r}}_i(V\gamma) = \widetilde{\mathsf{r}}_i(V)\gamma \neq \mathbf{0}$ for all $\gamma \in \mathrm{SL}_d(\mathbb{Z})$. □

Lemma 5.4. *Let $\psi = (j, i) \in \Psi$ and $\boldsymbol{q} \in \mathbb{R}^d$. If $\boldsymbol{q} \in \mathcal{L}_\psi$ then $\widetilde{\mathsf{r}}_{i\,*}\,\omega_j^{(\boldsymbol{q})} = \delta_{\mathbf{0}}$, while if $\boldsymbol{q} \notin \mathcal{L}_\psi$ then $\big(\widetilde{\mathsf{r}}_{i\,*}\,\omega_j^{(\boldsymbol{q})}\big)(\{\mathbf{0}\}) = 0$.*

Proof. Note that $\boldsymbol{q} \in \mathcal{L}_\psi$ holds if and only if $\mathsf{r}_i(U_j^{(\boldsymbol{q})}) \in \mathbb{Z}^d$ (cf. (2.5) and (2.19)), that is, if and only if $\widetilde{\mathsf{r}}_i(\pi(U_j^{(\boldsymbol{q})})) = \mathbf{0}$ in $(\mathbb{R}/\mathbb{Z})^d$. Hence the lemma follows from Lemma 5.3. □

Lemma 5.5. *For every $\psi \in \Psi$ we have $\widetilde{\mathsf{r}}_{i*}\,\omega_j^{\psi} = \delta_{\mathbf{0}}$ when $(j,i) = \psi$, while $\big(\widetilde{\mathsf{r}}_{i*}\,\omega_j^{\psi}\big)(\{\mathbf{0}\}) = 0$ for every $(j,i) \in \Psi \setminus \{\psi\}$.*

Proof. The first statement follows from Lemma 5.4, since ω_j^{ψ} can be obtained as a limit of measures $\omega_j^{(\boldsymbol{q})}$ for a sequence of points $\boldsymbol{q}$ in $\mathcal{L}_{\psi}$ (cf. Remark 4.2), and since the map $\widetilde{\mathsf{r}}_{i*}$: $P(\mathbb{T}_j^d) \to P((\mathbb{R}/\mathbb{Z})^d)$ is continuous. Alternatively it is easy to give a direct verification: In (4.1) and (4.2) one notes that the ith coordinate of $\boldsymbol{c}_j^{\psi}$ equals 1 thus $\mathsf{r}_i(W_j^{\psi}) = \mathbf{0}$, and so by (4.4), $L_j^{\psi} \perp \boldsymbol{e}_i$, meaning that every point in $\mathbb{S}_j^{\psi}$ has a vanishing ith coordinate. We know from the proof of Lemma 5.4 that the same fact holds for every point in $\mathbb{S}_j^{(\boldsymbol{q})}$, for any $\boldsymbol{q} \in \mathcal{L}_{\psi}$; hence by (4.17) it also holds for every point in $\widetilde{\mathbb{S}}_j^{\psi}$, and so $\widetilde{\mathsf{r}}_i((\widetilde{\mathbb{S}}_j^{\psi})^d) = \{\mathbf{0}\}$ and $\widetilde{\mathsf{r}}_{i*}\,\omega_j^{\psi} = \delta_{\mathbf{0}}$.

We turn to the proof of the second statement (this is similar to the proof of the second half of Lemma 5.3). Thus let $(j,i) \in \Psi$, $(j,i) \neq \psi$. If $L_j^{\psi} \not\perp \boldsymbol{e}_i$ then $\widetilde{\mathsf{r}}_i\big((\mathbb{S}_j^{\psi})^d\big) = (\mathbb{R}/\mathbb{Z})^d$ and hence $\omega_j^{\psi}\big(\widetilde{\mathsf{r}}_i^{-1}(\{\mathbf{0}\})\big) = 0$ (cf. (4.17)). Now assume $L_j^{\psi} \perp \boldsymbol{e}_i$. Then $j = j_{\psi}$, by (4.4), and also $c_{\psi}/c_{j,i} \in \mathbb{Q}$, by (4.1) and (4.4). Hence by (2.7), $c_{\psi} = c_{j,i}$ and $\boldsymbol{w}_{\psi} - \boldsymbol{w}_{j,i} \notin \mathbb{Z}^d$. Now pick an arbitrary point $\boldsymbol{q} \in \mathcal{L}_{\psi}$, so that (4.17) holds. It follows from $c_{\psi} = c_{j,i}$ and $\boldsymbol{w}_{\psi} - \boldsymbol{w}_{j,i} \notin \mathbb{Z}^d$ that $\mathsf{r}_i(U_j^{(\boldsymbol{q})}) \notin \mathbb{Z}^d$, and hence $\mathsf{r}_i(\pi(U_j^{(\boldsymbol{q})})\gamma) \neq \mathbf{0}$ for all $\gamma \in \mathrm{SL}_d(\mathbb{Z})$. Also $\widetilde{\mathsf{r}}_i\big(\mathbb{S}_j^{\psi}\big) = \{\mathbf{0}\}$, since $L_j^{\psi} \perp \boldsymbol{e}_i$. Hence again $\omega_j^{\psi}\big(\widetilde{\mathsf{r}}_i^{-1}(\{\mathbf{0}\})\big) = 0$ (see (4.17)). □

5.2. **The space of marks Σ and the associated maps.** Let $\mathcal{P}$ be a finite union of grids in $\mathbb{R}^d$, and fix an *admissible* presentation of $\mathcal{P}$ as in (2.6), (2.5). (This is possible by Proposition 4.14.)

We are now finally in a position to introduce our precise choice of space of marks Σ and the associated maps and measure.

To prepare for these definitions, recall that

$$\Psi = \{(j,i) \;:\; j \in \{1,\ldots,N\},\; i \in \{1,\ldots,r_j\}\}. \tag{5.3}$$

Set

$$\Omega = \prod_{j=1}^{N} P(\mathbb{T}_j^d)'. \tag{5.4}$$

For any $\boldsymbol{q} \in \mathbb{R}^d$ we define

$$\omega^{(\boldsymbol{q})} := \big(\omega_1^{(\boldsymbol{q})},\ldots,\omega_N^{(\boldsymbol{q})}\big) \in \Omega, \tag{5.5}$$

and for any $\psi \in \Psi$ we define

$$\omega^{\psi} := \big(\omega_1^{\psi},\ldots,\omega_N^{\psi}\big) \in \Omega. \tag{5.6}$$

Now we define our space of marks Σ through:

$$\Sigma := \Big\{((j,i),\omega) \in \Psi \times \Omega \;:\; \widetilde{\mathsf{r}}_{i*}\,\omega_j = \delta_{\mathbf{0}}\Big\}, \tag{5.7}$$

where $\omega_j \in P(\mathbb{T}_j^d)'$ is the jth entry of ω, and where $\delta_{\mathbf{0}}$ is the Dirac measure at the point $\mathbf{0} \in (\mathbb{R}/\mathbb{Z})^d$. It is to be understood that in (5.7), Ψ is equipped with the discrete topology, and $\Psi \times \Omega$ with the product topology; this makes Σ a closed and hence compact subset of $\Psi \times \Omega$.

Remark 5.1. The reason that we cannot simply choose Σ to be $\Psi \times \Omega$ is that we want the map $\varsigma \mapsto \mu_{\varsigma}$ below to be continuous.

Next we define our marking ς through

$$\varsigma : \mathcal{P} \to \Sigma, \qquad \varsigma(\boldsymbol{q}) = (\psi(\boldsymbol{q}), \omega^{(\boldsymbol{q})}), \tag{5.8}$$

where $\psi : \mathcal{P} \to \Psi$ is as in (2.10), i.e. a fixed function such that $\boldsymbol{q} \in \mathcal{L}_{\psi(\boldsymbol{q})}$ for all $\boldsymbol{q} \in \mathcal{P}$. It follows from Lemma 5.4 that ς is indeed a map into Σ.

For each $\psi \in \Psi$ we set

$$\sigma^{\psi} := (\psi, \omega^{\psi}) \in \Sigma. \tag{5.9}$$

It follows from Lemma 5.5 that σ^{ψ} indeed lies in Σ. Next we define the Borel probability measure m on Σ to be the atomic measure on Σ supported on the points σ^{ψ}, with

$$\mathsf{m}\big(\sigma^{\psi}\big) := \frac{\overline{n}_{\psi}}{c_{\mathcal{P}}} \qquad (\psi \in \Psi). \tag{5.10}$$

Cf. (2.11) and (2.12); in particular note that (2.12) implies that m is indeed a probability measure. Note that m is supported on the *finite* subset

$$\Sigma' := \{\sigma^{\psi} \,:\, \psi \in \Psi\}. \tag{5.11}$$

As in Section 1.1 we set

$$\mathcal{X} = \mathbb{R}^d \times \Sigma \qquad \text{and} \qquad \mu_{\mathcal{X}} = \mathrm{vol} \times \mathsf{m}.$$

Finally we will make our choice of the map $\varsigma \mapsto \mu_{\varsigma}$ from Σ to $P(N(\mathcal{X}))$. To prepare for this, note that we have a natural map $\omega \mapsto \overline{\omega}$, from Ω to $P(\mathbb{X})$, defined by

$$\overline{\omega} := \overline{\omega_1} \otimes \cdots \otimes \overline{\omega_N} \in P(\mathbb{X}) \qquad (\omega \in \Omega). \tag{5.12}$$

(This makes sense since $\overline{\omega_j} \in P(\mathbb{X}_j)$ for each j; cf. (3.8).) Next we introduce the following map:

$$J : \mathbb{X} \to N_s(\mathcal{X}), \qquad J(\Gamma g) = \bigcup_{\psi \in \Psi} c_{\psi}\big(\mathbb{Z}^d\, p_{\psi}(g)\big) \times \{\sigma^{\psi}\} \qquad (g \in G). \tag{5.13}$$

This map extends the map J_0 defined in (2.18) in an obvious sense.

Lemma 5.6. *J is continuous.*

Proof. Recall that $N_s(\mathcal{X})$ is equipped with the vague topology; hence the task is to prove that if $x_1, x_2, \ldots$ is a sequence in $\mathbb{X}$ converging to $x \in \mathbb{X}$, then for any $f \in \mathrm{C}_c(\mathcal{X})$ we have $\sum_{p \in J_{\psi}(x_k)} f(p) \to \sum_{p \in J_{\psi}(x)} f(p)$. This is immediate using the formula

$$\sum_{p \in J(\Gamma g)} f(p) = \sum_{\psi \in \Psi} \sum_{\boldsymbol{m} \in \mathbb{Z}^d} f\Big(c_{\psi}\big(\boldsymbol{m}\, p_{\psi}(g)\big), \sigma^{\psi}\Big), \qquad \forall g \in G.$$

□

Next, for every $\psi \in \Psi$ we introduce the following modification of the map J:

$$J_{\psi} : \mathbb{X}^{\psi} \to N_s(\mathcal{X}); \qquad J_{\psi}(\Gamma g) := J(\Gamma g) \setminus \{(\mathbf{0}, \sigma^{\psi})\}. \tag{5.14}$$

This map J_{ψ} is also continuous; this is proved in the same way as Lemma 5.6, using also the fact that $(\mathbf{0}, \sigma^{\psi}) \in J(x)$ for all $x \in \mathbb{X}^{\psi}$ (cf. (5.1) and (5.13)). At last, we now define the map $\varsigma \mapsto \mu_{\varsigma}$ from Σ to $P(N_s(\mathcal{X}))$ by setting

$$\mu_{\varsigma} = J_{\psi\,*}\, \overline{\omega} \in P(N_s(\mathcal{X})) \qquad \text{for } \varsigma = (\psi, \omega) \in \Sigma. \tag{5.15}$$

To see that μ_{ς} is indeed a probability measure, note that

$$\overline{\omega}(\mathbb{X}^{\psi}) = 1, \qquad \text{for all } \varsigma = (\psi, \omega) \in \Sigma; \tag{5.16}$$

cf. (5.2), Lemma 5.2, (5.7) and (5.12).

Lemma 5.7. *The map $\varsigma \mapsto \mu_{\varsigma}$ is continuous.*

Proof. Let $\varsigma_1, \varsigma_2, \ldots$ be an arbitrary sequence in Σ converging to a point $\varsigma \in \Sigma$. Write $\varsigma_k = (\psi^{(k)}, \omega^{(k)})$ with $\omega^{(k)} = (\omega_1^{(k)}, \ldots, \omega_N^{(k)}) \in \Omega$; also write $\varsigma = (\psi, \omega)$. Throwing away finitely many initial points from the sequence we may assume that $\psi^{(k)} = \psi$ for all k. By (5.16) we then have $\overline{\omega}(\mathbb{X}^{\psi}) = 1$ and $\overline{\omega^{(k)}}(\mathbb{X}^{\psi}) = 1$ for all k. Hence we may just as well regard $\overline{\omega}$ and all $\overline{\omega^{(k)}}$ as elements in $P(\mathbb{X}^{\psi})$. For each fixed $j \in \{1, \ldots, N\}$ we have $\omega_j^{(k)} \to \omega_j$ in

$P(\mathbb{T}_j^d)'$ as $k \to \infty$; hence by (5.12), Lemma 3.6 and [1, Thm. 2.8(ii)], $\overline{\omega^{(k)}} \to \overline{\varpi}$ in $P(\mathbb{X})$. Hence also $\overline{\omega^{(k)}} \to \overline{\varpi}$ in $P(\mathbb{X}^{\psi})$ [12, Lemma 4.26], and by the continuous mapping theorem, $J_{\psi *}\overline{\omega^{(k)}} \to J_{\psi *}\overline{\varpi}$, in $P(N_s(\mathcal{X}))$, viz., $\mu_{\varsigma_k} \to \mu_{\varsigma}$. □

We conclude this section by determining the intensity measure of a point process with distribution μ_ν, for any $\nu \in \Sigma$.

Lemma 5.8. *Let $\nu = (\psi', \omega) \in \Sigma$. Then for any Borel set $B \subset \mathcal{X}$ we have*

$$\int_{N_s(\mathcal{X})} \#(B \cap Y)\, d\mu_\nu(Y) = c_{\mathcal{P}} \mu_{\mathcal{X}}(B) + \sum_{\psi \in \Psi \setminus \{\psi'\}} \omega_{j_\psi}\big(\widetilde{\mathsf{r}}_{i_\psi}^{-1}(\{\mathbf{0}\})\big) \cdot I\big((\mathbf{0}, \sigma^{\psi}) \in B\big). \tag{5.17}$$

In particular, for every $\psi' \in \Psi$, a point process with distribution $\mu_{\sigma^{\psi'}}$ has intensity measure $c_{\mathcal{P}}\mu_{\mathcal{X}}$.

Proof. By (5.15), (5.14) and (5.13), the left hand side of (5.17) equals

$$-I\big((\mathbf{0}, \sigma^{\psi'}) \in B\big) + \int_{\mathbb{X}^{\psi'}} \sum_{\psi \in \Psi} \sum_{\boldsymbol{m} \in \mathbb{Z}^d} I\Big(\big(c_\psi \cdot (\boldsymbol{m} p_\psi(g)), \sigma^{\psi}\big) \in B\Big)\, d\overline{\varpi}(\Gamma g).$$

By (5.16), the integration may just as well be taken over all $\mathbb{X}$. Moving out the sum over ψ and then using (5.12) and (2.17), we get

$$-I\big((\mathbf{0}, \sigma^{\psi'}) \in B\big) + \sum_{\psi \in \Psi} \int_{\mathbb{X}_{j_\psi}} \sum_{\boldsymbol{m} \in \mathbb{Z}^d} I\Big(\big(c_\psi \cdot (\boldsymbol{m}\mathsf{a}_{i_\psi}(g)), \sigma^{(\psi)}\big) \in B\Big)\, d\overline{\varpi}_{j_\psi}(\Gamma_{j_\psi} g),$$

and by Proposition 3.8, this equals

$$-I\big((\mathbf{0}, \sigma^{\psi'}) \in B\big) + \sum_{\psi \in \Psi} c_\psi^{-d} \int_{\mathbb{R}^d} I\big((\boldsymbol{x}, \sigma^{\psi}) \in B\big)\, d\boldsymbol{x} + \sum_{\psi \in \Psi} \omega_{j_\psi}\big(\widetilde{\mathsf{r}}_{i_\psi}^{-1}(\{\mathbf{0}\})\big) \cdot I\big((\mathbf{0}, \sigma^{\psi}) \in B\big).$$

Using $\omega_{j_{\psi'}}\big(\widetilde{\mathsf{r}}_{i_{\psi'}}^{-1}(\{\mathbf{0}\})\big) = 1$, which holds since $(\psi', \omega) \in \Sigma$, together with $\mu_{\mathcal{X}} = \mathrm{vol} \times \mathsf{m}$, (5.10) and (2.11), we obtain the right hand side of (5.17).

To obtain the last statement of the lemma, we apply (5.17) for $\nu = \sigma^{\psi'} = (\psi', \omega^{\psi'})$; in this case the sum over $\Psi \setminus \{\psi'\}$ in (5.17) vanishes, by Lemma 5.5; hence the right hand side of (5.17) equals $c_{\mathcal{P}}\, \mu_{\mathcal{X}}(B)$. □

5.3. **Example: The grids $\mathcal{L}_\psi$ being pairwise incommensurable.** In this section we consider the special case when $\mathcal{P}$ is a finite union of grids which are *pairwise incommensurable.* This case was previously studied in [19]. In this case we can express $\mathcal{P}$ as in (2.5)–(2.7) *with* $r_1 = \cdots = r_N = 1$. It should be noted that both the condition (2.7) and the admissibility condition in Definition 4.1 are *automatically* satisfied for any such finite union of grids.

Note that now each $\psi \in \Psi$ has the form $\psi = (j, 1)$ for some $j \in \{1, \ldots, N\}$, and we may just as well *identify* this ψ with j, so that from now on we write

$$\Psi = \{1, \ldots, N\}.$$

We have $\mathbb{T}_j = \mathbb{R}/\mathbb{Z}$, $G_j = \mathrm{S}_1(\mathbb{R}) = \mathrm{ASL}_d(\mathbb{R})$ and $\Gamma_j = \mathrm{S}_1(\mathbb{Z}) = \mathrm{ASL}_d(\mathbb{Z})$ for all $j \in \Psi$. Also, by (4.1) we now have $\boldsymbol{c}_j^{j'} = c_{j'}/c_j$ (a vector in $\mathbb{R}^1$) for all $j, j' \in \Psi$.

For any $j \neq j'$ in Ψ we have $L_j^{j'} = \mathbb{R}$ by (4.4); hence $\mathbb{S}_j^{j'} = \mathbb{T}_j$, and $\mathcal{O}_j^{j'} = \mathbb{T}_j^d$ by (4.17), meaning that $\omega_j^{j'}$ equals the Haar measure of $\mathbb{T}_j^d$. By the definition below (3.8), this implies that $\overline{\omega_j^{j'}}$ equals the $\mathrm{ASL}_d(\mathbb{R})$-invariant probability measure on $\mathbb{X}_j = \mathrm{ASL}_d(\mathbb{Z})\backslash \mathrm{ASL}_d(\mathbb{R})$; let us denote this measure by $\widetilde{\nu}$.

On the other hand when $j = j'$ we get $\boldsymbol{c}_j^j = c_j/c_j = 1$ and $W_j^j = 0$ in $\mathrm{M}_{1\times d}(\mathbb{R})$. Thus $L_j^j = \{0\} \subset \mathbb{R}^1$ by (4.4), and so $\mathbb{S}_j^j = \{0\} \subset \mathbb{T}_j$. Furthermore, $U_j^{(\boldsymbol{q})}$ runs through $\mathrm{M}_{1\times d}(\mathbb{Z})$ as $\boldsymbol{q}$ runs through $\mathcal{L}_j$, by (4.3). Hence by (4.17), $\mathcal{O}_j^j = \{\mathbf{0}\}$ in $\mathbb{T}_j^d$, meaning that ω_j^j is the

Dirac measure at the point $\mathbf{0}$. By the definition below (3.8), this implies that $\overline{\omega_j^j}$ equals ν, the $\mathrm{SL}_d(\mathbb{R})$-invariant probability measure on $\mathrm{SL}_d(\mathbb{Z})\backslash\mathrm{SL}_d(\mathbb{R})$, embedded as a submanifold of $\mathbb{X}_j = \mathrm{ASL}_d(\mathbb{Z})\backslash\mathrm{ASL}_d(\mathbb{R})$ in the usual way.

Recalling also (5.6) and (5.12), it now follows that for every $j \in \Psi$,

$$\overline{\omega^j} = \widetilde{\nu} \otimes \cdots \otimes \nu \otimes \cdots \otimes \widetilde{\nu} \in P(\mathbb{X}), \tag{5.18}$$

where the jth factor equals ν and all other factors equals $\widetilde{\nu}$. When combined with (5.15), (5.18) leads to an explicit description of a point process in $\mathcal{X}$ having the distribution μ_{σ^j} for a given $j \in \Psi$. For the statement of this description, let us agree that for any $\overline{n} > 0$, a *"random grid of density $\overline{n}$"* is a random point set in $\mathbb{R}^d$ distributed as $\overline{n}^{-1/d}\mathbb{Z}^d g$ with $\mathrm{ASL}_d(\mathbb{Z})g$ being a random point in $\mathrm{ASL}_d(\mathbb{Z})\backslash\mathrm{ASL}_d(\mathbb{R})$ distributed according to $\widetilde{\nu}$, and similarly, a *"random lattice of density $\overline{n}$"* is a random point set in $\mathbb{R}^d$ distributed as $\overline{n}^{-1/d}\mathbb{Z}^d g$ with $\mathrm{SL}_d(\mathbb{Z})g$ being a random point in $\mathrm{SL}_d(\mathbb{Z})\backslash\mathrm{SL}_d(\mathbb{R})$ distributed according to ν. Now for each $j \in \Psi$ let Λ_j (resp., Λ_j^0) be a random grid (resp., a random lattice) of density $\overline{n}_j$, with $\Lambda_1, \ldots, \Lambda_N$ and $\Lambda_1^0, \ldots, \Lambda_N^0$ being mutually independent. Then set

$$\Xi_j = \Big(\bigcup_{\substack{j' \in \Psi \\ (j' \neq j)}} \Lambda_{j'} \times \{\sigma^{j'}\} \Big) \bigcup \Big(\big(\Lambda_j^0 \setminus \{\mathbf{0}\}\big) \times \{\sigma^j\} \Big). \tag{5.19}$$

By (5.18) and (5.15), Ξ_j is a random point set in $\mathcal{X}$ with distribution μ_{σ^j}.

Feeding the above description of μ_{σ^j} into the general formulas (1.13) and (1.14), we obtain explicit expressions for the transitions kernels k^{g} and k in terms of of the transition kernels for the case of a *single* grid, which was treated in [15, 16]. To explain this derivation in the case of k (i.e., (1.14)) we first need to recall some definitions from [15, Sec. 7.1][8]. We set $H = \{h \in \mathrm{SL}_d(\mathbb{R}) : \boldsymbol{e}_1 h = \boldsymbol{e}_1\}$ and $H_{\mathbb{Z}} := H \cap \mathrm{SL}_d(\mathbb{Z})$, and for any given $j \in \Psi$ and $\boldsymbol{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, we fix matrices $M_{\boldsymbol{x}}^{(1,j)}, M_{\boldsymbol{x}}^{(2,j)}, \ldots \in \mathrm{SL}_d(\mathbb{R})$ such that $k\boldsymbol{e}_1 M_{\boldsymbol{x}}^{(k,j)} = \overline{n}_j^{1/d}\boldsymbol{x}$ for each $k \in \mathbb{Z}^+$. Now let k be a random positive integer with the distribution $\mathbb{P}(k = k_0) = \zeta(d)^{-1}k_0^{-d}$ ($\forall k_0 \in \mathbb{Z}^+$)[9]; let $H_{\mathbb{Z}}h$ be a random point in $H_{\mathbb{Z}}\backslash H$, independent from k and distributed according to the unique H-invariant probability measure on $H_{\mathbb{Z}}\backslash H$, and finally set:

$$\widetilde{\Lambda}_j^{(\boldsymbol{x})} := \overline{n}_j^{-1/d}\mathbb{Z}^d h M_{\boldsymbol{x}}^{(k,j)}.$$

This random point set $\widetilde{\Lambda}_j^{(\boldsymbol{x})}$ represents in a natural way a *"random lattice of density $\overline{n}_j$ conditioned to contain $\boldsymbol{x}$";* see [15, Sec. 7.1] and [28, Sec. 5]; in particular the distribution of $\widetilde{\Lambda}_j^{(\boldsymbol{x})}$ is independent of the choice of $M_{\boldsymbol{x}}$. We insist that the random elements k and $H_{\mathbb{Z}}h$ in the above construction be independent from the random point sets $\Lambda_1, \ldots, \Lambda_N, \Lambda_1^0, \ldots, \Lambda_N^0$ introduced earlier; then the same is true for $\widetilde{\Lambda}_j^{(\boldsymbol{x})}$.

For any given $j \in \Psi$, using the notation introduced above, a version $\nu_{\sigma^j} : \mathcal{X} \times \mathcal{N} \to [0,1]$ of the Palm distributions (see footnote 5) of the random point set Ξ_j in (5.19) can now be explicitly described as follows: For any fixed $\boldsymbol{x}_0 \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ and $j_0 \in \Psi$, $\nu_{\sigma^j}\big((\boldsymbol{x}_0, j_0);\,\cdot\,\big) \in P(N_s(\mathcal{X}))$ is the distribution of the following random point set:

$$\Theta_j^{(\boldsymbol{x}_0,j_0)} := \begin{cases} \Big(\bigcup_{\substack{j' \in \Psi \\ (j' \neq j)}} \Lambda_{j'} \times \{\sigma^{j'}\} \Big) \cup \Big(\big(\widetilde{\Lambda}_j^{(\boldsymbol{x}_0)} \setminus \{\mathbf{0}\}\big) \times \{\sigma^j\} \Big) & \text{if } j = j_0; \\ \Big(\bigcup_{\substack{j' \in \Psi \\ (j' \neq j, j_0)}} \Lambda_{j'} \times \{\sigma^{j'}\} \Big) \cup \Big(\big(\Lambda_{j_0}^0 + \boldsymbol{x}_0\big) \times \{\sigma^{j_0}\} \Big) \cup \Big(\big(\Lambda_j^0 \setminus \{\mathbf{0}\}\big) \times \{\sigma^j\} \Big) & \text{if } j \neq j_0. \end{cases}$$

[8]Here formulated using a slightly different notation.

[9]In other words, k is a zeta-distributed random variable with parameter d.

Now let Φ and $\Phi_{\mathbf{0}}$ be the transition kernels in the single-grid case, as explicitly defined in [15, 16].[10] Also as in these references we set

$$\Phi(\xi) = \int_{\mathcal{B}_1^{d-1}} \Phi(\xi, \boldsymbol{w})\, d\boldsymbol{w} = \int_{\xi}^{\infty} \int_{\mathcal{B}_1^{d-1}} \int_{\mathcal{B}_1^{d-1}} \Phi_{\mathbf{0}}(\xi', \boldsymbol{w}, \boldsymbol{z})\, d\boldsymbol{w}\, d\boldsymbol{z}\, d\xi'; \tag{5.20}$$

this is the density of the limiting distribution of the free path length from a generic initial point inside the billiard domain. It now follows from the formulas (1.13), (1.14), and and the above descriptions of the measures μ_{σ^j} and $\nu_{\sigma^j}\big((\boldsymbol{x}_0, j_0);\,\cdot\,\big)$, that the transitions kernels k^{g} and k are given by the following formulas: For any $\xi > 0$, $\boldsymbol{w} \in \mathcal{B}_1^{d-1}$, $j \in \Psi$,

$$k^{\mathsf{g}}(\xi, (\boldsymbol{w}, \sigma^j)) = v_{d-1}\, c_{\mathcal{P}}\, \Phi(\overline{n}_j \xi, \boldsymbol{w}) \prod_{j' \in \Psi \setminus \{j\}} \int_{\overline{n}_{j'}\xi}^{\infty} \Phi(\xi')\, d\xi'; \tag{5.21}$$

and for any $\xi > 0$, $\boldsymbol{w}, \boldsymbol{z} \in \mathcal{B}_1^{d-1}$, $\ell, j \in \Psi$,

$$k\big((\boldsymbol{w}, \sigma^\ell), \xi, (\boldsymbol{z}, \sigma^j)\big) = \begin{cases} v_{d-1} c_{\mathcal{P}}\, \Phi_{\mathbf{0}}(\overline{n}_j \xi, \boldsymbol{z}, -\boldsymbol{w}) \displaystyle\prod_{j' \in \Psi \setminus \{j\}} \int_{\overline{n}_{j'}\xi}^{\infty} \Phi(\xi')\, d\xi' & \text{if } \ell = j \\ v_{d-1} c_{\mathcal{P}}\, \Phi(\overline{n}_\ell \xi, \boldsymbol{w})\, \Phi(\overline{n}_j \xi, \boldsymbol{z}) \displaystyle\prod_{j' \in \Psi \setminus \{\ell, j\}} \int_{\overline{n}_{j'}\xi}^{\infty} \Phi(\xi')\, d\xi' & \text{if } \ell \neq j. \end{cases} \tag{5.22}$$

The formulas (5.21) and (5.22) agree with [19, (5.5), (5.6), (5.12)]. Indeed, the extra factor v_{d-1} in (5.21) and (5.22) corresponds to the different normalizations in the relation between the collision kernel and the transition kernel (compare [19, (4.4)] with (1.12) above); also the difference between the factor "$c_{\mathcal{P}}$" in (5.21) and (5.22) versus "$\overline{n}_j$" in [19, (5.5), (5.6), (5.12)] corresponds to the fact that we integrate with respect to the measure m in (1.9), (1.10), (1.15); recall here that $\mathsf{m}(\{\sigma^j\}) = \overline{n}_j / c_{\mathcal{P}}$ for each $j \in \Psi$. In particular, the generalized linear Boltzmann equation in (1.15) is, in the special case discussed here, equivalent with [19, (4.8)] when setting $f_t^{(j)}(\boldsymbol{Q}, \boldsymbol{V}_0, \xi, \boldsymbol{V}_+) = f(t, \boldsymbol{Q}, \boldsymbol{V}, \xi, \sigma^j, \boldsymbol{V}_+)$.

6. Verification of [Q1],[Q2],[Q3],[P1],[P3], and initial discussion regarding [P2]

In order to prove the main result of the paper, Theorem 1.1, we now wish to prove that all the conditions [P1]–[P3] and [Q1]–[Q3] are satisfied for the maps $\varsigma : \mathcal{P} \to \Sigma$ and $\varsigma \mapsto \mu_\varsigma$ and measure m which we have defined in the previous section. In the present section we will prove all of these conditions except [P2]; we will also reduce the verification of [P2] to a certain statement about uniform equidistribution in the homogeneous space $\mathbb{X}$, Theorem 6.6 below.

6.1. **Verification of [Q1], [Q2], [Q3].**

We start with the conditions [Q1]–[Q3].

The following lemma shows that [Q1] holds, in a much stronger form.

Lemma 6.1. *For every $\varsigma \in \Sigma$, μ_ς is $\mathrm{SL}_d(\mathbb{R})$-invariant.*

Proof. It is immediate from the definition in (3.9) that for any $\omega_j \in P(\mathbb{T}_j^d)'$, the measure $\overline{\omega}_j$ on $\mathbb{X}_j$ is right $\mathrm{SL}_d(\mathbb{R})$ invariant (this was also used in the proof of Proposition 3.7); hence for any $\omega \in \Omega$, the measure $\overline{\omega}$ on $\mathbb{X}$ is right $\mathrm{SL}_d(\mathbb{R})^N$-invariant, and in particular it is right $\varphi(\mathrm{SL}_d(\mathbb{R}))$-invariant, where $\varphi : \mathrm{SL}_d(\mathbb{R}) \to G$ is the diagonal embedding. Next we note that, for any $\psi \in \Psi$,

$$J_\psi(\Gamma g \varphi(h)) = J_\psi(\Gamma g) h \qquad \text{for all } \Gamma g \in \mathbb{X}^\psi,\ h \in \mathrm{SL}_d(\mathbb{R}).$$

[10] The relation between these Φ, $\Phi_{\mathbf{0}}$ and the k^{g}, k in Section 1.2 and [20] is as follows: If the scatterer configuration $\mathcal{P}$ is a single grid of density $c_{\mathcal{P}}$, then $k^{\mathsf{g}}(\xi, \boldsymbol{w}) \equiv v_{d-1} c_{\mathcal{P}} \Phi(c_{\mathcal{P}} \xi, \boldsymbol{w})$ and $k(\boldsymbol{w}, \xi, \boldsymbol{z}) \equiv v_{d-1} c_{\mathcal{P}} \Phi_{\mathbf{0}}(c_{\mathcal{P}} \xi, \boldsymbol{z}, -\boldsymbol{w})$. Here the factor v_{d-1} as well as the minus sign in front of $\boldsymbol{z}$ corresponds to the fact that in the present paper as well as in [20], the relation between the collision kernel and the transition kernel is normalized slightly differently than in [15, 16, 17, 19]; indeed, compare (1.11), (1.12) with [17, (1.8),(1.9)].

It follows that the measure $J_{\psi *}\overline{\omega}$ on $N_s(\mathcal{X})$ is $\mathrm{SL}_d(\mathbb{R})$-invariant, for any $\omega \in \Omega$. This implies the lemma, via the definition (5.15). □

Next, because of the following general fact, also [Q2] is an immediate consequence of Lemma 6.1:

Lemma 6.2. *If $\mu \in P(N_s(\mathcal{X}))$ is invariant under the action of $\mathrm{SO}(d)$ then*

$$\mu(\{\nu \in N(\mathcal{X}) \,:\, \exists x_1 \in \mathbb{R} \text{ s.t. } \nu(\{x_1\} \times \mathbb{R}^{d-1} \times \Sigma) > 1\}) = 0. \tag{6.1}$$

Proof. For any $\boldsymbol{v} \in \mathrm{S}_1^{d-1}$, set

$$A_{\boldsymbol{v}} = \{\nu \in N(\mathcal{X}) \,:\, \exists x_1 \in \mathbb{R} \text{ s.t. } \nu((x_1\boldsymbol{v} + \boldsymbol{v}^{\perp}) \times \Sigma) > 1\}. \tag{6.2}$$

Then our task is to prove $\mu(A_{\boldsymbol{e}_1}) = 0$. The fact that μ is $\mathrm{SO}(d)$-invariant implies that $\mu(A_{\boldsymbol{v}}) = \mu(A_{\boldsymbol{e}_1})$ for all $\boldsymbol{v} \in \mathrm{S}_1^{d-1}$. Hence we have, with λ_1 being the uniform probability measure on S_1^{d-1}:

$$\mu(A_{\boldsymbol{e}_1}) = \int_{\mathrm{S}_1^{d-1}} \mu(A_{\boldsymbol{v}})\, d\lambda_1(\boldsymbol{v}) = \int_{N_s(\mathcal{X})} \int_{\mathrm{S}_1^{d-1}} I(\nu \in A_{\boldsymbol{v}})\, d\lambda_1(\boldsymbol{v})\, d\mu(\nu) = 0.$$

Here the second equality holds by Fubini's Theorem and since $\mu(N(\mathcal{X}) \setminus N_s(\mathcal{X})) = 0$ (because of $\mu \in P(N_s(\mathcal{X}))$), and the last equality holds since for any $\nu \in N_s(\mathcal{X})$ we have $\int_{\mathrm{S}_1^{d-1}} I(\nu \in A_{\boldsymbol{v}})\, d\lambda_1(\boldsymbol{v}) = 0$, since the set $\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} \,:\, \nu \in A_{\boldsymbol{v}}\}$ is a countable union of subspheres of S_1^{d-1} of codimension one. □

Next we turn to the condition [Q3].

Lemma 6.3. *[Q3] holds.*

Proof. (Cf. the proof of [20, Lemma 5.3.13].) Set $\Lambda = \mathrm{SL}_d(\mathbb{Z})$ and $\mathbb{Y} = \Lambda\backslash \mathrm{SL}_d(\mathbb{R})$. Let us also fix a choice of $\psi = (j, i) \in \Psi$. For $R > 0$ we set

$$\mathbb{Y}(R) = \{\Lambda h \in \mathbb{Y} \,:\, c_\psi(\mathbb{Z}^d h) + \mathcal{B}_{R/2}^d = \mathbb{R}^d\}.$$

Note that the set $\mathbb{Y}(R)$ is increasing with respect to R. Also, for every $\Lambda h \in \mathbb{Y}$, the set $c_\psi(\mathbb{Z}^d h)$ is a lattice in $\mathbb{R}^d$ and hence there exists some $R = R(h) > 0$ such that $c_\psi(\mathbb{Z}^d h) + \mathcal{B}_{R/2}^d = \mathbb{R}^d$. Hence $\cup_{R>0}\mathbb{Y}(R) = \mathbb{Y}$, and it follows that for any given $\varepsilon > 0$ we can choose $R > 0$ so that $\eta(\mathbb{Y}(R)) > 1 - \varepsilon$, where η is the $\mathrm{SL}_d(\mathbb{R})$-invariant probability measure on $\mathbb{Y}$.

With this choice of R, we now claim that for any $\varsigma = (\psi', \omega) \in \Sigma$ and $\boldsymbol{x} \in \mathbb{R}^d$, (1.7) holds. By (5.15), this is equivalent to the following:

$$\overline{\omega}\big(\big\{\Gamma g \in \mathbb{X}^{\psi'} \,:\, J_{\psi'}(\Gamma g) \cap (\mathcal{B}_R^d(\boldsymbol{x}) \times \Sigma) = \emptyset\big\}\big) < \varepsilon. \tag{6.3}$$

Choose $\boldsymbol{y} \in \mathbb{R}^d$ so that $\mathcal{B}_{R/2}^d(\boldsymbol{y}) \subset \mathcal{B}_R^d(\boldsymbol{x})$ and $\mathbf{0} \notin \mathcal{B}_{R/2}^d(\boldsymbol{y})$. For any $g \in G$, letting $h = \iota(p_j(g)) \in \mathrm{SL}_d(\mathbb{R})$ we have that the grid $c_\psi(\mathbb{Z}^d p_\psi(g))$ is a translate of the lattice $c_\psi(\mathbb{Z}^d h)$; and in particular if $\Lambda h \in \mathbb{Y}(R)$ then $c_\psi(\mathbb{Z}^d p_\psi(g))$ must contain a point in $\mathcal{B}_{R/2}^d(\boldsymbol{y})$. This implies that for every $\Gamma g \in \mathbb{X}$ satisfying $\widetilde{\iota}(\widetilde{p}_j(\Gamma g)) \in \mathbb{Y}(R)$, the point set $J(\Gamma g)$ (cf. (5.13)) must contain a point in $\mathcal{B}_{R/2}^d(\boldsymbol{y}) \times \Sigma$. Using also $\mathbf{0} \notin \mathcal{B}_{R/2}^d(\boldsymbol{y})$ and $\mathcal{B}_{R/2}^d(\boldsymbol{y}) \subset \mathcal{B}_R^d(\boldsymbol{x})$, it follows that the measure in the left hand side of (6.3) is bounded above by

$$\overline{\omega}\big(\big\{\Gamma g \in \mathbb{X}^{\psi'} \,:\, \widetilde{\iota}(\widetilde{p}_j(\Gamma g)) \notin \mathbb{Y}(R)\big\}\big). \tag{6.4}$$

However, writing $\omega = (\omega_1, \ldots, \omega_j)$ we have $\widetilde{p}_{j*}(\overline{\omega}) = \overline{\omega}_j$, which is an $\mathrm{SL}_d(\mathbb{R})$-invariant probability measure on $\mathbb{X}_j$ (this is immediate from the definition (3.8), as we have noted previously). Hence the pushforward of $\overline{\omega}$ by $\widetilde{\iota} \circ \widetilde{p}_j$ equals η, and so the measure in (6.4) equals $\eta(\mathbb{Y} \setminus \mathbb{Y}(R))$, which by our choice of R is less than ε. Hence (6.3), and thereby the lemma, is proved. □

6.2. **Verification of [P1] (uniform density).**

Proposition 6.4. *[P1] holds, i.e. for any bounded subset $B \subset \mathcal{X}$ with $\mu_{\mathcal{X}}(\partial B) = 0$, we have*

$$\lim_{T\to\infty} \frac{\#(\widetilde{\mathcal{P}} \cap TB)}{T^d} = c_{\mathcal{P}} \mu_{\mathcal{X}}(B). \tag{6.5}$$

Proof. Note that $\mathcal{X}$ decomposes as the disjoint union $\sqcup_{\psi\in\Psi} \mathcal{X}_\psi$, where $\mathcal{X}_\psi = \mathbb{R}^d \times \Sigma_\psi$ with $\Sigma_\psi := (\{\psi\} \times \Omega) \cap \Sigma$; note also that each set $\mathcal{X}_\psi$ is both open and closed is $\mathcal{X}$. It follows that it suffices to prove (6.5) under the extra assumption that $B \subset \mathcal{X}_\psi$ for some fixed ψ. Using also the fact that the set $\{\boldsymbol{q} \in \mathcal{L}_\psi \,:\, \psi(\boldsymbol{q}) \neq \psi\}$ has density zero (cf. Remark 2.2), it follows that our task is to prove the following, for any bounded set $B \subset \mathcal{X}_\psi$ with $\mu_{\mathcal{X}}(\partial B) = 0$:

$$\lim_{T\to\infty} \frac{\#\{\boldsymbol{q} \in \mathcal{L}_\psi \,:\, (\boldsymbol{q}, (\psi, \omega^{(\boldsymbol{q})})) \in TB\}}{T^d} = c_{\mathcal{P}} \mu_{\mathcal{X}}(\mathcal{X}_\psi). \tag{6.6}$$

Let us first verify that (6.6) holds for any set B of the form

$$B = \Big(\prod_{i=1}^d [\alpha_i, \beta_i)\Big) \times U, \tag{6.7}$$

for any real numbers $\alpha_i < \beta_i$ ($i = 1, \ldots, d$) and any open neighbourhood U of σ^ψ in Σ_ψ. Indeed, given such a U, there exist open neighbourhoods U_j of ω_j^ψ in $P(\mathbb{T}_j^d)$ for $j = 1, \ldots, N$ such that $\Sigma_\psi \cap (\{\psi\} \times \prod_{j=1}^N U_j) \subset U$. Applying now Proposition 4.12 to the set U_j, for each $j = 1, \ldots, N$, it follows that there exists a subset $\mathcal{Z}' \subset \mathcal{L}_\psi$ of density zero such that $(\psi, \omega^{(\boldsymbol{q})}) \in U$ for all $\boldsymbol{q} \in \mathcal{L}_\psi \setminus \mathcal{Z}'$. Furthermore, for B as in (6.7), we have $c_{\mathcal{P}} \mu_{\mathcal{X}}(B) = \overline{n}_\psi \prod_{i=1}^d (\beta_i - \alpha_i)$, and so (6.6) follows from the fact that the grid $\mathcal{L}_\psi$ has asymptotic density $\overline{n}_\psi$ in $\mathbb{R}^d$.

Next, if B is any bounded *open* subset of $\mathcal{X}_\psi$, then there exists a sequence $B_1 \subset B_2 \subset \cdots$ of subsets of B such that each B_k is a finite disjoint union of sets of the form (6.7), and $B \cap (\mathbb{R}^d \times \{\sigma^\psi\}) \subset \cup_{k=1}^\infty B_k$. Using $\mu_{\mathcal{X}}(B) = \mu_{\mathcal{X}}(B \cap (\mathbb{R}^d \times \{\sigma^\psi\}))$, it follows that $\mu_{\mathcal{X}}(B_k) \to \mu_{\mathcal{X}}(B)$ as $k \to \infty$, and hence since (6.6) holds for each of our sets B_k, it follows that

$$\liminf_{T\to\infty} \frac{\#\{\boldsymbol{q} \in \mathcal{L}_\psi \,:\, (\boldsymbol{q}, (\psi, \omega^{(\boldsymbol{q})})) \in TB\}}{T^d} \geq c_{\mathcal{P}} \mu_{\mathcal{X}}(B). \tag{6.8}$$

Next if $\widetilde{B}$ is any bounded *closed* subset of $\mathcal{X}_\psi$, then by taking $R > 0$ so that $\widetilde{B} \subset B' := \mathcal{B}_R^d \times \Sigma_\psi$, and noticing that (6.6) holds for B' (since the grid $\mathcal{L}_\psi$ has asymptotic density $\overline{n}_\psi$ in $\mathbb{R}^d$), and also (6.8) holds for the bounded open set $B' \setminus \widetilde{B}$, it follows that

$$\limsup_{T\to\infty} \frac{\#\{\boldsymbol{q} \in \mathcal{L}_\psi \,:\, (\boldsymbol{q}, (\psi, \omega^{(\boldsymbol{q})})) \in T\widetilde{B}\}}{T^d} \leq c_{\mathcal{P}} \mu_{\mathcal{X}}(\widetilde{B}). \tag{6.9}$$

Finally consider an arbitrary bounded subset $B \subset \mathcal{X}_\psi$ with $\mu_{\mathcal{X}}(\partial B) = 0$. Let B° and $\overline{B}$ be the interior and the closure of B, respectively. Then (6.8) holds for B° and (6.9) holds for $\overline{B}$, and furthermore $\mu_{\mathcal{X}}(B^\circ) = \mu_{\mathcal{X}}(\overline{B})$, since $\mu_{\mathcal{X}}(\partial B) = 0$. Hence (6.6) holds. □

6.3. **Initial discussion regarding [P2] (uniform spherical equidistribution).** We have the following result:

Theorem 6.5. *[P2] holds, i.e. there exists a subset $\mathcal{E} \subset \mathcal{P}$ of density zero such that for any fixed $T \geq 1$ and $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$, we have $\mu_{\boldsymbol{q},\rho}^{(\lambda)} \xrightarrow{\mathrm{w}} \mu_{\varsigma(\boldsymbol{q})}$ as $\rho \to 0$, uniformly for $\boldsymbol{q} \in \mathcal{P} \cap \mathcal{B}_{T\rho^{1-d}}^d \setminus \mathcal{E}$.*

In this section we will prove that Theorem 6.5 follows from the following theorem on uniform equidistribution in the homogeneous space $\mathbb{X}$, the proof of which is the main goal of the later sections in this paper.

Let $\varphi : \mathrm{SL}_d(\mathbb{R}) \to G$ be the diagonal embedding.

Theorem 6.6. *Given any $\psi \in \Psi$ and any decreasing function $\mathcal{T} : (0,1) \to \mathbb{R}^+$, there exists a subset $\mathcal{E} \subset \mathcal{L}_\psi$ of density zero such that for any fixed $f \in \mathrm{C}_b(\mathbb{X}^\psi)$ and $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$, we have*

$$\int_{\mathrm{S}_1^{d-1}} f\big(\Gamma g_0^{(\boldsymbol{q})}\varphi(R(\boldsymbol{v})D_\rho)\big)\,d\lambda(\boldsymbol{v}) - \int_{\mathbb{X}^\psi} f\,d\overline{\omega^{(\boldsymbol{q})}} \to 0 \tag{6.10}$$

as $\rho \to 0$, uniformly over all $\boldsymbol{q} \in \mathcal{L}_\psi \cap \mathcal{B}^d_{\mathcal{T}(\rho)} \setminus \mathcal{E}$.

To see that the statement of Theorem 6.6 makes sense, note that for every $\boldsymbol{q} \in \mathcal{L}_\psi$ we have $\Gamma g_0^{(\boldsymbol{q})} \in \mathbb{X}^\psi$ by Lemma 5.1, and so $\Gamma g_0^{(\boldsymbol{q})}\varphi(R(\boldsymbol{v})D_\rho) \in \mathbb{X}^\psi$ for all $\boldsymbol{v} \in \mathrm{S}_1^{d-1}$ and $\rho > 0$; furthermore, $\overline{\omega^{(\boldsymbol{q})}}(\mathbb{X}^\psi) = 1$ by Lemmas 5.4 and 5.2, meaning that $\overline{\omega^{(\boldsymbol{q})}} \in P(\mathbb{X})$ gives a probability measure on the subset $\mathbb{X}^\psi$.

We will now give the proof of Theorem 6.5, assuming Theorem 6.6. As the very first step, let us apply Theorem 6.6 with $\mathcal{T}(\rho) = \rho^{-d}$ and for each $\psi \in \Psi$; this gives the existence of subsets $\mathcal{E}_\psi \subset \mathcal{L}_\psi$ of density zero such that for any fixed $f \in \mathrm{C}_b(\mathbb{X}^\psi)$ and $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$,

$$\int_{\mathrm{S}_1^{d-1}} f(\Gamma g_0^{(\boldsymbol{q})}\varphi(R(\boldsymbol{v})D_\rho))\,d\lambda(\boldsymbol{v}) - \int_{\mathbb{X}^\psi} f\,d\overline{\omega^{(\boldsymbol{q})}} \to 0 \tag{6.11}$$

as $\rho \to 0$, uniformly over all $\boldsymbol{q} \in \mathcal{L}_\psi \cap \mathcal{B}^d_{\rho^{-d}} \setminus \mathcal{E}_\psi$. Let us now set

$$\mathcal{E} := \bigcup_{\psi \in \Psi} \mathcal{E}_\psi \;\cup \bigcup_{\psi \neq \psi' \in \Psi} (\mathcal{L}_\psi \cap \mathcal{L}_{\psi'}). \tag{6.12}$$

By Lemma 2.2, this set $\mathcal{E}$ is a subset of $\mathcal{P}$ of density zero. We will keep this set $\mathcal{E}$ *fixed* in the rest of the section, and we will prove that the statement of Theorem 6.5 holds with *this* set $\mathcal{E}$. For any $T \geq 1$ and $\rho > 0$ we write, as in (1.6),

$$\mathcal{P}_T(\rho) := \mathcal{P} \cap \mathcal{B}^d_{T\rho^{1-d}} \setminus \mathcal{E}.$$

Note that for any given T we have $T\rho^{1-d} \leq \rho^{-d}$ for all sufficiently small ρ. Recall also that $\mathcal{P} = \cup_{\psi \in \Psi}\mathcal{L}_\psi$. Hence the statement around (6.11) now implies that the following holds:

$$\begin{cases} \text{for any fixed } T \geq 1,\ f \in \mathrm{C}_b(\mathbb{X}) \text{ and } \lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1}), \\ \qquad \displaystyle\int_{\mathrm{S}_1^{d-1}} f(\Gamma g_0^{(\boldsymbol{q})}\varphi(R(\boldsymbol{v})D_\rho))\,d\lambda(\boldsymbol{v}) - \int_{\mathbb{X}} f\,d\overline{\omega^{(\boldsymbol{q})}} \to 0 \\ \text{as } \rho \to 0, \text{ uniformly over all } \boldsymbol{q} \in \mathcal{P}_T(\rho). \end{cases} \tag{6.13}$$

Lemma 6.7. *Let $k \in \mathbb{Z}_{>0}$ and let B be a bounded subset of $\mathbb{R}^d$ with $\mathrm{vol}(\partial B) = 0$. Then for any $V > \mathrm{vol}(B)$, $T \geq 1$ and $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$, there exists $\rho_0 \in (0,1)$ such that*

$$\lambda(\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} \,:\, \#((\mathcal{P} - \boldsymbol{q})R(\boldsymbol{v})D_\rho \cap B \setminus \{\boldsymbol{0}\}) \geq k\}) < c_{\mathcal{P}}V/k \tag{6.14}$$

for all $\rho \in (0, \rho_0)$ and $\boldsymbol{q} \in \mathcal{P}_T(\rho)$.

Proof. The assumptions imply that B is Jordan measurable, and hence there is a function $f \in \mathrm{C}_c(\mathbb{R}^d)$ such that $f = 1$ on B, $0 \leq f \leq 1$ everywhere, and $V_f := \int_{\mathbb{R}^d} f\,d\mathrm{vol} < V$. For each $\psi = (j,i) \in \Psi$, let $\widehat{f}_\psi \in \mathrm{C}(\mathbb{X}_j)$ be the "ψth Siegel transform" of f, as defined in (3.14). The function $\widehat{f}_\psi$ is typically unbounded; therefore we set $\widetilde{f}_\psi = \min(k+1, \widehat{f}_\psi)$; this is a nonnegative function in $\mathrm{C}_b(\mathbb{X}_j)$, and hence $\widetilde{f}_\psi \circ p_j \in \mathrm{C}_b(\mathbb{X})$. Hence by (6.13), and since $p_j(g_0^{(\boldsymbol{q})}) = \mathrm{I}_{U_j^{(\boldsymbol{q})}}\,M_j$,

$$\int_{\mathrm{S}_1^{d-1}} \widetilde{f}_\psi\big(\Gamma_j\,\mathrm{I}_{U_j^{(\boldsymbol{q})}}\,M_j R(\boldsymbol{v})D_\rho\big)\,d\lambda(\boldsymbol{v}) - \int_{\mathbb{X}} \widetilde{f}_\psi \circ p_j\,d\overline{\omega^{(\boldsymbol{q})}} \to 0$$

as $\rho \to 0$, uniformly over all $\boldsymbol{q} \in \mathcal{P}_T(\rho)$. Here $p_{j*}\big(\overline{\omega^{(\boldsymbol{q})}}\big) = \overline{\omega_j^{(\boldsymbol{q})}}$ (cf. (5.12)); hence by Proposition 3.8 and Lemma 5.4,

$$\int_{\mathbb{X}} \widetilde{f}_\psi \circ p_j \, d\overline{\omega^{(\boldsymbol{q})}} \leq \int_{\mathbb{X}} \widehat{f}_\psi \circ p_j \, d\overline{\omega^{(\boldsymbol{q})}} = c_\psi^{-d} V_f + \delta_{\boldsymbol{q}\in\mathcal{L}_\psi} \cdot f(\mathbf{0}).$$

Adding the above over all ψ and using (2.12), it follows that there exists some $\rho_0 \in (0,1)$ such that

$$\int_{\mathrm{S}_1^{d-1}} \sum_{\psi\in\Psi} \Big(\widetilde{f}_\psi(\Gamma_j \, \mathrm{I}_{U_j^{(\boldsymbol{q})}} \, M_j R(\boldsymbol{v}) D_\rho) - \delta_{\boldsymbol{q}\in\mathcal{L}_\psi} \cdot f(\mathbf{0}) \Big) \, d\lambda(\boldsymbol{v}) < c_{\mathcal{P}} V \tag{6.15}$$

for all $\rho \in (0,\rho_0)$ and $\boldsymbol{q} \in \mathcal{P}_T(\rho)$. Here, by (3.14) and (2.24), we have for every $\boldsymbol{v}$ and ψ:

$$\widetilde{f}_\psi\big(\Gamma_j \, \mathrm{I}_{U_j^{(\boldsymbol{q})}} \, M_j R(\boldsymbol{v}) D_\rho\big) - \delta_{\boldsymbol{q}\in\mathcal{L}_\psi} \cdot f(\mathbf{0}) = \min\Bigg(k+1, \sum_{\boldsymbol{p}\in(\mathcal{L}_\psi - \boldsymbol{q})R(\boldsymbol{v})D_\rho} f(\boldsymbol{p})\Bigg) - \delta_{\boldsymbol{q}\in\mathcal{L}_\psi} \cdot f(\mathbf{0})$$
$$\geq \min\Bigg(k, \sum_{\boldsymbol{p}\in(\mathcal{L}_\psi - \boldsymbol{q})R(\boldsymbol{v})D_\rho\setminus\{\mathbf{0}\}} f(\boldsymbol{p})\Bigg).$$

Recalling also that $\cup_{\psi\in\Psi}\mathcal{L}_\psi = \mathcal{P}$, it follows that

$$\int_{\mathrm{S}_1^{d-1}} \min\Bigg(k, \sum_{\boldsymbol{p}\in(\mathcal{P} - \boldsymbol{q})R(\boldsymbol{v})D_\rho\setminus\{\mathbf{0}\}} f(\boldsymbol{p})\Bigg) \, d\lambda(\boldsymbol{v}) < c_{\mathcal{P}} V$$

for all $\rho \in (0,\rho_0)$ and $\boldsymbol{q} \in \mathcal{P}_T(\rho)$. Here, for every $\boldsymbol{v}$ such that $\#((\mathcal{P} - \boldsymbol{q})R(\boldsymbol{v})D_\rho \cap B \setminus \{\mathbf{0}\}) \geq k$, the integrand equals k. Hence we obtain the statement of the lemma. □

Lemma 6.8. *Let $\mathcal{Z}$ be any subset of $\mathcal{P}$ of density zero, and let $T \geq 1$, $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$ and $S > 0$. Then*

$$\lambda(\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} \, : \, \mathcal{Z} \cap (\boldsymbol{q} + \mathcal{B}_S^d D_\rho^{-1} R(\boldsymbol{v})^{-1}) \setminus \{\boldsymbol{q}\} \neq \emptyset\}) \to 0 \tag{6.16}$$

as $\rho \to 0$, uniformly over all $\boldsymbol{q} \in \mathcal{P}_T(\rho)$.

Proof. (This is similar to the proof of [20, Lemma 2.4.11].) Let λ_1 be the normalized Lebesgue measure on S_1^{d-1}. By a standard approximation argument, using the fact that $C_c(\mathrm{S}_1^{d-1})$ is dense in $L^1(\mathrm{S}_1^{d-1})$, it suffices to prove (6.16) for those λ which have a continuous density with respect to λ_1; and thus in fact it suffices to prove (6.16) for the single case $\lambda = \lambda_1$.

Let $T \geq 1$, $S > 0$ and $\varepsilon > 0$ be given. Take $0 < r < S$ so small that $c_{\mathcal{P}} \operatorname{vol}(\mathcal{B}_r^d) < \varepsilon$. Set $k = 2S/r > 2$ and $T' = k^{d-1}T$. By Lemma 6.7, there exists $\rho_0 \in (0,1)$ such that

$$\lambda_1(\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} \, : \, (\mathcal{P} - \boldsymbol{q})R(\boldsymbol{v})D_\rho \cap \mathcal{B}_r^d \setminus \{\mathbf{0}\} \neq \emptyset\}) < \varepsilon \tag{6.17}$$

for all $\rho \in (0,\rho_0)$ and $\boldsymbol{q} \in \mathcal{P}_{T'}(\rho)$. Set $\widetilde{B} := \mathcal{B}_r^d D_k^{-1}$. Replacing ρ by $k\rho$ in (6.17), it follows that for all $\rho \in (0,\rho_0/k)$ and $\boldsymbol{q} \in \mathcal{P}_{T'}(k\rho) = \mathcal{P}_T(\rho)$ we have

$$\lambda_1(\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} \, : \, (\mathcal{P} - \boldsymbol{q})R(\boldsymbol{v})D_\rho \cap \widetilde{B} \setminus \{\mathbf{0}\} \neq \emptyset\}) < \varepsilon. \tag{6.18}$$

One verifies that $|x_1| \geq k_1 := (r/2)^d S^{1-d}$ for all $\boldsymbol{x} \in \mathcal{B}_S^d \setminus \widetilde{B}$, and hence

$$(\mathcal{B}_S^d \setminus \widetilde{B})D_\rho^{-1} \subset A(\rho) := \mathcal{B}_{S\rho^{1-d}}^d \setminus \mathcal{B}_{k_1\rho^{1-d}}^d, \qquad \forall \rho > 0. \tag{6.19}$$

Now for any $\rho \in (0,\rho_0/k)$ and $\boldsymbol{q} \in \mathcal{P}_T(\rho)$ we have, using (6.18) and (6.19):

$$\lambda_1(\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} \, : \, (\mathcal{Z} - \boldsymbol{q})R(\boldsymbol{v})D_\rho \cap \mathcal{B}_S^d \setminus \{\mathbf{0}\} \neq \emptyset\}) < \varepsilon + \sum_{\boldsymbol{p}\in\mathcal{Z}\cap(\boldsymbol{q}+A(\rho))} \lambda_1\big(\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} \, : \, (\boldsymbol{p} - \boldsymbol{q})R(\boldsymbol{v})D_\rho \in \mathcal{B}_S^d\}\big). \tag{6.20}$$

But if $(\boldsymbol{p} - \boldsymbol{q})R(\boldsymbol{v})D_\rho \in \mathcal{B}_S^d$, or equivalently $\boldsymbol{p} \in \boldsymbol{q} + \mathcal{B}_S^d D_\rho^{-1} R(\boldsymbol{v})^{-1}$, then $\boldsymbol{p}$ has a distance less than $S\rho$ to the line $\boldsymbol{q} + \mathbb{R}\boldsymbol{v}$; and if also $\boldsymbol{p} \in \boldsymbol{q} + A(\rho)$ then the angle $\varphi(\boldsymbol{v}, \boldsymbol{p} - \boldsymbol{q})$ between the vectors $\boldsymbol{v}$ and $\boldsymbol{p} - \boldsymbol{q}$ satisfies $\sin\varphi(\boldsymbol{v}, \boldsymbol{p} - \boldsymbol{q}) < (S/k_1)\rho^d$. The measure of the set of such points

$\boldsymbol{v} \in \mathrm{S}_1^{d-1}$ with respect to λ_1 is bounded above by $C_1\rho^{d(d-1)}$, where C_1 depends on d, S, r but not on ρ or $\boldsymbol{p}$. Hence (6.20) is

$$\leq \varepsilon + \#(\mathcal{Z} \cap (\boldsymbol{q} + \mathcal{B}^d_{S\rho^{1-d}})) \cdot C_1\rho^{d(d-1)} \leq \varepsilon + \#(\mathcal{Z} \cap \mathcal{B}^d_{(T+S)\rho^{1-d}}) \cdot C_1\rho^{d(d-1)},$$

and since $\mathcal{Z}$ has density zero, the last term is less than ε for ρ sufficiently small. □

Recall that $\mu^{(\lambda)}_{\boldsymbol{q},\rho}$ is the distribution of $\mathcal{Q}_\rho(\boldsymbol{q},\boldsymbol{v})$ for $\boldsymbol{v}$ random in $(\mathrm{S}_1^{d-1},\lambda)$. We now introduce a certain approximation $\mathcal{Q}'_\rho(\boldsymbol{q},\boldsymbol{v})$ to $\mathcal{Q}_\rho(\boldsymbol{q},\boldsymbol{v})$, which will be easier to handle. We set

$$\widetilde{\mathcal{P}}' = \bigcup_{\psi\in\Psi} \{(\boldsymbol{p},\sigma^\psi) \,:\, \boldsymbol{p} \in \mathcal{L}_\psi\}. \tag{6.21}$$

Note that, unlike the projection $\widetilde{\mathcal{P}} \to \mathcal{P}$, the projection $\widetilde{\mathcal{P}}' \to \mathcal{P}$ is not necessarily injective! (However, by Remark 2.2, it becomes injective after removing a set of density zero from $\widetilde{\mathcal{P}}'$.) For any $\boldsymbol{q} \in \mathcal{P}$, we set

$$\widetilde{\mathcal{P}}'_{\boldsymbol{q}} = \begin{cases} \widetilde{\mathcal{P}}' \setminus \{(\boldsymbol{q},\sigma^{\psi(\boldsymbol{q})})\} & (\boldsymbol{q}\in\mathcal{P}) \\ \widetilde{\mathcal{P}}' & (\boldsymbol{q}\notin\mathcal{P}) \end{cases} \tag{6.22}$$

and

$$\mathcal{Q}'_\rho(\boldsymbol{q},\boldsymbol{v}) = (\widetilde{\mathcal{P}}'_{\boldsymbol{q}} - \boldsymbol{q})R(\boldsymbol{v})D_\rho. \tag{6.23}$$

Lemma 6.9. *For every $\boldsymbol{q} \in \mathcal{P}$ we have $\mathcal{Q}'_\rho(\boldsymbol{q},\boldsymbol{v}) = J_{\psi(\boldsymbol{q})}(\Gamma g_0^{(\boldsymbol{q})}\varphi(R(\boldsymbol{v})D_\rho))$.*

Proof. By parsing the definitions (6.22), (6.23) and (5.14), we see that it suffices to prove

$$\widetilde{\mathcal{P}}' - \boldsymbol{q} = J(\Gamma g_0^{(\boldsymbol{q})}). \tag{6.24}$$

However, comparing (5.13) and (6.21), we see that (6.24) is an immediate consequence of (2.24) (just as is (2.25)). □

The following lemma shows that $\mathcal{Q}'_\rho(\boldsymbol{q},\boldsymbol{v})$ approximates $\mathcal{Q}_\rho(\boldsymbol{q},\boldsymbol{v})$ in a sense that is appropriate for us.

Lemma 6.10. *Let $f \in \mathrm{C}_c(\mathcal{X})$, $T \geq 1$, $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$ and $\varepsilon > 0$. Then*

$$\lambda\bigg(\bigg\{\boldsymbol{v}\in\mathrm{S}_1^{d-1} \,:\, \bigg|\sum_{x\in\mathcal{Q}_\rho(\boldsymbol{q},\boldsymbol{v})} f(x) - \sum_{x\in\mathcal{Q}'_\rho(\boldsymbol{q},\boldsymbol{v})} f(x)\bigg| > \varepsilon\bigg\}\bigg) \to 0 \tag{6.25}$$

as $\rho \to 0$, uniformly over all $\boldsymbol{q} \in \mathcal{P}_T(\rho)$.

Proof. Choose $S > 0$ so that $\mathrm{supp}(f) \subset \mathcal{B}^d_S \times \Sigma$. Note that for every $\boldsymbol{q} \in \mathcal{P}\setminus\mathcal{E}$ and every $\boldsymbol{v} \in \mathrm{S}_1^{d-1}$ we have, using the fact that $\boldsymbol{q} \notin \mathcal{L}_{\psi'}$ for all $\psi' \neq \psi(\boldsymbol{q})$ (which follows from $\boldsymbol{q} \notin \mathcal{E}$ and (6.12)):

$$\begin{aligned} \sum_{x\in\mathcal{Q}_\rho(\boldsymbol{q},\boldsymbol{v})} f(x) - &\sum_{x\in\mathcal{Q}'_\rho(\boldsymbol{q},\boldsymbol{v})} f(x) \\ &= \sum_{\boldsymbol{p}\in\mathcal{P}\setminus\{\boldsymbol{q}\}} \bigg(f\big((\boldsymbol{p}-\boldsymbol{q})R(\boldsymbol{v})D_\rho, \varsigma(\boldsymbol{p})\big) - \sum_{\substack{\psi\in\Psi \\ (\boldsymbol{p}\in\mathcal{L}_\psi)}} f\big((\boldsymbol{p}-\boldsymbol{q})R(\boldsymbol{v})D_\rho,\sigma^\psi\big)\bigg). \end{aligned} \tag{6.26}$$

Set

$$A(\boldsymbol{q},\boldsymbol{v},\rho) := \mathcal{P} \cap (\boldsymbol{q} + \mathcal{B}^d_S D_\rho^{-1}R(\boldsymbol{v})^{-1}) \setminus \{\boldsymbol{q}\}.$$

Note that for every $\boldsymbol{p} \in \mathcal{P}\setminus A(\boldsymbol{q},\boldsymbol{v},\rho)$ we have $(\boldsymbol{p}-\boldsymbol{q})R(\boldsymbol{v})D_\rho \notin \mathcal{B}^d_S$, so that the corresponding term in (6.26) vanishes. Also for every $\boldsymbol{p} \notin \mathcal{E}$ we have $\boldsymbol{p} \notin \mathcal{L}_{\psi'}$ for all $\psi' \neq \psi(\boldsymbol{p})$ (cf. (6.12)), which implies that the corresponding term in (6.26) is bounded in absolute value by $d(\varsigma(\boldsymbol{p}))$, where the function $d : \Sigma \to \mathbb{R}_{\geq 0}$ is defined by

$$d(\psi,\omega) = \sup\{|f(\boldsymbol{x},(\psi,\omega)) - f(\boldsymbol{x},\sigma^\psi)| \,:\, \boldsymbol{x}\in\mathbb{R}^d\} \qquad (\psi,\omega)\in\Sigma. \tag{6.27}$$

Hence for every $\boldsymbol{q} \in \mathcal{P} \setminus \mathcal{E}$ and $\boldsymbol{v} \in \mathrm{S}_1^{d-1}$ such that $\mathcal{E} \cap A(\boldsymbol{q}, \boldsymbol{v}, \rho) = \emptyset$, we have

$$\Bigg| \sum_{x \in \mathcal{Q}_\rho(\boldsymbol{q},\boldsymbol{v})} f(x) - \sum_{x \in \mathcal{Q}'_\rho(\boldsymbol{q},\boldsymbol{v})} f(x) \Bigg| \leq \sum_{\boldsymbol{p} \in A(\boldsymbol{q},\boldsymbol{v},\rho)} d(\varsigma(\boldsymbol{p})). \tag{6.28}$$

Now let $\varepsilon' > 0$ be given. Take $K \in \mathbb{Z}^+$ and $\rho_0 \in (0,1)$ such that

$$\lambda(\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} \, : \, \#((\mathcal{P} - \boldsymbol{q})R(\boldsymbol{v})D_\rho \cap \mathcal{B}_S^d) > K\}) < \varepsilon' \tag{6.29}$$

for all $\rho \in (0, \rho_0)$ and all $\boldsymbol{q} \in \mathcal{P}_T(\rho)$. This is possible by Lemma 6.7. Next set

$$\mathcal{Z} := \{\boldsymbol{p} \in \mathcal{P} \, : \, d(\varsigma(\boldsymbol{p})) \geq \varepsilon/K\}.$$

We claim that the set $\mathcal{Z}$ has density zero. To prove this, set

$$U_\psi := \{\omega \in \Omega \, : \, (\psi, \omega) \in \Sigma \text{ and } d(\psi, \omega) < \varepsilon/K\},$$

so that $\mathcal{Z} \subset \cup_{\psi \in \Psi}\{\boldsymbol{q} \in \mathcal{L}_\psi \, : \, \omega^{(\boldsymbol{q})} \in U_\psi\}$. Note that $\{\psi\} \times U_\psi$ is an open neighbourhood of σ^ψ in Σ, since the function d is continuous. Hence as in the proof of Proposition 6.4, the set $\{\boldsymbol{q} \in \mathcal{L}_\psi \, : \, \omega^{(\boldsymbol{q})} \notin U_\psi\}$ has density zero. Hence also $\mathcal{Z}$ has density zero, as claimed.

It follows that also $\mathcal{E} \cup \mathcal{Z}$ has density zero, and so by Lemma 6.8, after possibly shrinking ρ_0, we have

$$\lambda(\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} \, : \, (\mathcal{E} \cup \mathcal{Z}) \cap A(\boldsymbol{q}, \boldsymbol{v}, \rho) \neq \emptyset\}) < \varepsilon' \tag{6.30}$$

for all $\rho \in (0, \rho_0)$ and all $\boldsymbol{q} \in \mathcal{P}_T(\rho)$. Now note that for any $\boldsymbol{q} \in \mathcal{P} \setminus \mathcal{E}$ and for any $\boldsymbol{v} \in \mathrm{S}_1^{d-1}$ which belongs to neither of the two sets in (6.29) and (6.30), the set $A(\boldsymbol{q}; \boldsymbol{v}, \rho)$ has cardinality at most K and is disjoint from $\mathcal{E} \cup \mathcal{Z}$; therefore the inequality in (6.28) holds, and the right hand side in that inequality is $\leq \varepsilon$. It follows that for all $\rho \in (0, \rho_0)$ and $\boldsymbol{q} \in \mathcal{P}_T(\rho)$, the measure in (6.25) is less than $2\varepsilon'$. □

Proof of Theorem 6.5. Let $T \geq 1$ and $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$ be given. Let $\rho_1, \rho_2 \ldots$ be an arbitrary sequence in $(0,1)$ with $\rho_n \to 0$, and let $\boldsymbol{q}_n \in \mathcal{P}_T(\rho_n)$ for $n = 1, 2, \ldots$ be such that the limit $\varsigma = (\psi, \omega) := \lim_{n\to\infty} \varsigma(\boldsymbol{q}_n) \in \Sigma$ exists. By [20, Lemma 2.1.2], it suffices to prove that in this situation we have

$$\mu^{(\lambda)}_{\boldsymbol{q}_n,\rho_n} \xrightarrow{\mathrm{w}} \mu_\varsigma \qquad \text{as } n \to \infty. \tag{6.31}$$

Since $\varsigma(\boldsymbol{q}_n) \to (\psi, \omega)$ implies that $\psi(\boldsymbol{q}_n) = \psi$ for all large n, we may without loss of generality assume that $\psi(\boldsymbol{q}_n) = \psi$ for *all* n. This means that $\boldsymbol{q}_n \in \mathcal{L}_\psi$ for all n.

For any $\boldsymbol{q} \in \mathcal{P}$, $\rho > 0$ and $\lambda \in P(\mathrm{S}_1^{d-1})$, let $\nu^{(\lambda)}_{\boldsymbol{q},\rho} \in P(\mathbb{X})$ be the distribution of $\Gamma g_0^{(\boldsymbol{q})} \varphi(R(\boldsymbol{v})D_\rho)$ for $\boldsymbol{v}$ random in $(\mathrm{S}_1^{d-1}, \lambda)$. As a first step, let us note that (6.13) implies that

$$\nu^{(\lambda)}_{\boldsymbol{q}_n,\rho_n} \xrightarrow{\mathrm{w}} \overline{\omega} \qquad \text{as } n \to \infty. \tag{6.32}$$

Indeed, let $f \in \mathrm{C}_b(\mathbb{X})$ be given. Then by (6.13) we have $\nu^{(\lambda)}_{\boldsymbol{q}_n,\rho_n}(f) - \overline{\omega^{(\boldsymbol{q}_n)}}(f) \to 0$ as $n \to \infty$. Also $\varsigma(\boldsymbol{q}_n) \to (\psi, \omega)$ implies that $\omega_j^{(\boldsymbol{q}_n)} \xrightarrow{\mathrm{w}} \omega_j$ in $P(\mathbb{T}_j^d)'$ for each $j \in \{1, \ldots, N\}$; hence by Lemma 3.6 and [1, Thm. 2.8(ii)], we have $\overline{\omega^{(\boldsymbol{q}_n)}} \xrightarrow{\mathrm{w}} \overline{\omega}$ in $P(\mathbb{X})$, and thus $\overline{\omega^{(\boldsymbol{q}_n)}}(f) \to \overline{\omega}(f)$. Hence $\nu^{(\lambda)}_{\boldsymbol{q}_n,\rho_n}(f) \to \overline{\omega}(f)$, and (6.32) is proved.

Next, for each n we have $\Gamma g_0^{(\boldsymbol{q}_n)} \in \mathbb{X}^\psi$ by Lemma 5.1, and hence $\nu^{(\lambda)}_{\boldsymbol{q}_n,\rho_n}(\mathbb{X}^\psi) = 1$; also $\overline{\omega}(\mathbb{X}^\psi) = 1$ since $(\psi, \omega) \in \Sigma$; cf. (5.16). Hence all $\nu^{(\lambda)}_{\boldsymbol{q}_n,\rho_n}$ as well as $\overline{\omega}$ may be regarded as elements in $P(\mathbb{X}^\psi)$, and (6.32) implies that $\nu^{(\lambda)}_{\boldsymbol{q}_n,\rho_n} \xrightarrow{\mathrm{w}} \overline{\omega}$ also in $P(\mathbb{X}^\psi)$ [12, Lemma 4.26]. Hence by the continuous mapping theorem,

$$J_{\psi\,*}\, \nu^{(\lambda)}_{\boldsymbol{q}_n,\rho_n} \xrightarrow{\mathrm{w}} J_{\psi\,*}\, \overline{\omega} \qquad \text{as } n \to \infty. \tag{6.33}$$

Here $J_{\psi\,*}\, \overline{\omega} = \mu_\varsigma$, by (5.15). Now let $f \in \mathrm{C}_c(\mathcal{X})$ be given, and let π_f be the continuous map from $N_s(\mathcal{X})$ to $\mathbb{R}$ given by $\pi_f(Q) = \sum_{x \in Q} f(x)$. Then (6.33) implies that

$$\pi_{f*}\, J_{\psi\,*}\, \nu^{(\lambda)}_{\boldsymbol{q}_n,\rho_n} \xrightarrow{\mathrm{w}} \pi_{f*}\, \mu_\varsigma \qquad \text{as } n \to \infty. \tag{6.34}$$

But note that for each $\boldsymbol{q} \in \mathcal{P}$, by Lemma 6.9, $J_{\psi(\boldsymbol{q})*}\nu_{\boldsymbol{q},\rho}^{(\lambda)}$ is the distribution of $\mathcal{Q}'_\rho(\boldsymbol{q},\boldsymbol{v})$ in $N_s(\mathcal{X})$ for $\boldsymbol{v}$ random in $(\mathrm{S}_1^{d-1},\lambda)$. Hence $\pi_{f*}J_{\psi*}\nu_{\boldsymbol{q}_n,\rho_n}^{(\lambda)}$ is the distribution of the real-valued random variable $F'(\boldsymbol{v}) = \sum_{x\in\mathcal{Q}'_{\rho_n}(\boldsymbol{q}_n,\boldsymbol{v})} f(x)$, for $\boldsymbol{v}$ random in $(\mathrm{S}_1^{d-1},\lambda)$. Similarly, $\pi_{f*}\mu_{\boldsymbol{q}_n,\rho_n}^{(\lambda)}$ is the distribution of the real-valued random variable $F(\boldsymbol{v}) = \sum_{x\in\mathcal{Q}_{\rho_n}(\boldsymbol{q}_n,\boldsymbol{v})} f(x)$. By Lemma 6.10, $|F(\boldsymbol{v}) - F'(\boldsymbol{v})|$ converges in probability to 0. Hence by [1, Thm. 3.1], (6.34) implies that

$$\pi_{f*}\mu_{\boldsymbol{q}_n,\rho_n}^{(\lambda)} \xrightarrow{\mathrm{w}} \pi_{f*}\mu_\varsigma \qquad \text{as } n\to\infty. \tag{6.35}$$

We have proved that this holds for any given $f \in \mathrm{C}_c(\mathcal{X})$. By [12, Thm. 16.16(ii)⇒(i)], this implies that (6.31) holds. □

6.4. **Verification of [P3], and the macroscopic limit.**

Proposition 6.11. *[P3] holds, i.e. for every bounded Borel set $B \subset \mathbb{R}^d$ we have*

$$\lim_{\xi\to\infty}\limsup_{\rho\to 0}\ [\mathrm{vol}\times\sigma]\big(\big\{(\boldsymbol{q},\boldsymbol{v})\in B\times \mathrm{S}_1^{d-1}\ :\ \mathcal{Q}_\rho(\rho^{1-d}\boldsymbol{q},\boldsymbol{v})\cap(\mathfrak{Z}_\xi\times\Sigma)=\emptyset\big\}\big)=0. \tag{6.36}$$

Proof. Using the fact that $\mathcal{P}$ contains at least one grid, the proposition follows from the existence of a limit distribution on $\mathbb{R}_{>0}$ (with zero mass at $+\infty$) for the macroscopic free path length in the Boltzmann-Grad limit of the Lorentz gas on a lattice scatterer configuration, [15, Theorem 1.2]. Indeed, fix an arbitrary $\psi \in \Psi$. For any $\rho > 0$ and $\boldsymbol{q} \in \mathbb{R}^d$, $\boldsymbol{v} \in \mathrm{S}_1^{d-1}$, set

$$Q'_\rho(\boldsymbol{q},\boldsymbol{v}) = (\mathcal{L}_\psi - \boldsymbol{q})R(\boldsymbol{v})D_\rho \qquad (\subset \mathbb{R}^d).$$

Comparing with the definition of $\mathcal{Q}_\rho(\boldsymbol{q},\boldsymbol{v})$ in (1.4), (1.3), and using $\mathcal{L}_\psi \subset \mathcal{P}$ and $\mathbf{0} \notin \mathfrak{Z}_\xi$, one verifies that for any $\xi > 0$, $\mathcal{Q}_\rho(\boldsymbol{q},\boldsymbol{v}) \cap (\mathfrak{Z}_\xi \times \Sigma) = \emptyset$ forces $Q'_\rho(\boldsymbol{q},\boldsymbol{v}) \cap \mathfrak{Z}_\xi = \emptyset$. Hence to prove the proposition it suffices to prove that

$$\lim_{\xi\to\infty}\limsup_{\rho\to 0}\ [\mathrm{vol}\times\sigma]\big(\big\{(\boldsymbol{q},\boldsymbol{v})\in B\times \mathrm{S}_1^{d-1}\ :\ Q'_\rho(\rho^{1-d}\boldsymbol{q},\boldsymbol{v})\cap\mathfrak{Z}_\xi=\emptyset\big\}\big)=0. \tag{6.37}$$

By a simple translation and rescaling argument we may reduce to the case when $\mathcal{L}_\psi$ has covolume one and is a lattice, and then (6.37) is a simple consequence of [15, Theorem 1.2]. □

Recall that in [20, Sec. 2.5], the condition [P3] is used to prove, for an arbitrary fixed point set $\mathcal{P} \subset \mathbb{R}^d$ satisfying the hypotheses in Section 1.1, the existence of a canonical measure $\mu^{\mathrm{g}} \in P(N_s(\mathcal{X}))$ [11] giving the limit distribution of $\mathcal{Q}_\rho(\boldsymbol{q},\boldsymbol{v})$ in the case of a *macroscopic* initial condition. That is, μ^{g} equals the limit distribution of $\mathcal{Q}_\rho(\rho^{1-d}\boldsymbol{q},\boldsymbol{v})$ for $(\boldsymbol{q},\boldsymbol{v})$ random in $(\mathrm{T}^1(\mathbb{R}^d),\Lambda)$, for any fixed probability measure $\Lambda \in P(\mathrm{T}^1(\mathbb{R}^d))$ absolutely continuous with respect to the Liouville measure vol $\times\sigma$ [20, Theorem 2.19]. This measure μ^{g} also appears in the original definition of the transition kernel for generic initial data, k^{g}; see [20, (3.7), (3.8)]. The formula which we stated for k^{g} in (1.13) follows from the fact that the *Palm distributions* of a point process with distribution μ^{g} can be given explicitly in terms of the measures μ_ς; see [20, Prop. 3.21].

In our case of $\mathcal{P}$ being a finite union of grids as in (2.6), the macroscopic limit measure μ^{g} can be explicitly defined as follows: Set

$$\omega^{\mathrm{g}} := \big(\omega_1^{\mathrm{g}},\ldots,\omega_N^{\mathrm{g}}\big) \in \Omega, \tag{6.38}$$

and then let

$$\mu^{\mathrm{g}} = J_*(\overline{\omega^{\mathrm{g}}}), \tag{6.39}$$

with $J : \mathbb{X} \to N_s(\mathcal{X})$ being the map in (5.13).

We next state without proof a limit result which significantly strengthens the above mentioned [20, Theorem 2.19] for our special class of $\mathcal{P}$. For any $\Lambda \in P(\mathrm{T}^1(\mathbb{R}^d))$, $s > 0$ and $\rho \in (0,1)$, let $\mu_\rho^{(\Lambda,s)}$ be the distribution of $\mathcal{Q}_\rho(s\boldsymbol{q},\boldsymbol{v})$ for $(\boldsymbol{q},\boldsymbol{v})$ random in $(\mathrm{T}^1(\mathbb{R}^d),\Lambda)$.

[11] In [20] this measure is called simply "μ".

Theorem 6.12. *For any $\Lambda \in P(\mathrm{T}^1(\mathbb{R}^d))$ which is absolutely continuous with respect to Liouville measure, and any $s_0 > 0$, we have $\mu_\rho^{(\Lambda,s)} \xrightarrow{\mathrm{w}} \mu^{\mathrm{g}}$ as $\rho \to 0$, uniformly over all $s \geq s_0$.*

Note that [20, Theorem 2.19] corresponds to the particular choice $s = \rho^{1-d}$ in Theorem 6.12. The formulation of Theorem 6.12 is inspired by [18, Theorem 1.1].

As mentioned, we will not give the proof of Theorem 6.12 in the present paper. However we remark that Theorem 6.12 can be deduced, by similar arguments as in Section 6.3, from the following equidistribution result in the homogeneous space $\mathbb{X}$, which is a kind of macroscopic analogue of Theorem 3.2.

Theorem 6.13. *For any $\Lambda \in P(\mathrm{T}^1(\mathbb{R}^d))$ which is absolutely continuous with respect to Liouville measure, and any $f \in \mathrm{C}_b(\mathbb{X})$ and $s_0 > 0$, we have*

$$\int_{\mathrm{T}^1(\mathbb{R}^d)} f\big(\Gamma g_0^{(s\boldsymbol{q})}\varphi(R(\boldsymbol{v})D_\rho)\big)\,d\Lambda(\boldsymbol{q},\boldsymbol{v}) \to \int_{\mathbb{X}} f\,d\overline{\omega^{\mathrm{g}}} \tag{6.40}$$

as $\rho \to 0$, uniformly over all $s \geq s_0$.

We will not give the proof of Theorem 6.13 either; however we note that it is to a large extent similar to the proof of Theorem 6.6 which we give in Section 8 below.

7. Application of the classification of invariant measures of unipotent flows

In this section we will state and prove a result, Theorem 7.1, on the equidistribution of certain expanding unipotent orbits in a slightly generalized version of the homogeneous space $\mathbb{X}$ introduced in Section 2.2. This theorem is tailor made to serve as the main ingredient in the proof of Theorem 6.6 which we give in Section 8 below; in particular, it will be crucial for us to have a certain uniformity with respect to the position of the initial point in the torus fiber (that is, uniformity with respect to the variable "V" in Theorem 7.1 below). The proof of Theorem 7.1 builds on Ratner's classification of ergodic measures invariant under unipotent flows [24] and further characterization results by Shah and Mozes [22].

We start by introducing some notation. We stress that in this Section 7, some of our notation (for example, "$\mathbb{X}$", "G", "Γ", "Γ_j" and "$\widetilde{M}$") will be used in a slightly different and more general way than in all the other sections of the paper. The reason is that the results of the present section will be applied, in Section 8, to certain homogeneous *submanifolds* of our original space "$\mathbb{X}$" (see the proofs of Theorems 8.1 and 8.2). To start with, similarly as before, we set

$$G = G_1 \times \cdots \times G_N = \mathrm{S}_{r_1}(\mathbb{R}) \times \cdots \times \mathrm{S}_{r_N}(\mathbb{R});$$

however now we allow $r_1, \ldots, r_N$ to be arbitrary (fixed) *non-negative* integers. That is, unlike all the other sections, we allow one or several of the r_js to be *zero,* with the natural convention that $\mathrm{S}_0(\mathbb{R}) := \mathrm{SL}_d(\mathbb{R})$. Next, we fix $\Gamma_1', \ldots, \Gamma_N'$ to be arbitrary, fixed, finite index subgroups of $\mathrm{SL}_d(\mathbb{Z})$, and set

$$\Gamma_j = \Gamma_j' \ltimes \mathrm{M}_{r_j \times d}(\mathbb{Z}) = \big\{(M,U) \in \mathrm{S}_{r_j}(\mathbb{Z}) \,:\, M \in \Gamma_j'\big\} \qquad (j = 1, \ldots, N) \tag{7.1}$$

(if $r_j = 0$, this should be understood as $\Gamma_j = \Gamma_j'$), and

$$\Gamma = \Gamma_1 \times \cdots \times \Gamma_N.$$

Then, as before, we set $\mathbb{X}_j := \Gamma_j \backslash G_j$ and

$$\mathbb{X} := \Gamma\backslash G = \mathbb{X}_1 \times \cdots \times \mathbb{X}_N,$$

and write $p_j : G \to \mathrm{S}_{r_j}(\mathbb{R})$ and $\widetilde{p}_j : \mathbb{X} \to \mathbb{X}_j$ $(j = 1, \ldots, N)$ for the projection maps.

Recall that we consider $\mathrm{SL}_d(\mathbb{R})$ to be an embedded subgroup of each group G_j, through $M \mapsto (M, 0)$. Now we also set $G' := \mathrm{SL}_d(\mathbb{R})^N$; this is an embedded subgroup of G. We also set

$$\Gamma' := \Gamma_1' \times \cdots \times \Gamma_N' \subset G'$$

and

$$\mathbb{X}' = \Gamma' \backslash G'.$$

Recall that we have a projection morphism $\iota : G_j \to \mathrm{SL}_d(\mathbb{R})$ for each j, and ι induces a projection map

$$\widetilde{\iota} : \mathbb{X}_j \to \Gamma'_j \backslash \mathrm{SL}_d(\mathbb{R}), \qquad \widetilde{\iota}(\Gamma_j g) = \Gamma'_j \, \iota(g) \quad (g \in G_j), \tag{7.2}$$

generalizing (2.26). In the present section we will also write ι for the product morphism from G to G', and write $\widetilde{\iota}$ for the induced projection map from $\mathbb{X}$ to $\mathbb{X}'$.

As before we set

$$\mathbb{T}_j := \mathbb{R}^{r_j}/\mathbb{Z}^{r_j} \qquad \text{and} \qquad \mathbb{T}_j^d := \underbrace{\mathbb{T}_j \times \cdots \times \mathbb{T}_j}_{d \text{ copies}};$$

if $r_j = 0$ this should be understood as $\mathbb{T}_j = \mathbb{T}_j^d = \{\mathbf{0}\}$, the trivial group. The definition in (2.28) of the embedding $x : \mathbb{T}_j^d \to \mathbb{X}_j$ carries over unchanged to our present setting, although our "$\mathbb{X}_j$" is now more general. (If $r_j = 0$ then we set $x(\{\mathbf{0}\}) := \Gamma_j \in \mathbb{X}_j$.) We now also set

$$\widetilde{\mathbb{T}} := \mathbb{T}_1^d \times \mathbb{T}_2^d \times \cdots \times \mathbb{T}_N^d, \tag{7.3}$$

and let $\widetilde{p}_j : \widetilde{\mathbb{T}} \to \mathbb{T}_j^d$ ($j = 1, \ldots, N$) be the projection maps; and we will write "x" also for the map $\widetilde{\mathbb{T}} \to \mathbb{X}$ which is the product of the maps $x : \mathbb{T}_j^d \to \mathbb{X}_j$. The fact that both "$x$" and "$\widetilde{p}_j$" now denote more than one map should not cause any confusion; in particular note that with this abuse of notation we have $x \circ \widetilde{p}_j = \widetilde{p}_j \circ x : \widetilde{\mathbb{T}} \to \mathbb{X}_j$ for each j.

We now come to the statement of the main result of the present section, Theorem 7.1. It concerns the equidistribution of pieces of expanding unipotent orbits in $\mathbb{X}$ of the form

$$\big\{ x(V) \widetilde{M} \, \varphi(n_-(\boldsymbol{u}) D_\rho) \; : \; \boldsymbol{u} \in \mathbb{R}^{d-1} \big\}, \tag{7.4}$$

where $V \in \widetilde{\mathbb{T}}$; $\widetilde{M}$ is an arbitrary element in G' not belonging to the subset

$$\mathfrak{D}_{\mathcal{S}} := \bigcup_{i<j} \big\{ (M_1, \ldots, M_N) \in G' \; : \; M_i M_j^{-1} \in \mathcal{S} \big\}, \tag{7.5}$$

with $\mathcal{S}$ as in (2.9); φ is the diagonal embedding of $\mathrm{SL}_d(\mathbb{R})$ in G; and finally

$$n_-(\boldsymbol{u}) := \begin{pmatrix} 1 & \boldsymbol{u} \\ 0 & I_{d-1} \end{pmatrix} \in \mathrm{SL}_d(\mathbb{R}) \tag{7.6}$$

(block diagonal notation). The equidistribution is with respect to the G-invariant probability measure on $\mathbb{X}$, which we call μ.

In order for orbits of the form (7.4) to equidistribute in $(\mathbb{X}, \mu)$ as $\rho \to 0$, we have to assume that V avoids a certain 'singular' subset $\Delta_k^{(\eta)}$ of $\widetilde{\mathbb{T}}$, which we now introduce. For each $j \in \{1, \ldots, N\}$ with $r_j \neq 0$, let us write π_j for the projection from $(\mathbb{R}^{r_j})^d$ to $\mathbb{T}_j^d$ (it was called "π" in (2.27)). Then for any $q \in \mathbb{Z}^+$ and $\boldsymbol{m} \in \mathbb{Z}^{r_j} \setminus \{\mathbf{0}\}$, we set

$$\Delta_{j,q,\boldsymbol{m}} := \pi_j((q^{-1}\mathbb{Z}^{r_j} + \boldsymbol{m}^\perp)^d) \subset \mathbb{T}_j^d, \tag{7.7}$$

where $\boldsymbol{m}^\perp$ is the orthogonal complement of $\boldsymbol{m}$ in $\mathbb{R}^{r_j}$. Also, for any $k \in \mathbb{Z}^+$, we set

$$\Delta_{j,k} := \bigcup_{q=1}^{k} \bigcup_{\substack{\boldsymbol{m} \in \mathbb{Z}^{r_j} \\ 0 < \|\boldsymbol{m}\| \leq k}} \Delta_{j,q,\boldsymbol{m}}. \tag{7.8}$$

Note that $\Delta_{j,q,\boldsymbol{m}}$ and $\Delta_{j,k}$ are only defined when $r_j \neq 0$, in which case they are both closed regular submanifolds of $\mathbb{T}_j^d$ of codimension d. Next, for any $\eta > 0$ we define $\Delta_{j,k}^{(\eta)}$ to be the open

η-neighbourhood of $\Delta_{j,k}$ in $\mathbb{T}_j^d$, with respect to the metric induced by the standard Euclidean metric on $\mathrm{M}_{r_j\times d}(\mathbb{R}) = (\mathbb{R}^{r_j})^d$. Finally, we set

$$\Delta_k^{(\eta)} = \bigcup_{\substack{j=1\\(r_j\neq 0)}}^{N} \widetilde{p}_j^{\,-1}\big(\Delta_{j,k}^{(\eta)}\big) \subset \widetilde{\mathbb{T}}. \tag{7.9}$$

Theorem 7.1. *Let $f \in \mathrm{C}_b(\mathbb{X})$ and $\varepsilon > 0$ be given. Then there exists some $k \in \mathbb{Z}^+$ such that for every $\lambda \in P_{\mathrm{ac}}(\mathbb{R}^{d-1})$, $\eta > 0$ and $\widetilde{M} \in G' \setminus \mathfrak{D}_{\mathcal{S}}$, there exists some $\rho_0 \in (0,1)$ such that*

$$\left| \int_{\mathbb{R}^{d-1}} f\Big(x(V)\widetilde{M}\varphi(n_-(\boldsymbol{u})D_\rho)\Big)\, d\lambda(\boldsymbol{u}) - \int_{\mathbb{X}} f\, d\mu \right| < \varepsilon \tag{7.10}$$

for all $\rho \in (0, \rho_0)$ and all $V \in \widetilde{\mathbb{T}} \setminus \Delta_k^{(\eta)}$.

The rest of this section is devoted to the proof of Theorem 7.1.

Definition 7.1. For each $k \in \mathbb{Z}^+$, we let P_k be the set of all measures $\nu \in P(\mathbb{X})$ which can be obtained as a weak limit of a sequence of probability measures $\nu_1, \nu_2, \ldots$ given by

$$\nu_m : \quad f \mapsto \int_{\mathbb{R}^{d-1}} f\big(x(V_m)\widetilde{M}\varphi(n_-(\boldsymbol{u})D_{\rho_m})\big)\, d\lambda(\boldsymbol{u}) \qquad (f \in \mathrm{C}_b(\mathbb{X})), \tag{7.11}$$

for some $\lambda \in P_{\mathrm{ac}}(\mathbb{R}^{d-1})$, $\widetilde{M} \in G' \setminus \mathfrak{D}_{\mathcal{S}}$, real numbers $\rho_1 > \rho_2 > \cdots \to 0$, and points $V_1, V_2, \ldots$ in $\widetilde{\mathbb{T}}$ such that $[\exists \eta > 0\colon \forall m \in \mathbb{Z}^+\colon V_m \notin \Delta_k^{(\eta)}]$.

Note that $P_1 \supset P_2 \supset \cdots$, since $\Delta_k^{(\eta)} \subset \Delta_{k'}^{(\eta)}$ whenever $k < k'$.

Throughout the rest of this section, we will let W denote the following subgroup of G:

$$W := \big\{\varphi(n_-(\boldsymbol{w}))\ :\ \boldsymbol{w} \in \mathbb{R}^{d-1}\big\}.$$

Lemma 7.2. *Every $\nu \in P_k$ is W-invariant.*

Proof. This is a (very) standard consequence of the fact that for $\rho < 1$, the action (from the right) of $\varphi(D_\rho)$ on $\mathbb{X}$ expands any W-orbit. The details are as follows. Let $\nu \in P_k$ be given. Then the task is to prove that for any given $\boldsymbol{w} \in \mathbb{R}^{d-1}$ and $f \in \mathrm{C}_b(\mathbb{X})$ we have $\nu(f\circ R_{\boldsymbol{w}}) = \nu(f)$, where $R_{\boldsymbol{w}} : \mathbb{X} \to \mathbb{X}$ denotes right multiplication by $\varphi(n_-(\boldsymbol{w}))$. Choose $\lambda, \widetilde{M}, (\rho_m), (V_m)$ as in Definition 7.1, so that ν is the weak limit of the measures ν_m given by (7.11). Using the relation $D_\rho\, n_-(\boldsymbol{w}) = n_-(\rho^d \boldsymbol{w})\, D_\rho$, and writing $\lambda' \in \mathrm{L}^1(\mathbb{R}^{d-1})$ for the density of λ with respect to Lebesgue measure, we now have:

$$\nu_m(f \circ R_{\boldsymbol{w}}) = \int_{\mathbb{R}^{d-1}} f\Big(x(V_m)\widetilde{M}\varphi(n_-(\boldsymbol{u})D_{\rho_m})\Big)\, \lambda'(\boldsymbol{u} - \rho_m^d \boldsymbol{w})\, d\boldsymbol{u}.$$

Hence $|\nu_m(f\circ R_{\boldsymbol{w}}) - \nu_m(f)| \leq \|f\|_{\mathrm{L}^\infty} \cdot \|\tau_{\rho_m^d \boldsymbol{w}}\lambda' - \lambda'\|_{\mathrm{L}^1(\mathbb{R}^{d-1})}$, and so by [7, Prop. 8.5] we have $\lim_{m\to\infty} \nu_m(f\circ R_{\boldsymbol{w}}) = \lim_{m\to\infty}\nu_m(f)$, that is, $\nu(f\circ R_{\boldsymbol{w}}) = \nu(f)$. □

Recall that we write $\widetilde{\iota}$ for the natural projection map from $\mathbb{X}$ to $\mathbb{X}'$; in particular $\widetilde{\mu} := \widetilde{\iota}_*\,\mu$ is the unique G'-invariant probability measure on $\mathbb{X}'$.

Lemma 7.3. *Every $\nu \in P_k$ satisfies $\widetilde{\iota}_*\,\nu = \widetilde{\mu}$.*

Proof. Let $\nu \in P_k$ be given, and let (ν_m) be a sequence as in Definition 7.1, tending weakly to ν. For any $f \in \mathrm{C}_b(\mathbb{X}')$ we have

$$\nu_m(f\circ\widetilde{\iota}) = \int_{\mathbb{R}^{d-1}} f\big(\widetilde{\iota}\big(x(V_m)\widetilde{M}\varphi(n_-(\boldsymbol{u})D_{\rho_m})\big)\big)\, d\lambda(\boldsymbol{u}) = \int_{\mathbb{R}^{d-1}} f\big(\Gamma'\widetilde{M}\varphi(n_-(\boldsymbol{u})D_{\rho_m})\big)\, d\lambda(\boldsymbol{u}).$$

This integral tends to $\widetilde{\mu}(f)$ as $m \to \infty$, by [19, Thm. 5] applied to the function $g \mapsto f(\Gamma'\widetilde{M}g)$ (which is left $\prod_{j=1}^N (M_j^{-1}\Gamma_j' M_j)$–invariant). On the other hand, by the definition of ν we have $\nu_m(f\circ\widetilde{\iota}) \to \nu(f\circ\widetilde{\iota}) = (\widetilde{\iota}_*\,\nu)(f)$. Hence $(\widetilde{\iota}_*\,\nu)(f) = \widetilde{\mu}(f)$. □

Recall that a subgroup U of G is said to be *unipotent*, if the linear automorphism $\mathrm{Ad}(u)$ of the Lie algebra of G is unipotent for all $u \in U$. For any $h \in G$ let us write $R_h : \mathbb{X} \to \mathbb{X}$ for the map $\Gamma g \mapsto \Gamma gh$. For any $\alpha \in P(\mathbb{X})$, let us define H_α to be the identity component of the subgroup of G consisting of all g which preserve α;

$$H_\alpha := \{g \in G \,:\, R_{g*}\alpha = \alpha\}^\circ. \tag{7.12}$$

This is a closed connected Lie subgroup of G. We let $\mathcal{Q}(\mathbb{X})$ be the set of all $\alpha \in P(\mathbb{X})$ such that the group generated by all unipotent one-parameter subgroups of G contained in H_α acts ergodically on $\mathbb{X}$ with respect to α. (Note that this definition of $\mathcal{Q}(\mathbb{X})$ is equivalent to the one in [22, p. 150], although our H_α equals "$\Lambda(\alpha)^\circ$" in the notation of [22].)

A key ingredient in our proof of Theorem 7.1 will be Ratner's classification of invariant measures of unipotent flows, [24, Thm. 1]. Applied in our setting, this result says that for every $\alpha \in \mathcal{Q}(\mathbb{X})$, there is some $g_\alpha \in G$ such that $\alpha(\Gamma\backslash\Gamma g_\alpha H_\alpha) = 1$. Note that in this situation, $\Gamma \cap g_\alpha H_\alpha g_\alpha^{-1}$ is a lattice in $g_\alpha H_\alpha g_\alpha^{-1}$, and the support of α equals $\Gamma\backslash\Gamma g_\alpha H_\alpha$, which is a smooth embedded submanifold of $\mathbb{X}$.

Lemma 7.4. *For any $\alpha \in \mathcal{Q}(\mathbb{X})$ such that $\widetilde{\iota}_* \alpha = \widetilde{\mu}$, we have $\iota(H_\alpha) = G'$.*

Proof. (This generalizes [5, Lemma 6], and the proof is the same.) Using the fact that the map $\iota : \mathbb{X} \to \mathbb{X}'$ has compact fibers, we have $\widetilde{\iota}(\operatorname{supp}\alpha) = \operatorname{supp}\widetilde{\iota}_*\alpha = \operatorname{supp}\widetilde{\mu} = \mathbb{X}'$. But $\operatorname{supp}\alpha = \Gamma\backslash\Gamma g_\alpha H_\alpha$. Hence $\Gamma'\iota(g_\alpha)\iota(H_\alpha) = G'$, and thus $\iota(H_\alpha) = G'$. □

Next, using basic Lie group and Lie algebra theory, we will derive a completely explicit description of any Lie subgroup H_α as in Lemma 7.4; cf. Lemma 7.6 below.

For any $r \in \mathbb{Z}_{\geq 0}$, let $\mathfrak{s}_r(\mathbb{R})$ be the Lie algebra of $\mathrm{S}_r(\mathbb{R})$, which we represent as the set of pairs $(A, X) \in \mathfrak{sl}_d(\mathbb{R}) \times \mathrm{M}_{r\times d}(\mathbb{R})$, with the Lie bracket given by

$$[(A_1, X_1), (A_2, X_2)] = ([A_1, A_2], X_1A_2 - X_2A_1). \tag{7.13}$$

(For $r = 0$ we have $\mathfrak{s}_0(\mathbb{R}) = \mathfrak{sl}_d(\mathbb{R})$, and in (7.13) we should view $\mathrm{M}_{0\times d}(\mathbb{R})$ as a singleton set containing only the "empty matrix".) Just as for the Lie groups, we always consider $\mathfrak{sl}_d(\mathbb{R})$ to be embedded in $\mathfrak{s}_r(\mathbb{R})$ through $A \mapsto (A, 0)$. We also set $\mathfrak{g} = \mathfrak{s}_{r_1}(\mathbb{R}) \oplus \cdots \oplus \mathfrak{s}_{r_N}(\mathbb{R})$ and $\mathfrak{g}' = \mathfrak{sl}_d(\mathbb{R})^N$; these are the Lie algebras of G and of G', respectively. Next, as in (3.1), given any linear subspace L of $\mathbb{R}^r$, we let $\mathrm{S}_L(\mathbb{R})$ be the closed connected subgroup of $\mathrm{S}_r(\mathbb{R})$ given by

$$\mathrm{S}_L(\mathbb{R}) := \mathrm{SL}_d(\mathbb{R}) \ltimes L^d = \left\{(M, U) \in \mathrm{S}_r(\mathbb{R}) \,:\, U \in L^d\right\} \tag{7.14}$$

Recall here that via our identification $\mathrm{M}_{r\times d}(\mathbb{R}) = (\mathbb{R}^r)^d$, L^d is the set of matrices in $\mathrm{M}_{r\times d}(\mathbb{R})$ all of whose column vectors lie in L. For any matrix $X \in \mathrm{M}_{r\times d}(\mathbb{R})$, we also write:

$$\mathrm{S}_L^X(\mathbb{R}) := \mathrm{I}_X \,\mathrm{S}_L(\mathbb{R})\, \mathrm{I}_X^{-1}\,. \tag{7.15}$$

(For $r = 0$ we have $\mathbb{R}^r = \{\mathbf{0}\}$, the only linear subspace $L \subset \mathbb{R}^0$ is $L = \mathbb{R}^0$, and the only matrix in $\mathrm{M}_{0\times d}(\mathbb{R})$ is $X =$ the empty matrix, and for these L, X we have $\mathrm{S}_L(\mathbb{R}) = \mathrm{S}_L^X(\mathbb{R}) = \mathrm{SL}_d(\mathbb{R})$.) We write $\mathfrak{s}_L(\mathbb{R})$ and $\mathfrak{s}_L^X(\mathbb{R})$ for the Lie subalgebras of $\mathfrak{s}_r(\mathbb{R})$ corresponding to $\mathrm{S}_L(\mathbb{R})$ and $\mathrm{S}_L^X(\mathbb{R})$, respectively. Thus in particular,

$$\mathfrak{s}_L(\mathbb{R}) = \{(A, Y) \in \mathfrak{s}_r(\mathbb{R}) \,:\, Y \in L^d\}. \tag{7.16}$$

Recall that we write ι for the natural projection $\mathrm{S}_r(\mathbb{R}) \to \mathrm{SL}_d(\mathbb{R})$; hence $d\iota$ is the natural projection $\mathfrak{s}_r(\mathbb{R}) \to \mathfrak{sl}_d(\mathbb{R})$.

Lemma 7.5. *If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{s}_r(\mathbb{R})$ satisfying $d\iota(\mathfrak{h}) = \mathfrak{sl}_d(\mathbb{R})$, then there exist a linear subspace $L \subset \mathbb{R}^r$ and a matrix $X \in \mathrm{M}_{r\times d}(\mathbb{R})$ such that $\mathfrak{h} = \mathfrak{s}_L^X(\mathbb{R})$.*

Proof. This is a Lie algebra version of [5, Lemma 7], and essentially the same proof works. Therefore we here give a rather terse presentation, referring to the proof in [5] for further details. Of course if $r = 0$ then $d\iota(\mathfrak{h}) = \mathfrak{sl}_d(\mathbb{R})$ implies $\mathfrak{h} = \mathfrak{sl}_d(\mathbb{R})$ and the lemma is trivial; hence in the following we may assume $r > 0$.

Set $L' = \{X \in \mathrm{M}_{r\times d}(\mathbb{R}) \,:\, (0,X) \in \mathfrak{h}\}$; this is a linear subspace of $\mathrm{M}_{r\times d}(\mathbb{R})$. Using $d\iota(\mathfrak{h}) = \mathfrak{sl}_d(\mathbb{R})$ it follows that $XA \in L'$ for all $X \in L'$ and all $A \in \mathfrak{sl}_d(\mathbb{R})$; and this in turn implies that L' must be of the form $L' = L^d$ for some linear subspace $L \subset \mathbb{R}^r$. Let $L^\perp$ be the orthogonal complement of L in $\mathbb{R}^r$; then $\mathrm{M}_{r\times d}(\mathbb{R}) = L^d \oplus (L^\perp)^d$, and for each $A \in \mathfrak{sl}_d(\mathbb{R})$ there exists a unique $Y \in (L^\perp)^d$ such that $(A,Y) \in \mathfrak{h}$. This implies that the Lie subalgebra $\mathfrak{s}_{L^\perp}(\mathbb{R}) \cap \mathfrak{h}$ is a Levi subalgebra of $\mathfrak{s}_{L^\perp}(\mathbb{R})$, and so by Malcev's Theorem [10, Ch. III.9], there exists some $X \in (L^\perp)^d$ such that $\mathfrak{s}_{L^\perp}(\mathbb{R}) \cap \mathfrak{h} = \big(\mathrm{Ad}\, \mathrm{I}_X\big)(\mathfrak{sl}_d(\mathbb{R}))$. But $\mathfrak{h}$ is the vector space direct sum of $\mathfrak{s}_{L^\perp}(\mathbb{R}) \cap \mathfrak{h}$ and $\{(0,U) \,:\, U \in L^d\}$; hence in fact $\mathfrak{h} = \mathfrak{s}_L^X(\mathbb{R})$. □

Lemma 7.6. *Assume that H is a connected Lie subgroup of G satisfying $\iota(H) = G'$. Then there exist linear subspaces $L_j \subset \mathbb{R}^{r_j}$ and matrices $X_j \in \mathrm{M}_{r_j\times d}(\mathbb{R})$ such that*

$$H = \mathrm{S}_{L_1}^{X_1}(\mathbb{R}) \times \cdots \times \mathrm{S}_{L_N}^{X_N}(\mathbb{R}). \tag{7.17}$$

Proof. Let $\mathfrak{h}$ be the Lie subalgebra of $\mathfrak{g} = \mathfrak{s}_{r_1}(\mathbb{R}) \times \cdots \times \mathfrak{s}_{r_N}(\mathbb{R})$ corresponding to H. It follows from $\iota(H) = G'$ that $d\iota(\mathfrak{h}) = \mathfrak{g}'$. Recall that $p_j : G \to G_j = \mathrm{S}_{r_j}(\mathbb{R})$ denotes the projection onto the jth factor. It follows from $d\iota(\mathfrak{h}) = \mathfrak{g}'$ that, for each j, the Lie subalgebra $dp_j(\mathfrak{h})$ of $\mathfrak{s}_{r_j}(\mathbb{R})$ satisfies $d\iota(dp_j(\mathfrak{h})) = \mathfrak{sl}_d(\mathbb{R})$, and so by Lemma 7.5 there exist a linear subspace $L_j \subset \mathbb{R}^{r_j}$ and a matrix $X_j \in \mathrm{M}_{r_j\times d}(\mathbb{R})$ such that

$$dp_j(\mathfrak{h}) = \mathfrak{s}_{L_j}^{X_j}(\mathbb{R}) \qquad (\forall j \in \{1,\ldots,N\}). \tag{7.18}$$

This implies:

$$\mathfrak{h} \subset \mathfrak{s}_{L_1}^{X_1}(\mathbb{R}) \times \cdots \times \mathfrak{s}_{L_N}^{X_N}(\mathbb{R}). \tag{7.19}$$

We claim that the two sides of (7.19) are in fact equal. In order to prove this equality, it suffices to prove that $\varphi_j(\mathfrak{s}_{L_j}^{X_j}(\mathbb{R})) \subset \mathfrak{h}$ for each j, where

$$\varphi_j : \mathfrak{s}_{r_j}(\mathbb{R}) \to \mathfrak{g}$$

is the Lie group homomorphism mapping X to $(0,\cdots,X,\cdots,0)$ (0s in all positions except the jth). Let j be fixed, and set

$$\mathfrak{l} = \{Z \in \mathfrak{s}_{L_j}^{X_j}(\mathbb{R}) \,:\, \varphi_j(Z) \in \mathfrak{h}\}.$$

Using (7.18) it follows that $\mathfrak{l}$ is an ideal of $\mathfrak{s}_{L_j}^{X_j}(\mathbb{R})$. Hence also $d\iota(\mathfrak{l})$ is an ideal of $\mathfrak{sl}_d(\mathbb{R})$. But given any two elements $Y, Y' \in \mathfrak{sl}_d(\mathbb{R})$, it follows from $d\iota(\mathfrak{h}) = \mathfrak{g}'$ that there exist $Z, Z' \in \mathfrak{h}$ such that $d\iota(dp_j(Z)) = Y$, $d\iota(dp_j(Z')) = Y'$, and $d\iota(dp_i(Z)) = d\iota(dp_i(Z')) = 0$ for all $i \neq j$. Then also $[Z,Z'] \in \mathfrak{h}$, and one computes that $[Z,Z'] = \varphi_j(([Y,Y'],C))$ for some $C \in \mathrm{M}_{r_j\times d}(\mathbb{R})$ (and then in fact $([Y,Y'],C) \in \mathfrak{s}_{L_j}^{X_j}$, because of (7.19)). Hence we conclude that $[Y,Y'] \in d\iota(\mathfrak{l})$, for all $Y,Y' \in \mathfrak{sl}_d(\mathbb{R})$. Since $\mathfrak{sl}_d(\mathbb{R})$ is a simple Lie algebra, it follows that $d\iota(\mathfrak{l}) = \mathfrak{sl}_d(\mathbb{R})$.

Next fix some $Y \in \mathfrak{sl}_d(\mathbb{R})$ which is invertible as a $d\times d$ matrix. Because of $d\iota(\mathfrak{l}) = \mathfrak{sl}_d(\mathbb{R})$ there is some $C \in L_j^d$ such that $\varphi_j((Y,C)) \in \mathfrak{h}$. Using also (7.18) it follows that for any $C' \in L_j^d$ there exists some $Z \in \mathfrak{h}$ satisfying $p_j(Z) = (Y,C')$. Then $\mathfrak{h}$ also contains the Lie product $[\varphi_j((Y,C)), Z] = \varphi_j([(Y,C),(Y,C')]) = \varphi_j((0,(C-C')Y))$. Hence $(0,(C-C')Y) \in \mathfrak{l}$. Since C' is an arbitrary element in L_j^d and Y is invertible, it follows that $(0,C) \in \mathfrak{l}$ for *all* $C \in L_j^d$. Combining this fact with $d\iota(\mathfrak{l}) = \mathfrak{sl}_d(\mathbb{R})$, we finally conclude that $\mathfrak{l} = \mathfrak{s}_{L_j}^{X_j}(\mathbb{R})$. Hence $\varphi(\mathfrak{s}_{L_j}^{X_j}(\mathbb{R})) \subset \mathfrak{h}$. We have proved that this holds for all j; hence we finally conclude:

$$\mathfrak{h} = \mathfrak{s}_{L_1}^{X_1}(\mathbb{R}) \times \cdots \times \mathfrak{s}_{L_N}^{X_N}(\mathbb{R}), \tag{7.20}$$

and so (7.17)holds. □

Lemma 7.7. *Let H be as in* (7.17)*, and assume that $\Gamma \cap H$ is a lattice in H. Then for each j, L_j is a rational subspace of $\mathbb{R}^{r_j}$, and there exist $Y_j \in \mathrm{M}_{r_j\times d}(\mathbb{Q})$ such that*

$$H = \mathrm{S}_{L_1}^{Y_1}(\mathbb{R}) \times \cdots \times \mathrm{S}_{L_N}^{Y_N}(\mathbb{R}). \tag{7.21}$$

Proof. The assumption implies that $\Gamma_j \cap \mathrm{S}_{L_j}^{X_j}(\mathbb{R})$ is a lattice in $\mathrm{S}_{L_j}^{X_j}(\mathbb{R})$, for each j. As in [20, (5.58), (5.59)], this implies that $L_j \cap \mathbb{Z}^{r_j}$ is a lattice in L_j, i.e. L_j is a rational subspace of $\mathbb{R}^{r_j}$, and furthermore $X_j \in \mathrm{M}_{r_j \times d}(\mathbb{Q}) + L_j^d$. Choosing now any $Y_j \in \mathrm{M}_{r_j\times d}(\mathbb{Q})$ such that $X_j \in Y_j + L_j^d$, we have $\mathrm{S}_{L_j}^{X_j}(\mathbb{R}) = \mathrm{S}_{L_j}^{Y_j}(\mathbb{R})$. Carrying this out for each j, we obtain (7.21). □

The following very basic observation will also be useful for us:

Lemma 7.8. *Let L and L' be linear subspaces of $\mathbb{R}^r$, and $X, X' \in \mathrm{M}_{r\times d}(\mathbb{R})$. Then $\mathrm{S}_{L'}^{X'}(\mathbb{R}) \subset \mathrm{S}_L^X(\mathbb{R})$ holds if and only if $L' \subset L$ and $X' - X \in L$.*

Proof. Let $Y := X' - X$. Then $\mathrm{S}_{L'}^{X'}(\mathbb{R}) \subset \mathrm{S}_L^X(\mathbb{R})$ holds if and only if $\mathrm{I}_Y(M,U)\,\mathrm{I}_Y^{-1} \in \mathrm{S}_L(\mathbb{R})$ for all $(M,U) \in \mathrm{S}_{L'}(\mathbb{R})$, that is, $U + Y(M-I) \in L^d$ for all $M \in \mathrm{SL}_d(\mathbb{R})$ and all $U \in {L'}^d$. Assume that this holds. Then, considering first only $M = I$ it follows that $L' \subset L$, thus ${L'}^d \subset L^d$, and using this fact it follows that we must have $Y(M-I) \in L^d$ for all $M \in \mathrm{SL}_d(\mathbb{R})$. Considering only the first column of $M - I$ we conclude that $Y\boldsymbol{a} \in L$ for all column vectors $\boldsymbol{a} \in \mathbb{R}^d \setminus \{-\boldsymbol{e}_1\}$, and this in turn implies $Y \in L^d$, i.e. $X' - X \in L$. The converse direction is immediate. □

In the next lemma we derive an important consequence of the condition "$[\exists \eta > 0\colon \forall m \in \mathbb{Z}^+\colon V_m \notin \Delta_k^{(\eta)}]$" in Definition 7.1. Given any $j \in \{1,\ldots,N\}$ with $r_j \neq 0$ and $\boldsymbol{m} \in \mathbb{Z}^{r_j} \setminus \{\boldsymbol{0}\}$, we set:

$$K_{j,\boldsymbol{m}} := \Big\{\mathrm{I}_B\, A\, \mathrm{I}_Y \;:\; B \in \|\boldsymbol{m}\|^{-2}\,\mathrm{M}_{r_j\times d}(\mathbb{Z}),\ A \in \mathrm{SL}_d(\mathbb{R}),\ Y \in \boldsymbol{m}^\perp \times (\mathbb{R}^{r_j})^{d-1}\Big\}. \tag{7.22}$$

Note that $K_{j,\boldsymbol{m}}$ is left Γ_j-invariant F_σ set in $\mathrm{S}_{r_j}(\mathbb{R})$; hence $p_j^{-1}(K_{j,\boldsymbol{m}})$ is a left Γ-invariant F_σ set in G, and $\pi(p_j^{-1}(K_{j,\boldsymbol{m}}))$ is an F_σ set in $\mathbb{X}$.

Lemma 7.9. *For any $k \in \mathbb{Z}^+$, $\nu \in P_k$, $j \in \{1,\ldots,N\}$ with $r_j \neq 0$ and $\boldsymbol{m} \in \mathbb{Z}^{r_j} \setminus \{\boldsymbol{0}\}$, if $\|\boldsymbol{m}\|^2 \le k$ then $\nu(\pi(p_j^{-1}(K_{j,\boldsymbol{m}}))) = 0$.*

Proof. (This is similar to [6, pp. 114–115] and [5, Lemma 9].) Let $k, \nu, j, \boldsymbol{m}$ be given as in the statement of the lemma. Set $q := \|\boldsymbol{m}\|^2$ (thus $0 < q \le k$). Let us fix a vector $\boldsymbol{b} \in \mathbb{Z}^{r_j}$ satisfying $\boldsymbol{b}\cdot\boldsymbol{m} = \gcd(m_1,\ldots,m_{r_j})$. Then we have

$$\mathbb{Z}^{r_j} = (\boldsymbol{m}^\perp \cap \mathbb{Z}^{r_j}) \oplus \mathbb{Z}\boldsymbol{b}. \tag{7.23}$$

Let $p_{\boldsymbol{b}} : \mathbb{R}^{r_j} \to \mathbb{R}$ be the linear map such that $\boldsymbol{v} - p_{\boldsymbol{b}}(\boldsymbol{v})\boldsymbol{b} \in \boldsymbol{m}^\perp$ for all $\boldsymbol{v} \in \mathbb{R}^{r_j}$. For any matrix $Z \in \mathrm{M}_{r_j\times e}(\mathbb{R})$, we write $Z_1,\ldots,Z_e \in \mathbb{R}^{r_j}$ for its column vectors (in order), and we define $p_{\boldsymbol{b}}(Z) = (p_{\boldsymbol{b}}(Z_1),\ldots,p_{\boldsymbol{b}}(Z_e))$, i.e. $p_{\boldsymbol{b}}(Z)$ is the vector in $\mathbb{R}^e$ obtained by applying $p_{\boldsymbol{b}}$ individually to each column of Z. Furthermore, for any matrix $Z \in \mathrm{M}_{r_j\times d}(\mathbb{R})$ we will write $Z' = (Z_2,\ldots,Z_d)$ for the matrix in $\mathrm{M}_{r_j\times(d-1)}(\mathbb{R})$ formed by removing the first column vector from Z. For any $T > 0$ and $\delta > 0$ we now introduce the following subsets of $\mathrm{M}_{r_j\times d}(\mathbb{R})$:

$$\Omega_T := \{Z \in \mathrm{M}_{r_j\times d}(\mathbb{R}) \;:\; p_{\boldsymbol{b}}(Z_1) = 0,\ \|p_{\boldsymbol{b}}(Z')\| < T\};$$
$$\Omega_{T,\delta} := \{Z \in \mathrm{M}_{r_j\times d}(\mathbb{R}) \;:\; |p_{\boldsymbol{b}}(Z_1)| < \delta,\ \|p_{\boldsymbol{b}}(Z')\| < T\}.$$

We also introduce the following subsets of $\mathrm{S}_{r_j}(\mathbb{R})$:

$$K_T = \big\{\mathrm{I}_B\, A\, \mathrm{I}_Y \;:\; B \in q^{-1}\,\mathrm{M}_{r_j\times d}(\mathbb{Z}), A \in \mathrm{SL}_d(\mathbb{R}), Y \in \Omega_T\big\}$$

and

$$K_{T,\delta} = \big\{\mathrm{I}_B\, A\, \mathrm{I}_Y \;:\; B \in q^{-1}\,\mathrm{M}_{r_j\times d}(\mathbb{Z}), A \in \mathrm{SL}_d(\mathbb{R}), Y \in \Omega_{T,\delta}\big\}.$$

Note that both K_T and $K_{T,\delta}$ are left Γ_j-invariant; also $K_{T,\delta}$ is open; hence $p_j^{-1}(K_{T,\delta})$ is an open subset of G, and $\pi(p_j^{-1}(K_{T,\delta}))$ is an open subset of $\mathbb{X}$.

Now our goal is to prove that for any given $\eta > 0$, $\lambda \in P_{\mathrm{ac}}(\mathbb{R}^{d-1})$, $\widetilde{M} = (M_1, \ldots, M_N) \in G' \setminus \mathfrak{D}_{\mathcal{S}}$, real numbers $\rho_1 > \rho_2 > \cdots \to 0$, and points $V_1, V_2, \ldots$ in $\widetilde{\mathbb{T}} \setminus \Delta_k^{(\eta)}$, if $\nu_m \in P(\mathbb{X})$ is defined as in (7.11), then we have:

$$\forall T, \varepsilon > 0: \;\exists \delta, m_0 > 0: \;\forall m \geq m_0: \qquad \nu_m(\pi(p_j^{-1}(K_{T,\delta}))) < \varepsilon. \tag{7.24}$$

This will complete the proof of Lemma 7.9. Indeed, if ν is any weak limit of $\nu_1, \nu_2, \ldots$, then (7.24) together with the Portmanteau Theorem implies that for any $T, \varepsilon > 0$ there exists $\delta > 0$ such that $\nu(\pi(p_j^{-1}(K_{T,\delta}))) \leq \varepsilon$. This forces $\nu(\pi(p_j^{-1}(K_T))) = 0$ for all $T > 0$, and so $\nu(\pi(p_j^{-1}(K_{j,\boldsymbol{m}}))) = 0$, i.e. the lemma is proved.

Let us note that it suffices to prove (7.24) for special choices of λ. Indeed, since $\mathrm{C}_c(\mathbb{R}^{d-1})$ is dense in $L^1(\mathbb{R}^{d-1})$, it suffices to prove (7.24) for measures $\lambda \in P_{\mathrm{ac}}(\mathbb{R}^{d-1})$ of the form $d\lambda(\boldsymbol{x}) = \lambda'(\boldsymbol{x})\, d\boldsymbol{x}$ with $\lambda' \in \mathrm{C}_c(\mathbb{R}^{d-1})$. Next, using the fact that any such function λ' is bounded, it follows that it actually suffices to prove (7.24) when $\lambda = \mathrm{vol}\big|_{\mathcal{B}_R^{d-1}}$, i.e. Lebesgue measure restricted to a ball $\mathcal{B}_R^{d-1}$,[12] with $R > 1$ arbitrary and fixed. Hence from now on we assume that λ is of this form.

Let us write $V_m = (V_{m,1}, \ldots, V_{m,N})$ with $V_{m,j} \in \mathbb{T}_j^d$. Then for any T, δ, m,

$$\nu_m(\pi(p_j^{-1}(K_{T,\delta}))) = \mathrm{vol}\Big(\Big\{\boldsymbol{u} \in \mathcal{B}_R^{d-1} \;:\; x(V_{m,j}) M_j n_-(\boldsymbol{u}) D_{\rho_m} \in \pi(K_{T,\delta})\Big\}\Big). \tag{7.25}$$

Take $U_{m,j} \in \mathrm{M}_{r_j \times d}(\mathbb{R})$ with $\pi(U_{m,j}) = V_{m,j}$. Since $K_{T,\delta}$ is Γ_j-invariant, the condition $x(V_{m,j}) M_j n_-(\boldsymbol{u}) D_{\rho_m} \in \pi(K_{T,\delta})$ is equivalent with $\mathrm{I}_{U_{m,j}} M_j n_-(\boldsymbol{u}) D_{\rho_m} \in K_{T,\delta}$, which in turn is equivalent with

$$\big(U_{m,j} - q^{-1}\,\mathrm{M}_{r_j \times d}(\mathbb{Z})\big) M_j n_-(\boldsymbol{u}) D_{\rho_m} \bigcap \Omega_{T,\delta} \neq \emptyset. \tag{7.26}$$

But for any $Z \in \mathrm{M}_{r_j \times d}(\mathbb{R})$ the condition $Z n_-(\boldsymbol{u}) D_{\rho_m} \in \Omega_{T,\delta}$ holds if and only if the vector $\boldsymbol{z} := p_{\boldsymbol{b}}(Z)$ satisfies $|z_1| < \delta \rho_m^{1-d}$ and $\|z_1 \boldsymbol{u} + (z_2, \ldots, z_d)\| < T\rho_m$. We also have $p_{\boldsymbol{b}}(AM_j) = p_{\boldsymbol{b}}(A) M_j$ for all $A \in \mathrm{M}_{r_j \times d}(\mathbb{R})$; furthermore $p_{\boldsymbol{b}}(\mathbb{Z}^{r_j}) = \mathbb{Z}$, by (7.23), and thus $p_{\boldsymbol{b}}(\mathrm{M}_{r_j \times d}(\mathbb{Z})) = \mathbb{Z}^d$; therefore

$$\big\{p_{\boldsymbol{b}}(Z) \;:\; Z \in (U_{m,j} - q^{-1}\,\mathrm{M}_{r_j \times d}(\mathbb{Z})) M_j\big\} = \big(p_{\boldsymbol{b}}(U_{m,j}) + q^{-1}\mathbb{Z}^d\big) M_j =: L_m. \tag{7.27}$$

Note that this set L_m is a grid in $\mathbb{R}^d$. It now follows that the measure in (7.25) equals

$$\begin{aligned} &\mathrm{vol}\Big(\Big\{\boldsymbol{u} \in \mathcal{B}_R^{d-1} \;:\; \big[\exists \boldsymbol{z} \in L_m : \; |z_1| < \delta \rho_m^{1-d} \text{ and } \|z_1 \boldsymbol{u} + (z_2, \ldots, z_d)\| < T\rho_m\big]\Big\}\Big) \\ &\qquad \leq \sum_{\substack{\boldsymbol{z} \in L_m \\ (|z_1| < \delta \rho_m^{1-d})}} \mathrm{vol}\Big(\Big\{\boldsymbol{u} \in \mathcal{B}_R^{d-1} \;:\; \|z_1 \boldsymbol{u} + (z_2, \ldots, z_d)\| < T\rho_m\Big\}\Big). \end{aligned} \tag{7.28}$$

Recall that we are assuming $V_m \notin \Delta_k^{(\eta)}$ for all m; this implies that for all $q' \in \{1, \ldots, k\}$ and all $\boldsymbol{n} \in \mathbb{Z}^{r_j}$ with $0 < \|\boldsymbol{n}\| \leq k$, the point $V_{m,j}$ in $\mathbb{T}_j^d$ has distance $\geq \eta$ from the set $\Delta_{j,q',\boldsymbol{n}}$, and therefore $U_{m,j}$ has distance $\geq \eta$ from the set $({q'}^{-1}\mathbb{Z}^{r_j} + \boldsymbol{n}^\perp)^d$ in in $\mathrm{M}_{r_j \times d}(\mathbb{R}) = (\mathbb{R}^{r_j})^d$. In particular, $U_{m,j}$ has distance $\geq \eta$ from $(q^{-1}\mathbb{Z}^{r_j} + \boldsymbol{m}^\perp)^d$, and this is seen to be equivalent to the statement that $\|p_{\boldsymbol{b}}(U_{m,j}) - q^{-1}\boldsymbol{a}\| \geq \frac{\|\boldsymbol{m}\|}{\gcd(\boldsymbol{m})}\eta$ for all $\boldsymbol{a} \in \mathbb{Z}^{r_j}$. Combining this with the definition of L_m in (7.27), it follows that there exists a constant $\eta' > 0$, independent of m, such that

$$\forall m \in \mathbb{Z}^+: \;\forall \boldsymbol{z} \in L_m: \;\; \|\boldsymbol{z}\| \geq \eta'. \tag{7.29}$$

Now let $T > 0$ be given, and keep $m \in \mathbb{Z}^+$ so large that $T\rho_m < \eta'/6$. Consider any vector $\boldsymbol{z} \in L_m$ which gives a non-zero contribution in the sum in (7.28). This means that there exists some $\boldsymbol{u} \in \mathcal{B}_R^{d-1}$ such that $\|z_1 \boldsymbol{u} + (z_2, \ldots, z_d)\| < T\rho_m < \eta'/6$, and thus $\|(z_2, \ldots, z_d)\| < \eta'/6 + R|z_1|$. If $R|z_1| \leq \eta'/3$ then it would follow that $\|\boldsymbol{z}\| < \eta'$, which is impossible by (7.29).

[12] This measure should really be normalized by a factor $\mathrm{vol}(\mathcal{B}_R^{d-1})^{-1}$, to make λ and ν_m probability measures; however such a normalizing factor clearly does not affect the validity of (7.24), and so we will ignore it.

Hence we have proved that every $\boldsymbol{z} \in L_m$ which gives a non-zero contribution in the sum in (7.28) satisfies

$$|z_1| > \frac{\eta'}{3R} \quad \text{and} \quad \|(z_2, \ldots, z_d)\| < 2R|z_1|. \tag{7.30}$$

Let $L_{m,0}$ be the set of all $\boldsymbol{z} \in L_m$ satisfying (7.30) and $|z_1| \leq 2$, and for each $\ell \in \mathbb{Z}^+$ let $L_{m,\ell}$ be the set of all $\boldsymbol{z} \in L_m$ satisfying (7.30) and $2^\ell < |z_1| \leq 2^{\ell+1}$. Then the sum in (7.28) is

$$\leq \sum_{0 \leq \ell < \log_2(\delta \rho_m^{1-d})} \sum_{\boldsymbol{z} \in L_{m,\ell}} \mathrm{vol}(\mathcal{B}_1^{d-1}) \cdot \Big(\frac{T\rho_m}{|z_1|}\Big)^{d-1}.$$

But for every $\ell \geq 0$ and every $\boldsymbol{z} \in L_m$ we have

$$\|\boldsymbol{z}\| \leq |z_1| + \|(z_2, \ldots, z_d)\| < (1 + 2R)|z_1| < 2^{\ell+3}R.$$

Recall also that each grid L_m is a translate of the fixed lattice $q^{-1}\mathbb{Z}^d M_j$. It follows that there exists a constant $C > 0$ which is independent of m and ℓ (but which depends on R, q and M_j) such that $\#L_{m,\ell} < C2^{d\ell}$ for all $\ell \geq 0$. It follows that our sum is

$$\leq C \cdot \mathrm{vol}(\mathcal{B}_1^{d-1}) \cdot (T\rho_m)^{d-1} \cdot \bigg(\Big(\frac{\eta'}{3R}\Big)^{1-d} + \sum_{1 \leq \ell < \log_2(\delta\rho_m^{1-d})} 2^\ell\bigg) < C'\big(\rho_m^{d-1} + \delta\big),$$

where C' is a constant which is independent of m or δ (but which depends on R, T, η').

To sum up, we have proved that for any $T, \delta > 0$, and all $m \in \mathbb{Z}^+$ so large that $T\rho_m < \eta'/6$, we have $\nu_m(\pi(p_j^{-1}(K_{T,\delta}))) \leq C'(\rho_m^{d-1} + \delta)$, with a constant C' which may depend on T, but is independent of m and δ. This bound implies that (7.24) holds, and hence Lemma 7.9 is proved. □

Lemma 7.10. *Let $j \in \{1, \ldots, N\}$, let L be a rational subspace of $\mathbb{R}^{r_j}$, $L \neq \mathbb{R}^{r_j}$ (thus $r_j \neq 0$), and let $X \in \mathrm{M}_{r_j \times d}(\mathbb{Q})$. Then for any $\boldsymbol{m} \in \mathbb{Z}^{r_j} \cap L^\perp \setminus \{\boldsymbol{0}\}$ satisfying $X^{\mathrm{T}}\boldsymbol{m} \in \mathbb{Z}^d$, and any $Y \in \mathrm{M}_{r_j \times d}(\mathbb{R})$ satisfying $n_-(\mathbb{R}^{d-1}) \subset \mathrm{S}_L^{X-Y}(\mathbb{R})$, we have $\Gamma_j\, \mathrm{S}_L^X(\mathbb{R})\, \mathrm{I}_Y \subset K_{j,\boldsymbol{m}}$.*

Proof. Set $\widetilde{X} = \|\boldsymbol{m}\|^{-2}\boldsymbol{m}\,(X^{\mathrm{T}}\boldsymbol{m})^{\mathrm{T}} \in \|\boldsymbol{m}\|^{-2}\,\mathrm{M}_{r_j \times d}(\mathbb{Z})$, and note that $\widetilde{X}^{\mathrm{T}}\boldsymbol{m} = X^{\mathrm{T}}\boldsymbol{m}$. Let $Y \in \mathrm{M}_{r_j \times d}(\mathbb{R})$ be such that $n_-(\mathbb{R}^{d-1}) \subset \mathrm{S}_L^{X-Y}(\mathbb{R})$. This implies that $(Y-X)(n_-(\boldsymbol{w})-I) \subset L^d$ for all $\boldsymbol{w} \in \mathbb{R}^{d-1}$, which forces the first column of $Y - X$ lies in L.

Now consider an arbitrary element in $\Gamma_j\, \mathrm{S}_L^X(\mathbb{R})\, \mathrm{I}_Y$. This element can be expressed as follows, for some $(\gamma, B) \in \Gamma_j$ and $(A, U) \in \mathrm{S}_L(\mathbb{R})$:

$$(\gamma, B)\, \mathrm{I}_X(A, U)\, \mathrm{I}_{-X}\, \mathrm{I}_Y = \big(\gamma A, (B + X)A + U + Y - X\big) = \mathrm{I}_{(B+\widetilde{X})\gamma^{-1}}\, \gamma A\, \mathrm{I}_W, \tag{7.31}$$

where $W := (X - \widetilde{X})A + U + Y - X$. Here $(B + \widetilde{X})\gamma^{-1} \in \|\boldsymbol{m}\|^{-2}\,\mathrm{M}_{r_j \times d}(\mathbb{Z})$. Furthermore, $\widetilde{X}^{\mathrm{T}}\boldsymbol{m} = X^{\mathrm{T}}\boldsymbol{m}$ implies $X - \widetilde{X} \in (\boldsymbol{m}^\perp)^d$ and so $(X - \widetilde{X})A \in (\boldsymbol{m}^\perp)^d$. Also $U \in L^d \subset (\boldsymbol{m}^\perp)^d$, and finally the first column of $Y - X$ lies in L, hence in $\boldsymbol{m}^\perp$. It follows that the first column of W lies in $\boldsymbol{m}^\perp$. Hence the element in (7.31) lies in $K_{j,\boldsymbol{m}}$, and we have proved that $\Gamma_j\, \mathrm{S}_L^X(\mathbb{R})\, \mathrm{I}_Y \subset K_{j,\boldsymbol{m}}$. □

Lemma 7.11. *Given any $f \in \mathrm{C}_b(\mathbb{X})$ and $\varepsilon > 0$, there exists $k \in \mathbb{Z}^+$ such that $\big|\nu(f) - \mu(f)\big| < \varepsilon$ holds for all $\nu \in P_k$.*

Proof. Assume the opposite; this means (since $P_1 \supset P_2 \supset \cdots$) that there exist some $f \in \mathrm{C}_b(\mathbb{X})$, $\varepsilon > 0$, and measures $\nu_k \in P_k$ for $k = 1, 2, \ldots$, such that $|\nu_k(f) - \mu(f)| > \varepsilon$ for all k.

For each k, since ν_k is W-invariant by Lemma 7.2, we can apply ergodic decomposition to ν_k: Let $\mathcal{E}$ be the set of ergodic W-invariant probability measures on $\mathbb{X}$, provided with its usual Borel σ-algebra; then there exists a unique Borel probability measure ω_k on $\mathcal{E}$ such that

$$\nu_k = \int_{\mathcal{E}} \alpha \, d\omega_k(\alpha). \tag{7.32}$$

Cf., e.g., [29, Theorem 4.4]. Note that (7.32) together with Lemma 7.3 implies $\widetilde{\mu} = \widetilde{\iota}_* \nu_k = \int_{\mathcal{E}} \widetilde{\iota}_* \alpha \, d\omega_k(\alpha)$, and for each $\alpha \in \mathcal{E}$, $\widetilde{\iota}_* \alpha$ is an ergodic W-invariant measure on $\mathbb{X}'$. Hence in fact $\widetilde{\iota}_* \alpha = \widetilde{\mu}$ for ω_k-almost all $\alpha \in \mathcal{E}$, by uniqueness of the ergodic decomposition of $\widetilde{\mu}$. Furthermore, for every $j \in \{1, \ldots, N\}$ with $r_j \neq 0$ and every $\boldsymbol{m} \in \mathbb{Z}^{r_j}$ with $0 < \|\boldsymbol{m}\| \leq k$, we have $\int_{\mathcal{E}} \alpha(\pi(p_j^{-1}(K_{j,\boldsymbol{m}}))) \, d\omega_k(\alpha) = \nu_k(\pi(p_j^{-1}(K_{j,\boldsymbol{m}}))) = 0$, by Lemma 7.9, and hence $\alpha(\pi(p_j^{-1}(K_{j,\boldsymbol{m}}))) = 0$ for ω_k-almost all α. Note also that

$$\int_{\mathcal{E}} \left|\alpha(f) - \mu(f)\right| d\omega_k(\alpha) \geq \left| \int_{\mathcal{E}} \alpha(f) \, dP_{\nu_k}(\alpha) - \mu(f) \right| = \left|\nu_k(f) - \mu(f)\right| > \varepsilon,$$

and hence the set $\{\alpha \in \mathcal{E} : |\alpha(f) - \mu(f)| > \varepsilon\}$ must have positive measure with respect to ω_k.

It follows from the above discussion that for each $k \in \mathbb{Z}^+$, there must exist some $\alpha_k \in \mathcal{E}$ satisfying $\widetilde{\iota}_* \alpha_k = \widetilde{\mu}$ and $|\alpha_k(f) - \mu(f)| > \varepsilon$, and

$$\forall j \in \{1, \ldots, N\} : \ r_j \neq 0 \Rightarrow \ \ \forall \boldsymbol{m} \in \mathbb{Z}^{r_j} \setminus \{\boldsymbol{0}\} : \ \|\boldsymbol{m}\|^2 \leq k \Rightarrow \alpha_k(\pi(p_j^{-1}(K_{j,\boldsymbol{m}}))) = 0. \tag{7.33}$$

We will assume that such measures $\alpha_1, \alpha_2, \ldots$ have now been fixed, and we will derive a contradiction.

Clearly $\mathcal{E} \subset \mathcal{Q}(\mathbb{X})$, and hence for each k, we can apply Ratner's theorem, [24, Thm. 1], to α_k. As discussed below (7.12), this implies that there exists some element $g_k \in G$ such that, writing $H_k := H_{\alpha_k}$, $\Gamma \cap g_k H_k g_k^{-1}$ is a lattice in $g_k H_k g_k^{-1}$ and the support of α_k equals $\Gamma \backslash \Gamma g_k H_k$. The validity of the previous statements remain intact when replacing g_k by any other element from the double coset $\Gamma g_k H_k$. Hence, since $\iota(H_k) = G'$ by Lemma 7.4, after right multiplying g_k by an appropriate element in H_k we may assume that g_k is of the form $g_k = (\mathrm{I}_{Y_{k,1}}, \cdots, \mathrm{I}_{Y_{k,N}})$ for some matrices $Y_{k,j} \in \mathrm{M}_{r_j \times d}(\mathbb{R})$. Next, by left multiplying g_k by an appropriate element in Γ (in fact in $\Gamma \cap \iota^{-1}(\{I\})$), we may furthermore assume that every entry of every matrix $Y_{k,j}$ lies in the interval $[0, 1)$.

Note that the sequence $\alpha_1, \alpha_2, \ldots$ in $P(\mathbb{X})$ is tight, since $\widetilde{\iota}_* \alpha_k = \widetilde{\mu}$ for all k and the map $\widetilde{\iota} : \mathbb{X} \to \mathbb{X}'$ is proper. Hence by Prohorov's Theorem, there exists a subsequence, say $\alpha_{k_1}, \alpha_{k_2}, \ldots$ where $1 \leq k_1 < k_2 < \cdots$, which converges to some limit measure $\nu \in P(\mathbb{X})$. In view of our assumption on the entries of the matrices $Y_{k,j}$, we may *also* assume that g_{k_ℓ} converges to some element $\tilde{g}$ in G as $\ell \to \infty$. We have $\widetilde{\iota}_* \nu = \widetilde{\mu}$, by the continuous mapping theorem. Furthermore, by [22, Cor. 1.1] we have $\nu \in \mathcal{Q}(\mathbb{X})$, and hence by Ratner's [24, Thm. 1], there exists some $g_\nu \in G$ such that $\Gamma \cap g_\nu H_\nu g_\nu^{-1}$ is a lattice in $g_\nu H_\nu g_\nu^{-1}$ and $\mathrm{supp}(\nu) = \Gamma \backslash \Gamma g_\nu H_\nu$. Note also that $\iota(H_\nu) = G'$, by Lemma 7.4. Next we will apply [22, Thm. 1.1(2)]. As a preparation, note that by [22, Lemma 2.3], for each ℓ we can find a one-parameter subgroup $\{u_\ell(t)\}_{t \in \mathbb{R}}$ of W which acts ergodically with respect to α_{k_ℓ}, and we can then find an element $s_\ell \in G$ such that the trajectory $\{\Gamma g_\nu s_\ell u_\ell(t) \ : \ t > 0\}$ is uniformly distributed with respect to α_{k_ℓ}. Note that this last property remains valid if s_ℓ is replaced by $s_\ell u_\ell(t)$ for any $t > 0$, and in this way we may modify the elements $s_1, s_2, \ldots$ so that $s_\ell \to e$ in G as $\ell \to \infty$ (this is possible since $\alpha_{k_\ell} \to \nu$ in $P(\mathbb{X})$ and since the point Γg_ν lies in the support of ν). Hence by [22, Thm. 1.1(2)], for all sufficiently large ℓ we have $\mathrm{supp}(\alpha_{k_\ell}) \subset \mathrm{supp}(\nu) \cdot s_\ell$, or equivalently, $\Gamma \backslash \Gamma g_{k_\ell} H_{k_\ell} \subset \Gamma \backslash \Gamma g_\nu H_\nu s_\ell$, viz., $\Gamma(g_{k_\ell} H_{k_\ell} g_{k_\ell}^{-1}) g_{k_\ell} s_\ell^{-1} g_\nu^{-1} \subset \Gamma(g_\nu H_\nu g_\nu^{-1})$. But we know that $\Gamma(g_\nu H_\nu g_\nu^{-1})$ is a closed regular submanifold of G, and for each $\gamma \in \Gamma$, $\gamma(g_\nu H_\nu g_\nu^{-1})$ is a connected component of this submanifold (cf. [23, Theorem 1.13]). Hence for each large ℓ there exists some $\gamma_\ell \in \Gamma$ such that

$$(g_{k_\ell} H_{k_\ell} g_{k_\ell}^{-1}) g_{k_\ell} s_\ell^{-1} g_\nu^{-1} \subset \gamma_\ell (g_\nu H_\nu g_\nu^{-1}). \tag{7.34}$$

Recall that we also have $g_{k_\ell} \to \tilde{g}$ as $\ell \to \infty$; hence $g_{k_\ell} s_\ell^{-1} g_\nu^{-1} \to \tilde{g} g_\nu^{-1}$, and since $g_{k_\ell} s_\ell^{-1} g_\nu^{-1} \in \gamma_\ell(g_\nu H_\nu g_\nu^{-1})$ for all large ℓ it follows that there is some $\widetilde{\gamma} \in \Gamma$ such that $\gamma_\ell(g_\nu H_\nu g_\nu^{-1}) = \widetilde{\gamma}(g_\nu H_\nu g_\nu^{-1})$ for all sufficiently large ℓ. For these ℓ, (7.34) implies

$$g_{k_\ell} s_\ell^{-1} g_\nu^{-1} \widetilde{\gamma}^{-1} \in \widetilde{\gamma} g_\nu H_\nu g_\nu^{-1} \widetilde{\gamma}^{-1} \qquad \text{and} \qquad g_{k_\ell} H_{k_\ell} g_{k_\ell}^{-1} \subset \widetilde{\gamma} g_\nu H_\nu g_\nu^{-1} \widetilde{\gamma}^{-1}. \tag{7.35}$$

We have proved that (7.35) holds for all sufficiently large ℓ; however by removing the initial elements from the sequence $k_1 < k_2 < \cdots$, we may from now on assume that (7.35) holds for *all* $\ell \in \mathbb{Z}^+$.

Next let us apply Lemmas 7.6 and 7.7 to the group $\widetilde{\gamma} g_\nu H_\nu g_\nu^{-1} \widetilde{\gamma}^{-1}$ and the groups $g_{k_\ell} H_{k_\ell} g_{k_\ell}^{-1}$ for all ℓ. This gives that there exist rational subspaces L_j and $L_{\ell,j}$ of $\mathbb{R}^{r_j}$ and matrices X_j and $X_{\ell,j}$ in $\mathrm{M}_{r_j \times d}(\mathbb{Q})$ (for all $j \in \{1, \ldots, N\}$ and $\ell \in \mathbb{Z}^+$), such that

$$\widetilde{\gamma} g_\nu H_\nu g_\nu^{-1} \widetilde{\gamma}^{-1} = \mathrm{S}_{L_1}^{X_1}(\mathbb{R}) \times \cdots \times \mathrm{S}_{L_N}^{X_N}(\mathbb{R}). \tag{7.36}$$

and

$$g_{k_\ell} H_{k_\ell} g_{k_\ell}^{-1} = \mathrm{S}_{L_{\ell,1}}^{X_{\ell,1}}(\mathbb{R}) \times \cdots \times \mathrm{S}_{L_{\ell,N}}^{X_{\ell,N}}(\mathbb{R}) \qquad (\forall \ell \in \mathbb{Z}^+). \tag{7.37}$$

It now follows from (7.35) that $\mathrm{S}_{L_{\ell,j}}^{X_{\ell,j}}(\mathbb{R}) \subset \mathrm{S}_{L_j}^{X_j}(\mathbb{R})$, and hence by Lemma 7.8,

$$L_{\ell,j} \subset L_j \quad \text{and} \quad X_{\ell,j} - X_j \in L_j^d, \qquad \forall \ell \in \mathbb{Z}^+,\ j \in \{1, \ldots, N\}. \tag{7.38}$$

Let us also note that (7.37) and $g_k = (\mathrm{I}_{Y_{k,1}}, \cdots, \mathrm{I}_{Y_{k,N}})$ imply that

$$H_{k_\ell} = \mathrm{S}_{L_{\ell,j}}^{X_{\ell,1} - Y_{k_\ell,1}}(\mathbb{R}) \times \cdots \times \mathrm{S}_{L_{\ell,j}}^{X_{\ell,N} - Y_{k_\ell,N}}(\mathbb{R}). \tag{7.39}$$

Now we obtain a contradiction as follows: We have $\nu \neq \mu$, since $\nu(f) = \lim_{\ell \to \infty} \alpha_{k_\ell}(f)$ and $|\alpha_k(f) - \mu(f)| > \varepsilon$ for all k. Hence $H_\nu \neq G$, and so by (7.36) there is some $j \in \{1, \ldots, N\}$ such that $L_j \neq \mathbb{R}^{r_j}$ (this implies in particular $r_j > 0$). Hence we can choose some $\boldsymbol{m} \in \mathbb{Z}^{r_j} \cap L_j^\perp \setminus \{\mathbf{0}\}$ with $X_j^{\mathrm{T}} \boldsymbol{m} \in \mathbb{Z}^d$. Now by (7.38) we also have $\boldsymbol{m} \perp L_{\ell,j}$ and $X_{\ell,j}^{\mathrm{T}} \boldsymbol{m} = X_j^{\mathrm{T}} \boldsymbol{m} \in \mathbb{Z}^d$ for all ℓ. Let us apply this for some fixed choice of ℓ so large that $k_\ell \geq \|\boldsymbol{m}\|^2$. It is immediate from the definition of $H_{k_\ell} = H_{\alpha_{k_\ell}}$ (cf. (7.12)) that $W \subset H_{k_\ell}$; thus $n_-(\mathbb{R}^{d-1}) \subset \mathrm{S}_{L_{\ell,j}}^{X_{\ell,j} - Y_{k_\ell,j}}(\mathbb{R})$ (cf. (7.39)), and so by Lemma 7.10, the set $\Gamma_j \, \mathrm{I}_{Y_{k_\ell,j}} \mathrm{S}_{L_{\ell,j}}^{X_{\ell,j} - Y_{k_\ell,j}}(\mathbb{R})$ is contained in $K_{j,\boldsymbol{m}}$. By (7.39), this implies that $\Gamma \backslash \Gamma g_{k_\ell} H_{k_\ell}$, i.e. the support of α_{k_ℓ}, is contained in $\pi(p_j^{-1}(K_{j,\boldsymbol{m}}))$, and so $\alpha_{k_\ell}(\pi(p_j^{-1}(K_{j,\boldsymbol{m}}))) = 1$. This is a contradiction against (7.33), since $\|\boldsymbol{m}\|^2 \leq k_\ell$.

Hence the lemma is proved. □

Proof of Theorem 7.1. Given $f \in \mathrm{C}_b(\mathbb{X})$ and $\varepsilon > 0$, we choose $k \in \mathbb{Z}^+$ as in Lemma 7.11. Now also let arbitrary $\lambda \in P_{\mathrm{ac}}(\mathbb{R}^{d-1})$, $\eta > 0$, $\widetilde{M} \in G' \setminus \mathfrak{D}_{\mathcal{S}}$ be given. Assume that there does *not* exist any $\rho_0 \in (0,1)$ such that (7.10) holds for all $\rho \in (0, \rho_0)$ and $V \in \widetilde{\mathbb{T}} \setminus \Delta_k^{(\eta)}$. This means that there exist sequences $\rho_1 > \rho_2 > \cdots \to 0$ and $V_1, V_2, \ldots$ in $\widetilde{\mathbb{T}} \setminus \Delta_k^{(\eta)}$ satisfying

$$\left| \int_{\mathbb{R}^{d-1}} f\big(x(V_m) \widetilde{M} \varphi(n_-(\boldsymbol{u}) D_{\rho_m})\big) \, d\lambda(\boldsymbol{u}) - \int_{\mathbb{X}} f \, d\mu \right| \geq \varepsilon, \qquad \forall m \in \mathbb{Z}^+. \tag{7.40}$$

Define $\nu_m \in P(\mathbb{X})$ through $\nu_m(g) = \int_{\mathbb{R}^{d-1}} g\big(x(V_m) \widetilde{M} \varphi(n_-(\boldsymbol{u}) D_{\rho_m})\big) \, d\lambda(\boldsymbol{u})$ for all $g \in \mathrm{C}_b(\mathbb{X})$ (just as in (7.11)). By [19, Thm. 5] (applied in the same way as in the proof of Lemma 7.3) we have $\widetilde{\iota}_* \nu_m \to \widetilde{\mu}$ in $P(\mathbb{X}')$. Hence the sequence $\nu_1, \nu_2, \ldots$ in $P(\mathbb{X})$ is tight, and so by Prohorov's Theorem, after passing to a subsequence we may assume that $\nu_1, \nu_2, \ldots$ converges to some $\nu \in P(\mathbb{X})$. Then $\nu \in P_k$ (cf. Def. 7.1), and thus by our choice of k we have $\big|\nu(f) - \mu(f)\big| < \varepsilon$. But ν_m converges weakly to ν; in particular $\nu_m(f) \to \nu(f)$ as $m \to \infty$, and hence we conclude that $\big|\nu_m(f) - \mu(f)\big| < \varepsilon$ for all sufficiently large m. This is a contradiction against (7.40), and thus Theorem 7.1 is proved. □

Next we establish a variant of Theorem 7.1, where instead of the unipotent element $n_-(\boldsymbol{u})$ we have a rotation:

Theorem 7.12. *Let $f \in \mathrm{C}_b(\mathbb{X})$ and $\varepsilon > 0$ be given. Then there exists some $k \in \mathbb{Z}^+$ such that for every $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$, $\eta > 0$ and $\widetilde{M} \in G' \setminus \mathfrak{D}_{\mathcal{S}}$, there exists some $\rho_0 \in (0,1)$ such that*

$$\left| \int_{\mathrm{S}_1^{d-1}} f\Big(x(V) \widetilde{M} \varphi(R(\boldsymbol{v}) D_\rho)\Big) \, d\lambda(\boldsymbol{v}) - \int_{\mathbb{X}} f \, d\mu \right| < \varepsilon \tag{7.41}$$

for all $\rho \in (0,\rho_0)$ *and all* $V \in \widetilde{\mathbb{T}} \setminus \Delta_k^{(\eta)}$.

Proof. This follows from Theorem 7.1 via a fairly standard approximation argument; cf., e.g., [15, Thm. 5.3 and Cor. 5.4]. Some care is needed to ensure that we can obtain a uniform statement as in the theorem.

To start with, we restrict to functions f of compact support. Thus let $f \in \mathrm{C}_c(\mathbb{X})$ and $\varepsilon > 0$ be given. Fix a corresponding positive integer k as in Theorem 7.1. Now also let $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$, $\eta > 0$ and $\widetilde{M} \in G' \setminus \mathfrak{D}_{\mathcal{S}}$ be given.

For $\delta > 0$ we set $\mathcal{U}_\delta = \{T \in \mathrm{SL}_d(\mathbb{R}) \,:\, \|T - I\| < \delta\}$, where $\|\cdot\|$ denotes the entrywise maximum norm on $d\times d$ matrices. Thus $\mathcal{U}_\delta$ is an open neighborhood of the identity in $\mathrm{SL}_d(\mathbb{R})$. Since f has compact support, we can fix $0 < \delta < 1$ so small that

$$|f(x\varphi(T)) - f(x)| < \varepsilon, \qquad \forall x \in \mathbb{X},\, T \in \mathcal{U}_\delta. \tag{7.42}$$

Recall that R is continuous when restricted to S_1^{d-1} minus one point; it follows that there exists a compact subset $S \subset \mathrm{S}_1^{d-1}$ such that the restriction of R to S is continuous, and

$$\|f\|_\infty \cdot \lambda(\mathrm{S}_1^{d-1} \setminus S) \leq \varepsilon. \tag{7.43}$$

For each $\boldsymbol{v}_0 \in S$ we set $\Omega_{\boldsymbol{v}_0} = \{\boldsymbol{v} \in S \,:\, R(\boldsymbol{v}_0)^{-1}R(\boldsymbol{v}) \in \mathcal{U}_{\delta/2}\}$; this is a relatively open neighborhood of $\boldsymbol{v}_0$ in S. Since S is compact, we can fix a finite subset $\mathfrak{Q}_0 \subset \mathrm{S}_1^{d-1}$ such that the sets $\Omega_{\boldsymbol{v}_0}$ for $\boldsymbol{v}_0 \in \mathfrak{Q}_0$ cover S. Let us fix an arbitrary total order $\prec$ on $\mathfrak{Q}_0$, and set $\Omega'_{\boldsymbol{v}_0} := \Omega_{\boldsymbol{v}_0} \setminus \big(\cup_{\substack{\boldsymbol{v}_0' \in \mathfrak{Q}_0 \\ \boldsymbol{v}_0' \prec \boldsymbol{v}_0}} \Omega_{\boldsymbol{v}_0'}\big)$. Then the sets $\Omega'_{\boldsymbol{v}_0}$ for $\boldsymbol{v}_0 \in \mathfrak{Q}_0$ form a partition of S. Set $\mathfrak{Q}_0' = \{\boldsymbol{v}_0 \in \mathfrak{Q}_0 \,:\, \lambda(\Omega'_{\boldsymbol{v}_0}) > 0\}$, and for each $\boldsymbol{v}_0 \in \mathfrak{Q}_0'$ let $\lambda_{\boldsymbol{v}_0} := \lambda(\Omega'_{\boldsymbol{v}_0})^{-1}\, \lambda\big|_{\Omega'_{\boldsymbol{v}_0}} \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$. Note that we now have

$$\lambda|_S = \sum_{\boldsymbol{v}_0 \in \mathfrak{Q}_0'} \lambda(\Omega'_{\boldsymbol{v}_0})\lambda_{\boldsymbol{v}_0}. \tag{7.44}$$

Let us fix $\boldsymbol{v}_0 \in \mathfrak{Q}_0'$ temporarily, and consider the functions $E : \mathrm{S}_1^{d-1} \to \mathrm{M}_d(\mathbb{R})$, $a : \mathrm{S}_1^{d-1} \to \mathbb{R}$, $\boldsymbol{b}, \boldsymbol{c} : \mathrm{S}_1^{d-1} \to \mathbb{R}^{d-1}$, $D : \mathrm{S}_1^{d-1} \to \mathrm{M}_{d-1}(\mathbb{R})$ defined by

$$E(\boldsymbol{v}) = \begin{pmatrix} a(\boldsymbol{v}) & \boldsymbol{b}(\boldsymbol{v}) \\ \boldsymbol{c}(\boldsymbol{v})^{\mathrm{T}} & D(\boldsymbol{v}) \end{pmatrix} := R(\boldsymbol{v}_0)^{-1}R(\boldsymbol{v}) \qquad (\boldsymbol{v} \in \mathrm{S}_1^{d-1}).$$

Note that $(a(\boldsymbol{v}), \boldsymbol{c}(\boldsymbol{v})) = \boldsymbol{e}_1 E(\boldsymbol{v})^{\mathrm{T}} = \boldsymbol{v}R(\boldsymbol{v}_0)$ for all $\boldsymbol{v} \in \mathrm{S}_1^{d-1}$; in particular $a(\boldsymbol{v}) = \boldsymbol{v}\cdot\boldsymbol{v}_0$, and, since $\delta < 1$, it follows that $\Omega_{\boldsymbol{v}_0}$ is contained in the open disc $\mathcal{H}_{\boldsymbol{v}_0} = \{\boldsymbol{v} \in \mathrm{S}_1^{d-1} \,:\, \boldsymbol{v}\cdot\boldsymbol{v}_0 > \frac{1}{2}\}$. We introduce the function $\boldsymbol{x} : \mathcal{H}_{\boldsymbol{v}_0} \to \mathbb{R}^{d-1}$, $\boldsymbol{x}(\boldsymbol{v}) = -a(\boldsymbol{v})^{-1}\boldsymbol{c}(\boldsymbol{v})$; this is a diffeomorphism of $\mathcal{H}_{\boldsymbol{v}_0}$ onto the open ball $\mathcal{B}_{\sqrt{3}}^{d-1}$. We set $\widetilde{\lambda}_{\boldsymbol{v}_0} = \boldsymbol{x}_*(\lambda_{\boldsymbol{v}_0}) \in P_{\mathrm{ac}}(\mathbb{R}^{d-1})$.

Note that $\widetilde{M} \notin \mathfrak{D}_{\mathcal{S}}$ implies that $\widetilde{M}\varphi(R(\boldsymbol{v}_0)) \notin \mathfrak{D}_{\mathcal{S}}$ for every $\boldsymbol{v}_0 \in \mathfrak{Q}_0'$. Hence by Theorem 7.1 and our choice of k, there exists $\rho_0 \in (0,1)$ such that for all $\boldsymbol{v}_0 \in \mathfrak{Q}_0'$, $\rho \in (0,\rho_0)$ and $V \in \widetilde{\mathbb{T}} \setminus \Delta_k^{(\eta)}$, we have:

$$\left| \int_{\mathbb{R}^{d-1}} f\Big(x(V)\widetilde{M}\varphi(R(\boldsymbol{v}_0))\,\varphi(n_-(\boldsymbol{x})D_\rho)\Big)\, d\widetilde{\lambda}_{\boldsymbol{v}_0}(\boldsymbol{x}) - \int_{\mathbb{X}} f\, d\mu \right| < \varepsilon. \tag{7.45}$$

Here in the integral over $\mathbb{R}^{d-1}$, we substitute $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{v})$ and then use (7.42) combined with the fact that for every $\boldsymbol{v} \in \Omega'_{\boldsymbol{v}_0}$ and $\rho \in (0,\rho_0)$ we have $\left(\begin{smallmatrix} a(\boldsymbol{v})^{-1} & \boldsymbol{0} \\ \rho^d \boldsymbol{c}(\boldsymbol{v})^{\mathrm{T}} & D(\boldsymbol{v}) \end{smallmatrix}\right) \in \mathcal{U}_\delta$ (this is immediate from the fact that $E(\boldsymbol{v}) \in \mathcal{U}_{\delta/2}$, once we note that $|a-1| < \delta/2$ implies $|a^{-1}-1| < 2|1-a| < \delta$). This gives:

$$\begin{aligned} &\Bigg| \int_{\mathbb{R}^{d-1}} f(x(V)\widetilde{M}\varphi(R(\boldsymbol{v}_0)n_-(\boldsymbol{x})D_\rho))\, d\widetilde{\lambda}_{\boldsymbol{v}_0}(\boldsymbol{x}) \\ &\quad - \int_{\Omega'_{\boldsymbol{v}_0}} f\left(x(V)\widetilde{M}\varphi\left(R(\boldsymbol{v}_0)n_-(\boldsymbol{x}(\boldsymbol{v}))D_\rho \begin{pmatrix} a(\boldsymbol{v})^{-1} & \boldsymbol{0} \\ \rho^d \boldsymbol{c}(\boldsymbol{v})^{\mathrm{T}} & D(\boldsymbol{v}) \end{pmatrix}\right)\right) d\lambda_{\boldsymbol{v}_0}(\boldsymbol{v}) \Bigg| < \varepsilon. \end{aligned} \tag{7.46}$$

Here the last integral can be simplified using $R(\boldsymbol{v}_0)n_-(\boldsymbol{x}(\boldsymbol{v}))D_\rho\left(\begin{smallmatrix} a(\boldsymbol{v})^{-1} & \mathbf{0} \\ \rho^d \boldsymbol{c}(\boldsymbol{v})^{\mathrm{T}} & D(\boldsymbol{v}) \end{smallmatrix}\right) = R(\boldsymbol{v})D_\rho$. Hence, by (7.45) and (7.46), we have for any $\boldsymbol{v}_0 \in \mathfrak{Q}'_0$, $\rho \in (0, \rho_0)$ and $V \in \widetilde{\mathbb{T}} \setminus \Delta_k^{(\eta)}$:

$$\left| \int_{\Omega'_{\boldsymbol{v}_0}} f(x(V)\widetilde{M}\varphi(R(\boldsymbol{v})D_\rho))\, d\lambda_{\boldsymbol{v}_0}(\boldsymbol{v}) - \int_{\mathbb{X}} f\, d\mu \right| < 2\varepsilon. \tag{7.47}$$

Multiplying this inequality by $\lambda(\Omega'_{\boldsymbol{v}_0})$ and then adding over all $\boldsymbol{v}_0 \in \mathfrak{Q}'_0$ and using (7.44) and (7.43), we conclude that (7.41) holds with 4ε in place of ε, for all $\rho \in (0, \rho_0)$ and $V \in \widetilde{\mathbb{T}} \setminus \Delta_k^{(\eta)}$.

Thus the theorem is proved under the extra assumption that $f \in \mathrm{C}_c(\mathbb{X})$. Finally, the extension to the case of arbitrary functions $f \in \mathrm{C}_b(\mathbb{X})$ is achieved by a completely standard approximation argument. □

The following is an immediate corollary of Theorem 7.12:

Corollary 7.13. *Let V be an arbitrary, fixed point in $\widetilde{\mathbb{T}} \setminus \cup \widetilde{p}_j^{-1}(\Delta_{j,q,\boldsymbol{m}})$, where the union is taken over all triples $\langle j, q, \boldsymbol{m}\rangle$ with $j \in \{1, \ldots, N\}$, $r_j \neq 0$, $q \in \mathbb{Z}^+$ and $\boldsymbol{m} \in \mathbb{Z}^{r_j} \setminus \{\mathbf{0}\}$. Then for any $\widetilde{M} \in G'$, $f \in \mathrm{C}_b(\mathbb{X})$ and $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$, we have*

$$\int_{\mathrm{S}_1^{d-1}} f\Big(x(V)\widetilde{M}\varphi(R(\boldsymbol{v})D_\rho)\Big)\, d\lambda(\boldsymbol{v}) \to \int_{\mathbb{X}} f\, d\mu \qquad \text{as } \rho \to 0. \tag{7.48}$$

Proof. The assumption on V implies that for any $k \in \mathbb{Z}^+$ there is some $\eta > 0$ such that $V \notin \Delta_k^{(\eta)}$. Using this fact, the corollary is an immediate consequence of Theorem 7.12. □

8. Proof of [P2] (uniform spherical equidistribution)

We now return to using the same notation as in Sections 2–6; in particular $\mathcal{P}$ is a finite union of grids in $\mathbb{R}^d$, and we assume that an admissible presentation of $\mathcal{P}$ has been fixed (cf. (2.6), (2.5)), and corresponding to this presentation we let the homogeneous space $\mathbb{X} = \Gamma\backslash G$ be as defined in Section 2.2. In particular we have $r_1, \ldots, r_N > 0$ and

$$\Gamma = \mathrm{S}_{r_1}(\mathbb{Z}) \times \cdots \times \mathrm{S}_{r_N}(\mathbb{Z}),$$

which is a stricter requirement than what was imposed in Section 7.

Our goal in this section is to complete the proof of [P2]. Recall that by our initial discussion in Section 6.3, the task which remains is to prove Theorem 6.6. A key tool in our proof will be Theorem 7.12; note that this theorem will in general be applied to a certain homogeneous *submanifold* of our present homogeneous space $\mathbb{X}$.

8.1. Non-uniform equidistribution.

We will start by proving the following non-uniform result, which as we will see fairly easily implies Theorem 3.2, and which will also play an important role in our proof of Theorem 6.6.

As in Section 7, we let the subset $\mathfrak{D}_{\mathcal{S}} \subset G'$ be given by (7.5). Recall that $\Omega = \prod_{j=1}^N P(\mathbb{T}_j^d)'$; cf. (5.4). As in (7.3) we set $\widetilde{\mathbb{T}} = \mathbb{T}_1^d \times \mathbb{T}_2^d \times \cdots \times \mathbb{T}_N^d$, and we let $x : \widetilde{\mathbb{T}} \to \mathbb{X}$ be the natural embedding. Finally, for any $V = \langle V_1, \ldots, V_N\rangle \in \widetilde{\mathbb{T}}$ we define:

$$\omega^{(V)} := \langle \omega_1^{(V_1)}, \ldots, \omega_N^{(V_N)}\rangle \in \Omega. \tag{8.1}$$

Theorem 8.1. *For any $V \in \widetilde{\mathbb{T}}$, $\widetilde{M} \in G' \setminus \mathfrak{D}_{\mathcal{S}}$, $f \in \mathrm{C}_b(\mathbb{X})$ and $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$ we have*

$$\int_{\mathrm{S}_1^{d-1}} f(x(V)\widetilde{M}\varphi(R(\boldsymbol{v})D_\rho))\, d\lambda(\boldsymbol{v}) \to \int_{\mathbb{X}} f\, d\overline{\omega^{(V)}} \tag{8.2}$$

as $\rho \to 0$.

Proof. Let $V = \langle V_1, \ldots, V_N \rangle \in \widetilde{\mathbb{T}}$ be given. For each $j \in \{1, \ldots, N\}$ we write $L_j := L_j^{(V_j)}$, fix some $X_j \in \mathrm{M}_{r_j \times d}(\mathbb{Q})$ such that $V_j - \pi(X_j) \in \big(\mathbb{S}_j^{(V_j)}\big)^{\circ\, d}$, and fix some $\widetilde{V}_j \in X_j + L_j^d \subset \mathrm{M}_{r_j \times d}(\mathbb{R})$ with $\pi(\widetilde{V}_j) = V_j$ (cf. the discussion above (3.11)). Using the notation in (7.15), we then set:

$$H := \mathrm{S}_{L_1}^{X_1}(\mathbb{R}) \times \cdots \times \mathrm{S}_{L_N}^{X_N}(\mathbb{R}) = \mathrm{S}_{L_1}^{\widetilde{V}_1}(\mathbb{R}) \times \cdots \times \mathrm{S}_{L_N}^{\widetilde{V}_N}(\mathbb{R}).$$

As in (3.11) and the proof of Lemma 3.1, we have for each j that Γ_j intersects $\mathrm{S}_{L_j}^{\widetilde{V}_j}(\mathbb{R})$ in a lattice, and the orbit $x(V_j) \cdot \mathrm{S}_{L_j}(\mathbb{R}) = \Gamma_j \backslash \Gamma_j\, \mathrm{S}_{L_j}^{\widetilde{V}_j}(\mathbb{R})\, \mathrm{I}_{\widetilde{V}_j}$ is a closed embedded submanifold of $\mathbb{X}_j$ which carries a unique $\mathrm{S}_{L_j}(\mathbb{R})$-invariant probability measure; and by Proposition 3.7 this measure equals $\overline{\omega_j^{(V_j)}}$. Taking the product over all j, and writing $\widetilde{V} := \big(\widetilde{V}_1, \ldots, \widetilde{V}_N\big) \in \prod_{j=1}^N \mathrm{M}_{r_j \times d}(\mathbb{R})$, it follows that $\Gamma \backslash \Gamma H\, \mathrm{I}_{\widetilde{V}}$ is a closed embedded submanifold of $\mathbb{X}$ and, using also (8.1) and (5.12), that $\overline{\omega^{(V)}}$ is the unique $\prod_{j=1}^N \mathrm{S}_{L_j}(\mathbb{R})$-invariant probability measure on $\Gamma \backslash \Gamma H\, \mathrm{I}_{\widetilde{V}}$.

Let $\Gamma_H := \Gamma \cap H$, and let μ be the unique H-invariant probability measure on the homogeneous submanifold $\Gamma \backslash \Gamma H = \Gamma_H \backslash H$ of $\mathbb{X}$. In order to prove the theorem, we will prove that for any $\widetilde{M} \in G' \setminus \mathfrak{D}_{\mathcal{S}}$, $F \in \mathrm{C}_b(\Gamma_H \backslash H)$ and $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$, we have

$$\int_{\mathrm{S}_1^{d-1}} F\Big(\Gamma_H\, \mathrm{I}_{\widetilde{V}}\, \widetilde{M} \varphi(R(\boldsymbol{v}) D_\rho)\, \mathrm{I}_{\widetilde{V}}^{-1}\Big)\, d\lambda(\boldsymbol{v}) \to \int_{\Gamma_H \backslash H} F\, d\mu \tag{8.3}$$

as $\rho \to 0$. (To see that the integral to the left in (8.3) is well-defined, note that $\mathrm{I}_{\widetilde{V}}\, g\, \mathrm{I}_{\widetilde{V}}^{-1} \in H$ for all $g \in G'$.)

To see that the convergence in (8.3) implies the statement of the theorem, let $\tau : \mathbb{X} \to \mathbb{X}$ be right multiplication by $\mathrm{I}_{\widetilde{V}}$; then $\overline{\omega^{(V)}} = \tau_*(\mu)$, and the left side in (8.2) can be expressed as

$$\int_{\mathrm{S}_1^{d-1}} (f \circ \tau)\Big(\Gamma\, \mathrm{I}_{\widetilde{V}}\, \widetilde{M} \varphi(R(\boldsymbol{v}) D_\rho)\, \mathrm{I}_{\widetilde{V}}^{-1}\Big)\, d\lambda(\boldsymbol{v}).$$

Hence (8.2) follows from (8.3) if we let F be the restriction of $f \circ \tau$ to $\Gamma \backslash \Gamma H$.

We now turn to the proof of (8.3). We will start by fixing an isomorphism from H onto the Lie group

$$\widetilde{G} := \mathrm{S}_{s_1}(\mathbb{R}) \times \cdots \times \mathrm{S}_{s_N}(\mathbb{R}),$$

where $s_j := \dim L_j$. Given any linear bijection $\varphi : L \xrightarrow{\sim} \mathbb{R}^s$, where L is a linear subspace of $\mathbb{R}^r$ (for some $r \in \mathbb{Z}^+$) of dimension $s \geq 0$, we write S_φ for the following Lie group isomorphism:

$$\mathrm{S}_\varphi : \mathrm{S}_L(\mathbb{R}) \xrightarrow{\sim} \mathrm{S}_s(\mathbb{R}), \qquad \mathrm{S}_\varphi\big((M, U)\big) = (M, \varphi^d(U)), \tag{8.4}$$

where φ^d is the linear bijection from L^d onto $\mathrm{M}_{s \times d}(\mathbb{R})$ given by applying φ to each column of the matrix. (If $s = 0$ so that $L = \{\mathbf{0}\}$ and $\mathrm{S}_s(\mathbb{R}) = \mathrm{SL}_d(\mathbb{R})$, the definition in (8.4) should of course be interpreted to say $\mathrm{S}_\varphi\big((M, 0)\big) = M$.) One verifies immediately that S_φ is indeed a Lie group isomorphism. Also for any $X \in \mathrm{M}_{r \times d}(\mathbb{R})$, we introduce the following Lie group isomorphism:

$$\mathrm{S}_\varphi^X : \mathrm{S}_L^X(\mathbb{R}) \xrightarrow{\sim} \mathrm{S}_s(\mathbb{R}), \qquad \mathrm{S}_\varphi^X(g) = \mathrm{S}_\varphi\big(\mathrm{I}_X^{-1}\; g\; \mathrm{I}_X\big). \tag{8.5}$$

Next, for each j, we fix, once and for all, a linear bijection $\varphi_j : L_j \xrightarrow{\sim} \mathbb{R}^{s_j}$ with the property that $\varphi_j(L_j \cap \mathbb{Z}^{r_j}) = \mathbb{Z}^{s_j}$; this is possible since L_j is a rational subspace of $\mathbb{R}^{r_j}$. Finally we let Φ be the Lie group isomorphism

$$\Phi := \mathrm{S}_{\varphi_1}^{X_1} \times \cdots \times \mathrm{S}_{\varphi_N}^{X_N} : \quad H \xrightarrow{\sim} \widetilde{G}.$$

For each j, fix a positive integer q_j such that $X_j \in q_j^{-1}\mathbb{Z}$, and let Γ'_j be the principal congruence subgroup of $\mathrm{SL}_d(\mathbb{Z})$ of level q_j:

$$\Gamma'_j := \{M \in \mathrm{SL}_d(\mathbb{Z}) \,:\, M \equiv I \bmod q_j\}. \tag{8.6}$$

Then set

$$\widetilde{\Gamma}_j := \Gamma'_j \ltimes \mathrm{M}_{s_j\times d}(\mathbb{Z}) \quad (j=1,\ldots,N) \qquad \text{and} \quad \widetilde{\Gamma} := \widetilde{\Gamma}_1 \times \cdots \times \widetilde{\Gamma}_N. \tag{8.7}$$

We now claim that

$$\widetilde{\Gamma} \subset \Phi(\Gamma_H). \tag{8.8}$$

To verify this, it suffices to verify that $\widetilde{\Gamma}_j \subset \mathrm{S}_{\varphi_j}^{X_j}\big(\mathrm{S}_{r_j}(\mathbb{Z}) \cap \mathrm{S}_{L_j}^{X_j}(\mathbb{R})\big)$ for each j. To do so, note that given any $(M,U) \in \widetilde{\Gamma}_j$, we have $\big(\mathrm{S}_{\varphi_j}^{X_j}\big)^{-1}(M,U) \in \mathrm{S}_{L_j}^{X_j}(\mathbb{R})$ and

$$\big(\mathrm{S}_{\varphi_j}^{X_j}\big)^{-1}(M,U) = \big(M, (\varphi_j^d)^{-1}(U) + X_j(M-I)\big).$$

Here $(\varphi_j^d)^{-1}(U) \in \mathrm{M}_{r_j\times d}(\mathbb{Z})$ since $U \in \mathrm{M}_{s_j\times d}(\mathbb{Z})$ and $\varphi_j^{-1}(\mathbb{Z}^{s_j}) = L_j \cap \mathbb{Z}^{r_j}$; also $X_j(M-I) \in \mathrm{M}_{r_j\times d}(\mathbb{Z})$ since $M \in \Gamma'_j$; hence $(\mathrm{S}_{\varphi_j}^{X_j})^{-1}(M,U) \in \mathrm{S}_{r_j}(\mathbb{Z})$. This completes the proof of (8.8). Note that it follows from (8.8) that we have a well-defined covering map

$$J : \widetilde{\Gamma}\backslash\widetilde{G} \to \Gamma_H\backslash H, \qquad J(\widetilde{\Gamma}g) = \Gamma_H\,\Phi^{-1}(g). \tag{8.9}$$

Next, the result of Corollary 7.13, applied to the homogeneous space $\widetilde{\Gamma}\backslash\widetilde{G}$, can be stated as follows: Let $\widetilde{\mu}$ be the invariant probability measure on $\widetilde{\Gamma}\backslash\widetilde{G}$. Let $W = (W_1,\ldots,W_N)$ be an arbitrary element in $\prod_{j=1}^N \mathrm{M}_{s_j\times d}(\mathbb{R})$ such that for every $j \in \{1,\ldots,N\}$ and every rational subspace $L' \subsetneq \mathbb{R}^{s_j}$, we have

$$W_j \notin \mathrm{M}_{s_j\times d}(\mathbb{Q}) + (L')^d. \tag{8.10}$$

Then for any $\widetilde{M} \in G' \setminus \mathfrak{D}_{\mathcal{S}}$, $F_1 \in \mathrm{C}_b(\widetilde{\Gamma}\backslash\widetilde{G})$ and $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$, we have

$$\int_{\mathrm{S}_1^{d-1}} F_1\Big(\widetilde{\Gamma}\,\mathrm{I}_W\,\widetilde{M}\varphi(R(\boldsymbol{v})D_\rho)\Big)\,d\lambda(\boldsymbol{v}) \to \int_{\widetilde{\Gamma}\backslash\widetilde{G}} F_1\,d\widetilde{\mu} \qquad \text{as } \rho \to 0. \tag{8.11}$$

(Note that $\mathrm{I}_W \in \widetilde{G}$ since $W \in \prod_{j=1}^N \mathrm{M}_{s_j\times d}(\mathbb{R})$; cf. (2.21).) Starting from (8.11) and applying the continuous mapping theorem with the covering map J (cf. (8.9)), we conclude: For any element $W = (W_1,\ldots,W_N)$ in $\prod_{j=1}^N L_j^d$ such that for every $j \in \{1,\ldots,N\}$ and every rational subspace $L' \subsetneq \mathbb{R}^{s_j}$,

$$\varphi_j^d(W_j) \notin \mathrm{M}_{s_j\times d}(\mathbb{Q}) + (L')^d, \tag{8.12}$$

and for any $\widetilde{M} \in G' \setminus \mathfrak{D}_{\mathcal{S}}$, $F_2 \in \mathrm{C}_b(\Gamma_H\backslash H)$ and $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$,

$$\int_{\mathbb{R}^{d-1}} F_2\Big(\Gamma_H\,\mathrm{I}_W\,\mathrm{I}_X\,\widetilde{M}\varphi(R(\boldsymbol{v})D_\rho)\,\mathrm{I}_X^{-1}\Big)\,d\lambda(\boldsymbol{v}) \to \int_{\Gamma_H\backslash H} F_2\,d\mu \qquad \text{as } \rho \to 0, \tag{8.13}$$

where $X := (X_1,\ldots,X_N) \in \prod_{j=1}^N \mathrm{M}_{r_j\times d}(\mathbb{Q})$. In the above deduction we used the fact that $J_*(\widetilde{\mu}) = \mu$, the unique H-invariant probability measure on $\Gamma_H\backslash H$.

We wish to apply the last convergence relation with $W := \widetilde{V} - X$, i.e. $W_j := \widetilde{V}_j - X_j$ for each j. This W lies in $\prod_{j=1}^N L_j^d$, and we proceed to verify that also the condition (8.12) holds for every $j \in \{1,\ldots,N\}$ and every rational subspace $L' \subsetneq \mathbb{R}^{s_j}$. Assume the opposite, i.e. assume that there exists $j \in \{1,\ldots,N\}$ and a rational subspace $L' \subsetneq \mathbb{R}^{s_j}$ such that $\varphi_j^d(W_j) \in \mathrm{M}_{s_j\times d}(\mathbb{Q}) + (L')^d$. Using the fact that $\varphi_j^{-1}(\mathbb{Q}^{s_j}) \subset \mathbb{Q}^{r_j}$ (since $\varphi_j^{-1}(\mathbb{Z}^{s_j}) = L_j \cap \mathbb{Z}^{r_j}$), we conclude $W_j \in \mathrm{M}_{r_j\times d}(\mathbb{Q}) + (L'')^d$, where $L'' := \varphi_j^{-1}(L')$. Since $W_j := \widetilde{V}_j - X_j$ and $X_j \in \mathrm{M}_{r_j\times d}(\mathbb{Q})$, it follows that $\widetilde{V}_j \in \mathrm{M}_{r_j\times d}(\mathbb{Q}) + (L'')^d$, or equivalently

$$\widetilde{V}_{j,1},\ldots,\widetilde{V}_{j,d} \in \mathbb{Q}^{r_j} + L'', \tag{8.14}$$

where $\widetilde{V}_{j,1},\ldots,\widetilde{V}_{j,d}\in\mathbb{R}^{r_j}$ are the column vectors of $\widetilde{V}_j$. But we have $L''\subsetneq L_j$ and L'' is a rational subspace of $\mathbb{R}^{r_j}$; also $L_j=L_j^{(V_j)}=\mathfrak{J}(\{\widetilde{V}_{j,1},\ldots,\widetilde{V}_{j,d}\})$ (cf. (3.6)), and thus by Lemma 3.3, L_j is the smallest rational subspace of $\mathbb{R}^{r_j}$ with the property that $\widetilde{V}_{j,1},\ldots,\widetilde{V}_{j,d}\in\mathbb{Q}^{r_j}+L_j$. This is a contradiction against (8.14). This completes the proof that the condition (8.12) is fulfilled for our choice of W.

Note also that $\mathrm{I}_W\in H$ since $W\in\prod_{j=1}^N L_j^d$, and $\mathrm{I}_W\,\mathrm{I}_X=\mathrm{I}_{\widetilde{V}}$. Hence for any given $F\in\mathrm{C}_b(\Gamma_H\backslash H)$, also the function F_2 defined by $F_2(\Gamma_H h)=F(\Gamma_H h\,\mathrm{I}_W^{-1})$ lies in $\mathrm{C}_b(\Gamma_H\backslash H)$; and applying (8.13) to this function F_2 we conclude that (8.3) holds for the given function F. □

Remark 8.1. If the point $V=\langle V_1,\ldots,V_N\rangle\in\widetilde{\mathbb{T}}$ satisfies $\widetilde{\mathsf{r}}_{i_\psi}(V_{j_\psi})=\mathbf{0}$ for some $\psi\in\Psi$, then the statement of Theorem 8.1 also holds with "$\mathbb{X}^\psi$" in place of "$\mathbb{X}$", i.e. for any $\widetilde{M}\in G'\setminus\mathfrak{D}_{\mathcal{S}}$, $f\in\mathrm{C}_b(\mathbb{X}^\psi)$ and $\lambda\in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$ we have

$$\int_{\mathrm{S}_1^{d-1}} f(x(V)\widetilde{M}\varphi(R(\boldsymbol{v})D_\rho))\,d\lambda(\boldsymbol{v})\to\int_{\mathbb{X}^\psi} f\,d\overline{\omega^{(V)}} \tag{8.15}$$

as $\rho\to 0$.

Proof. By [12, Lemma 4.26], this follows from Theorem 8.1 if we can only verify that $\overline{\omega^{(V)}}(\mathbb{X}^\psi)=1$ and $x(V)\widetilde{M}\varphi(R(\boldsymbol{v})D_\rho)\in\mathbb{X}^\psi$ for all ρ and $\boldsymbol{v}$. The first of these statements is immediate from (5.2), Lemma 5.2 and Lemma 5.3. For the second statement, note that $\mathbb{X}^\psi$ is preserved by right multiplication of any G'-element; hence it suffices to verify that $x(V)\in\mathbb{X}^\psi$. But writing $\psi=(j,i)$, and taking $\widetilde{V}_j\in\mathrm{M}_{r_j\times d}(\mathbb{R})$ so that $V_j=\pi(\widetilde{V}_j)$, we have $\mathsf{r}_i(\widetilde{V}_j)\in\mathbb{Z}^d$ since $\widetilde{\mathsf{r}}_i(V_j)=\mathbf{0}$. Now $x(V_j)=\Gamma_j\,\mathrm{I}_{\widetilde{V}_j}$ and $\mathbb{Z}^d\,\mathsf{a}_i(\mathrm{I}_{\widetilde{V}_j})=\mathbb{Z}^d\,(\mathrm{I},\mathsf{r}_i(\widetilde{V}_j))=\mathbb{Z}^d$, which is a lattice containing $\mathbf{0}$. Hence, by (5.2), $x(V)\in\mathbb{X}^\psi$, and the proof is complete. □

Let us note that Theorem 3.2 is an immediate consequence of Theorem 8.1:

Proof of Theorem 3.2. By (2.23), $g_0^{(\boldsymbol{q})}:=\mathrm{I}_{U^{(\boldsymbol{q})}}\,\widetilde{M}$, where $\widetilde{M}=(M_1,\ldots,M_N)$ with the M_js coming from the fixed presentation of $\mathcal{P}$ in (2.5), (2.6). This $\widetilde{M}$ lies outside $\mathfrak{D}_{\mathcal{S}}$, by (2.8). Hence the left hand side of (8.2) equals the left hand side of (3.3) if we choose $V:=\pi(U^{(\boldsymbol{q})})$, i.e. $V=(V_1,\ldots,V_N)$ with $V_j=\pi(U_j^{(\boldsymbol{q})})$. With this choice, $\omega_j^{(\boldsymbol{q})}=\omega_j^{(V_j)}$ holds by definition, and hence by Proposition 3.7 and (3.2), (5.12), (8.1), we have $\overline{\omega^{(V)}}=\mu^{(\boldsymbol{q})}$, meaning that also the right hand sides of (8.2) and (3.3) agree. □

8.2. **A first uniform result.** We will now prove a uniform equidistribution result, Theorem 8.2 below, which, in combination with the non-uniform result of Theorem 8.1, will play a key role in our proof of Theorem 6.6.

For any $\psi\in\Psi$ and $j\in\{1,\ldots,N\}$ we pick an arbitrary point $\boldsymbol{q}\in\mathcal{L}_\psi$, and define

$$\mathbb{Y}_j^\psi:=\pi(U_j^{(\boldsymbol{q})})+(\mathbb{S}_j^\psi)^d\subset\mathbb{T}_j^d. \tag{8.16}$$

This is a connected component of the group $(\widetilde{\mathbb{S}}_j^\psi)^d$; cf. Lemma 4.7. Note that $\mathbb{Y}_j^\psi$ is independent of the choice of $\boldsymbol{q}$, since $\pi\big(U_j^{(\boldsymbol{q})}\big)-\pi\big(U_j^{(\boldsymbol{q}')}\big)\in(\mathbb{S}_j^\psi)^d$ for any two $\boldsymbol{q},\boldsymbol{q}'\in\mathcal{L}_\psi$, as was noted in the proof of Lemma 4.7. Let us also fix a matrix $X_j^\psi\in\mathrm{M}_{r_j\times d}(\mathbb{Q})$ with the property that

$$\mathbb{Y}_j^\psi=\pi\big(X_j^\psi+(L_j^\psi)^d\big). \tag{8.17}$$

(Proof of existence: Choose any $\boldsymbol{q}\in\mathcal{L}_\psi$; then by Lemma 3.3 we can choose $X_j^\psi\in\mathrm{M}_{r_j\times d}(\mathbb{Q})$ so that $U_j^{(\boldsymbol{q})}-X_j^\psi\in\big(L_j^{(\boldsymbol{q})}\big)^d$; using (8.16) and Lemma 4.2 it then follows that (8.17) holds.) Furthermore, we fix a linear bijection $\varphi_j^\psi:L_j^\psi\xrightarrow{\sim}\mathbb{R}^s$ (with $s=s(\psi,j)=\dim L_j^\psi$) with the property that $\varphi_j^\psi(L_j^\psi \cap \mathbb{Z}^{r_j}) = \mathbb{Z}^s$. These matrices X_j^ψ and bijections φ_j^ψ will be kept fixed throughout the present section. We also introduce the following map:

$$\widetilde{\varphi_j^\psi} : \mathbb{Y}_j^\psi \to \mathrm{M}_{s\times d}(\mathbb{R}/\mathbb{Z}); \qquad \widetilde{\varphi_j^\psi}\big(\pi(X_j^\psi + W)\big) = \pi'\big((\varphi_j^\psi)^d(W)\big) \quad \big(W \in (L_j^\psi)^d\big), \tag{8.18}$$

where π' is the projection map $\mathrm{M}_{s\times d}(\mathbb{R}) \to \mathrm{M}_{s\times d}(\mathbb{R}/\mathbb{Z})$. It follows from the defining property of φ_j^ψ that the map $\widetilde{\varphi_j^\psi}$ is well-defined, and that $\widetilde{\varphi_j^\psi}$ is a diffeomorphism from $\mathbb{Y}_j^\psi$ onto the torus $\mathrm{M}_{s\times d}(\mathbb{R}/\mathbb{Z})$.

Next, for any $\psi \in \Psi$, $j \in \{1, \ldots, N\}$, $k \in \mathbb{Z}^+$ and $\eta > 0$, we set

$$\Delta_{\psi,j,k}^{(\eta)} := \begin{cases} \big(\widetilde{\varphi_j^\psi}\big)^{-1}\big(\Delta_{j,k}^{(\eta)}\big) & \text{if } L_j^\psi \neq \{\mathbf{0}\}, \\ \emptyset & \text{if } L_j^\psi = \{\mathbf{0}\}, \end{cases} \tag{8.19}$$

where if $L_j^\psi \neq \{\mathbf{0}\}$, the set $\Delta_{j,k}^{(\eta)} \subset \mathrm{M}_{s\times d}(\mathbb{R}/\mathbb{Z})$ is defined as on p. 42, but using $s = \dim L_j^\psi$ (> 0) in the place of r_j, so that "$\mathbb{T}_j^d$" on p. 42 becomes $\mathrm{M}_{s\times d}(\mathbb{R}/\mathbb{Z})$.

Next, for each $\psi \in \Psi$, we define

$$\mathbb{Y}^\psi = \mathbb{Y}_1^\psi \times \cdots \times \mathbb{Y}_N^\psi \subset \widetilde{\mathbb{T}}$$

and for any $k \in \mathbb{Z}^+$ and $\eta > 0$:

$$\Delta_{\psi,k}^{(\eta)} := \big\{V = \langle V_1, \ldots, V_N\rangle \in \mathbb{Y}^\psi \;:\; V_j \in \Delta_{\psi,j,k}^{(\eta)} \text{ for some } j\big\}. \tag{8.20}$$

Theorem 8.2. *Let $\psi \in \Psi$, $f \in \mathrm{C}_b(\mathbb{X}^\psi)$ and $\varepsilon > 0$ be given. Then there exists some $k \in \mathbb{Z}^+$ such that for every $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$, $\eta > 0$ and $\widetilde{M} \in G' \setminus \mathfrak{D}_{\mathcal{S}}$, there exists some $\rho_0 \in (0,1)$ such that*

$$\left| \int_{\mathbb{R}^{d-1}} f\big(x(V)\widetilde{M}\varphi(R(\boldsymbol{v})D_\rho)\big)\, d\lambda(\boldsymbol{v}) - \int_{\mathbb{X}^\psi} f\, d\overline{\omega^\psi} \right| < \varepsilon \tag{8.21}$$

for all $\rho \in (0, \rho_0)$ and all $V \in \mathbb{Y}^\psi \setminus \Delta_{\psi,k}^{(\eta)}$.

To see that the statement of Theorem 8.2 makes sense, note that for every $V \in \mathbb{Y}^\psi$ we have $x(V) \in \mathbb{X}^\psi$ by the following Lemma 8.3; thus also $x(V)g \in \mathbb{X}^\psi$ for all $g \in G'$; and we also have $\overline{\omega^\psi}(\mathbb{X}^\psi) = 1$, by Lemma 5.2 and Lemma 5.5.

Lemma 8.3. *For any $\psi \in \Psi$ and $V \in \mathbb{Y}^\psi$, we have $\widetilde{\mathsf{r}}_{i_\psi}(V_{j_\psi}) = \mathbf{0}$ and $x(V) \in \mathbb{X}^\psi$.*

Proof. Assume $V = (V_1, \ldots, V_N) \in \mathbb{Y}^\psi$. Write $\psi = (j, i)$, and choose a point $\boldsymbol{q} \in \mathcal{L}_\psi$. Take $W \in \mathrm{M}_{r_j\times d}(\mathbb{R})$ so that $V_j = \pi(W)$. It follows from $V_j \in \mathbb{Y}_j^\psi$ that $W \in U_j^{(\boldsymbol{q})} + (L_j^\psi)^d + \mathrm{M}_{r_j\times d}(\mathbb{Z})$. It follows from (2.5) and (2.19) that $\mathsf{r}_i\big(U_j^{(\boldsymbol{q})}\big) \in \mathbb{Z}^d$, and we noted in the proof of Lemma 5.5 that $L_j^\psi \perp \boldsymbol{e}_i$. Hence $\mathsf{r}_i(W) \in \mathbb{Z}^d$. This shows that $\widetilde{\mathsf{r}}_i(V_j) = \mathbf{0}$, and it also implies that the grid $\mathbb{Z}^d\,\mathsf{a}_i(\mathrm{I}_W) = \mathbb{Z}^d + W$ contains $\mathbf{0}$, i.e. the point $x(V_j) = \Gamma_j\,\mathrm{I}_W$ lies in $\mathbb{X}_j^{(i)}$. Hence $x(V) \in \mathbb{X}^\psi$. $\square$

Proof of Theorem 8.2. Let $\psi \in \Psi$ be given. Let us set $X := (X_1^\psi, \ldots, X_N^\psi)$ and

$$H := \mathrm{S}_{L_1^\psi}^{X_1^\psi}(\mathbb{R}) \times \cdots \times \mathrm{S}_{L_N^\psi}^{X_N^\psi}(\mathbb{R}) = \mathrm{I}_X\Big(\mathrm{S}_{L_1^\psi}(\mathbb{R}) \times \cdots \times \mathrm{S}_{L_N^\psi}(\mathbb{R})\Big)\,\mathrm{I}_X^{-1}. \tag{8.22}$$

Recall that $\overline{\omega^\psi} = \overline{\omega_1^\psi} \otimes \cdots \otimes \overline{\omega_N^\psi}$. We claim that $\overline{\omega^\psi}$ equals the unique $\mathrm{I}_X^{-1}\,H\,\mathrm{I}_X$-invariant probability measure on $\Gamma\backslash\Gamma H\,\mathrm{I}_X$. To prove this, it suffices to prove that for each fixed j, $\overline{\omega_j^\psi}$ equals the unique $\mathrm{S}_{L_j^\psi}(\mathbb{R})$-invariant probability measure on $\Gamma_j\backslash\Gamma_j\,\mathrm{I}_{X_j^\psi}\,\mathrm{S}_{L_j^\psi}(\mathbb{R})$. To this end, fix an arbitrary matrix $W \in X_j^\psi + (L_j^\psi)^d$ which is generic in the sense that it lies outside $\mathrm{M}_{r_j\times d}(\mathbb{Q}) + L^d$ for every rational subspace $L \subsetneq L_j^\psi$, and set $V = \pi(W) \in \mathbb{T}_j^d$. Then $L_j^{(V)} = L_j^\psi$ by Lemma 3.3, and hence by Proposition 3.7, $\overline{\omega_j^{(V)}}$ is the unique $\mathrm{S}_{L_j^\psi}(\mathbb{R})$-invariant probability measure on $\Gamma_j\backslash\Gamma_j \, \mathrm{I}_W \, \mathrm{S}_{L_j^\psi}(\mathbb{R}) = \Gamma_j\backslash\Gamma_j \, \mathrm{I}_{X_j^\psi} \, \mathrm{S}_{L_j^\psi}(\mathbb{R})$. Furthermore, we have $\big(\mathbb{S}_j^{(V)}\big)^\circ = \pi\big(L_j^{(V)}\big) = \pi(L_j^\psi)$ and thus $V + \big(\mathbb{S}_j^{(V)}\big)^{\circ d} = \pi(X_j^\psi + (L_j^\psi)^d) = \mathbb{Y}_j^\psi = \pi\big(U_j^{(\boldsymbol{q})}\big) + (\mathbb{S}_j^\psi)^d$ for any $\boldsymbol{q} \in \mathcal{L}_\psi$. This implies $\mathcal{O}_j^{(V)} = \mathcal{O}_j^\psi$ (cf. (4.17) and (3.7)), and hence $\omega_j^{(V)} = \omega_j^\psi$. This completes the proof of the claim.

Set $\Gamma_H := \Gamma \cap H$, and let μ be the H-invariant probability measure on $\Gamma_H\backslash H$.

Next, for each j let us write $s_j := \dim L_j^\psi$, and recall that we have fixed a linear bijection $\varphi_j^\psi : L_j^\psi \xrightarrow{\sim} \mathbb{R}^{s_j}$ with the property that $\varphi_j^\psi(L_j^\psi \cap \mathbb{Z}^{r_j}) = \mathbb{Z}^{s_j}$. Set

$$\widetilde{G} := \mathrm{S}_{s_1}(\mathbb{R}) \times \cdots \times \mathrm{S}_{s_N}(\mathbb{R}),$$

and let Φ be the Lie group isomorphism

$$\Phi := \mathrm{S}_{\varphi_1^\psi}^{X_1^\psi} \times \cdots \times \mathrm{S}_{\varphi_N^\psi}^{X_N^\psi} : \quad H \xrightarrow{\sim} \widetilde{G}$$

(using the notation from (8.5)). For each j, choose $q_j \in \mathbb{Z}^+$ so that $X_j^\psi \in q_j^{-1}\mathbb{Z}$, and let Γ_j' be the principal congruence subgroup of $\mathrm{SL}_d(\mathbb{Z})$ of order q_j (cf. (8.6)); then define $\widetilde{\Gamma}_j$ and $\widetilde{\Gamma}$ as in (8.7). By an argument entirely similar to the discussion in the proof of Theorem 8.1 (leading up to (8.13)), one verifies that Theorem 7.12 applied to the homogeneous space $\widetilde{\Gamma}\backslash\widetilde{G}$, yields the following result: For any $f \in \mathrm{C}_b(\Gamma_H\backslash H)$ and $\varepsilon > 0$, there exists some $k \in \mathbb{Z}^+$ such that for every $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$, $\eta > 0$ and $\widetilde{M} \in G' \setminus \mathfrak{D}_{\mathcal{S}}$, there exists some $\rho_0 \in (0,1)$ such that

$$\left| \int_{\mathbb{R}^{d-1}} f\Big(\Gamma_H \, \mathrm{I}_W \, \mathrm{I}_X \, \widetilde{M}\varphi(R(\boldsymbol{v})D_\rho) \, \mathrm{I}_X^{-1}\Big) \, d\lambda(\boldsymbol{v}) - \int_{\Gamma_H\backslash H} f \, d\mu \right| < \varepsilon \tag{8.23}$$

for all $\rho \in (0,\rho_0)$ and all $W = \langle W_1,\ldots,W_N\rangle \in \prod_{j=1}^N (L_j^\psi)^d$ satisfying $\pi_j'((\varphi_j^\psi)^d(W_j)) \notin \Delta_{j,k}^{(\eta)}$ for every $j \in \{1,\ldots,N\}$ with $s_j > 0$. In the last condition, the set $\Delta_{j,k}^{(\eta)}$ is defined exactly as on p. 42 but using the dimension s_j in place of r_j (so that "$\mathbb{T}_j^d$" on p. 42 becomes $\mathrm{M}_{s_j\times d}(\mathbb{R}/\mathbb{Z})$), and π_j' is the projection $\mathrm{M}_{s_j\times d}(\mathbb{R}) \to \mathrm{M}_{s_j\times d}(\mathbb{R}/\mathbb{Z})$.

Finally, let us write $\widetilde{V} := W + X$ in the previous result; thus $\widetilde{V}_j := W_j + X_j^\psi \in X_j^\psi + (L_j^\psi)^d$ for each j, and also $\mathrm{I}_W \, \mathrm{I}_X = \mathrm{I}_{\widetilde{V}}$ in (8.23). Then in view of the definitions (8.19) and (8.20), the condition on W is equivalent to $\pi(\widetilde{V}) \notin \Delta_{\psi,k}^{(\eta)}$. Hence, by an argument completely similar to the proof that (8.3) suffices to give Theorem 8.1, the result stated around (8.23) implies the statement of Theorem 8.2. □

Next let us note that by combining Theorem 8.2 with Theorem 8.1, we immediately obtain a variant of Theorem 8.2, where the limit measure $\overline{\omega^\psi}$ in (8.21) is replaced by $\overline{\omega^{(V)}}$:

Theorem 8.4. *For any $\psi \in \Psi$, $f \in \mathrm{C}_b(\mathbb{X}^\psi)$ and $\varepsilon > 0$, there exists some $k \in \mathbb{Z}^+$ such that for every $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$, $\eta > 0$ and every $\widetilde{M} \in G' \setminus \mathfrak{D}_{\mathcal{S}}$, there exists some $\rho_0 \in (0,1)$ such that*

$$\left| \int_{\mathrm{S}_1^{d-1}} f\big(x(V)\widetilde{M}\varphi(R(\boldsymbol{v})D_\rho)\big) \, d\lambda(\boldsymbol{v}) - \int_{\mathbb{X}^\psi} f \, d\overline{\omega^{(V)}} \right| < \varepsilon \tag{8.24}$$

for all $\rho \in (0,\rho_0)$ and all $V \in \mathbb{Y}^\psi \setminus \Delta_{\psi,k}^{(\eta)}$.

Proof. Given ψ, f, ε, take k as in Theorem 8.2. Now also let $\lambda \in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$, $\eta > 0$ and $\widetilde{M} \in G' \setminus \mathfrak{D}_{\mathcal{S}}$ be given. Take $\rho_0 \in (0,1)$ as in Theorem 8.2, i.e. so that (8.21) holds for all $\rho \in (0,\rho_0)$ and all $V \in \mathbb{Y}^\psi \setminus \Delta_{\psi,k}^{(\eta)}$. Now for any fixed $V \in \mathbb{Y}^\psi$ we have $\widetilde{\mathsf{r}}_{i_\psi}(V_{j_\psi}) = \mathbf{0}$ by Lemma 8.3, and so by Theorem 8.1 and Remark 8.1, the convergence in (8.15) holds as $\rho \to 0$. Combining this fact with (8.21) gives

$$\left| \int_{\mathbb{X}^\psi} f \, d\overline{\omega^{(V)}} - \int_{\mathbb{X}^\psi} f \, d\overline{\omega^\psi} \right| \le \varepsilon, \qquad \forall V \in \mathbb{Y}^\psi \setminus \Delta_{\psi,k}^{(\eta)}. \tag{8.25}$$

Combining (8.21) and (8.25), we conclude that (8.24), with 2ε in place of ε, holds for all $\rho \in (0, \rho_0)$ and all $V \in \mathbb{Y}^\psi \setminus \Delta^{(\eta)}_{\psi,k}$. □

8.3. **Proof of the uniformity in [P2]; later steps.** Note that for any $\psi \in \Psi$ and $\boldsymbol{q} \in \mathcal{L}_\psi$ we have $\pi\big(U_j^{(\boldsymbol{q})}\big) \in \mathbb{Y}_j^\psi$ for all $j \in \{1, \ldots, N\}$, by (8.16), and hence $\pi\big(U^{(\boldsymbol{q})}\big) \in \mathbb{Y}^\psi$. The following proposition shows that by taking η small, we can ensure that the density of points $\boldsymbol{q} \in \mathcal{L}_\psi$ for which $\pi\big(U^{(\boldsymbol{q})}\big)$ falls inside the "singular" set $\Delta^{(\eta)}_{\psi,k}$, is small.

Proposition 8.5. *For every $\psi \in \Psi$ and $k \in \mathbb{Z}^+$ we have*

$$\lim_{\eta\to 0} \limsup_{T\to\infty} \frac{\#\big\{\boldsymbol{q} \in \mathcal{L}_\psi \cap \mathcal{B}_T^d \,:\, \pi\big(U^{(\boldsymbol{q})}\big) \in \Delta^{(\eta)}_{\psi,k}\big\}}{T^d} = 0. \tag{8.26}$$

Proof. It follows from the definitions in (8.19), (8.20) and on p. 42 that $\Delta^{(\eta)}_{\psi,k}$ is a union of sets of the form

$$\Delta^{(\eta)}_{j,A,L} := \big\{V = \langle V_1, \ldots, V_N\rangle \in \mathbb{Y}^\psi \,:\, V_j - \pi(X_j^\psi) \text{ is } \eta\text{-near } \pi(A + L^d)\big\}, \tag{8.27}$$

the union being taken over a finite set of triples $\langle j, A, L\rangle$ with $j \in \{1, \ldots, N\}$, $L_j^\psi \neq \mathbf{0}$, $A \in (L_j^\psi)^d \cap M_{r_j\times d}(\mathbb{Q})$ and L being a rational subspace of L_j^ψ, $L \neq L_j^\psi$. Note that in (8.27), "η-near" refers to the Riemannian metric on $\mathbb{Y}_j^\psi$ induced from the standard Euclidean metric on $\mathrm{M}_{s\times d}(\mathbb{R}/\mathbb{Z})$ (with $s = \dim L_j^\psi$) via the diffeomorphism in (8.18).

It follows that it suffices to prove that for any fixed such triple $\langle j, A, L\rangle$,

$$\lim_{\eta\to 0} \limsup_{T\to\infty} \frac{\#\big\{\boldsymbol{q} \in \mathcal{L}_\psi \cap \mathcal{B}_T^d \,:\, \pi\big(U_j^{(\boldsymbol{q})}\big) \in \Delta^{(\eta)}_{j,A,L}\big\}}{T^d} = 0. \tag{8.28}$$

But it follows from the formula for $U_j^{(\boldsymbol{q})}$ in (4.3) and Weyl equidistribution that if we let Z be the closed subgroup of $\mathbb{T}_j^d$ which is the closure of the set $\big\{\pi(U_j^{(\boldsymbol{q})} - W_j^\psi) \,:\, \boldsymbol{q} \in \mathcal{L}_\psi\big\}$, and if ν is the Haar measure on Z normalized so that $\nu(Z) = 1$, then for any fixed closed subset $C \subset Z$,

$$\limsup_{T\to\infty} \frac{\#\{\boldsymbol{q} \in \mathcal{L}_\psi \cap \mathcal{B}_T^d \,:\, \pi(U_j^{(\boldsymbol{q})} - W_j^\psi) \in C\}}{\overline{n}_\psi \operatorname{vol}(\mathcal{B}_T^d)} \leq \nu(C). \tag{8.29}$$

Note that it follows from (8.16) and (8.17) that $\pi(U_j^{(\boldsymbol{q})} - W_j^\psi) \in \pi(X_j^\psi - W_j^\psi) + (\mathbb{S}_j^\psi)^d$ for all $\boldsymbol{q} \in \mathcal{L}_\psi$; hence also $Z \subset \pi(X_j^\psi - W_j^\psi) + (\mathbb{S}_j^\psi)^d$, and since $0 \in Z$ it follows that $\pi(X_j^\psi - W_j^\psi) \in (\mathbb{S}_j^\psi)^d$ and $Z \subset (\mathbb{S}_j^\psi)^d$. Using (8.29) and (8.27), it follows that in order to prove (8.28), it suffices to prove that $\nu(C_\eta) \to 0$ as $\eta \to 0$, where C_η is the closed η-neighborhood of $\pi(X_j^\psi - W_j^\psi + A + L^d)$. But we have $\cap_{k=1}^\infty C_{1/k} = \pi(X_j^\psi - W_j^\psi + A + L^d)$; hence in fact it suffices to prove that

$$\nu(\pi(X_j^\psi - W_j^\psi + A + L^d)) = 0. \tag{8.30}$$

Fix $q \in \mathbb{Z}^+$ so that $X_j^\psi + A \in q^{-1}\,\mathrm{M}_{r_j\times d}(\mathbb{Z})$. Let Z° be the identity component of Z; then Z is a union of a finite number of Z°-cosets. Now if Z' is any of these cosets, we may argue as follows. Set

$$\mathcal{L}' := \{\boldsymbol{q} \in \mathcal{L}_\psi \,:\, \pi(U_j^{(\boldsymbol{q})} - W_j^\psi) \in Z'\};$$

this is a subgrid of $\mathcal{L}_\psi$. Hence by Lemma 4.1, $L_j^\psi = \mathfrak{L}(\{U_{j,\ell}^{(\boldsymbol{q})} \,:\, \boldsymbol{q} \in \mathcal{L}', \ell \in \{1, \ldots, d\}\})$, and so by Lemma 3.3, since L is a rational subspace of L_j^ψ and $L \neq L_j^\psi$, there exist some $\boldsymbol{q} \in \mathcal{L}'$ and

$\ell \in \{1, \ldots, d\}$ for which $U_{j,\ell}^{(\boldsymbol{q})} \notin q^{-1}\mathbb{Z}^{r_j} + L$. This implies that $U_j^{(\boldsymbol{q})} \notin q^{-1}\,\mathrm{M}_{r_j\times d}(\mathbb{Z}) + L^d$, and in particular $\pi(U_j^{(\boldsymbol{q})}) \notin \pi(X_j^{\psi} + A + L^d)$. But $\pi(U_j^{(\boldsymbol{q})} - W_j^{\psi}) \in Z'$. Hence we conclude that

$$Z' \not\subset \pi(X_j^{\psi} - W_j^{\psi} + A + L^d). \tag{8.31}$$

Now both Z' and $\pi(X_j^{\psi} - W_j^{\psi} + A + L^d)$ are translates of closed connected subtori of $(\mathbb{S}_j^{\psi})^d$; hence (8.31) implies that $Z' \cap \pi(X_j^{\psi} - W_j^{\psi} + A + L^d)$ is either empty or a submanifold of codimension ≥ 1 of Z'. Therefore

$$\nu(Z' \cap \pi(X_j^{\psi} - W_j^{\psi} + A + L^d)) = 0. \tag{8.32}$$

We have proved that (8.32) holds for every Z°-coset Z' in Z; hence (8.30) holds, and the lemma is proved. □

Next we prove an auxiliary lemma concerning the type of uniform convergence which we require. We define the *upper density* of a subset $\mathcal{Z} \subset \mathbb{R}^d$ to be the number[13]

$$\limsup_{T\to\infty} T^{-d}\#(\mathcal{Z} \cap \mathcal{B}_T^d). \tag{8.33}$$

Lemma 8.6. *Let $\mathcal{Q}$ be a locally finite subset of $\mathbb{R}^d$, let J be a countable set, and let a function $F : J \times \mathcal{Q} \times (0, 1) \to \mathbb{R}$ be given. Assume that*

(i) For any fixed $j \in J$ and $\boldsymbol{q} \in \mathcal{Q}$, $F(j, \boldsymbol{q}, \rho) \to 0$ as $\rho \to 0$,

and

(ii) for any $j \in J$ and $\varepsilon, \varepsilon' > 0$, there exist $\rho_0 \in (0, 1)$ and a subset $\mathcal{Z} \subset \mathcal{Q}$ of upper density $\leq \varepsilon'$, such that $|F(j, \boldsymbol{q}, \rho)| < \varepsilon$ for all $\rho \in (0, \rho_0)$ and all $\boldsymbol{q} \in \mathcal{Q} \setminus \mathcal{Z}$.

Then for any decreasing function $\mathcal{T} : (0, 1) \to \mathbb{R}^+$, there exists a subset $\mathcal{E} \subset \mathcal{Q}$ of density zero, such that for each fixed $j \in J$, we have $F(j, \boldsymbol{q}, \rho) \to 0$ as $\rho \to 0$, uniformly over all $\boldsymbol{q} \in \mathcal{Q} \cap \mathcal{B}_{\mathcal{T}(\rho)}^d \setminus \mathcal{E}$.

Proof. We assume, without loss of generality, that $J = \mathbb{Z}^+$. For any $j, k, m \in \mathbb{Z}^+$, by (ii) there exist $\rho_0' \in (0, 1)$ and a subset $\mathcal{Z}' \subset \mathcal{Q}$ of upper density $\leq 2^{-m-1}$ such that $|F(j, \boldsymbol{q}, \rho)| < 2^{-k}$ for all $\rho \in (0, \rho_0')$ and all $\boldsymbol{q} \in \mathcal{Q} \setminus \mathcal{Z}'$. We may then choose $T_0 > 0$ so that $T^{-d}\#(\mathcal{Z}' \cap \mathcal{B}_T^d) < 2^{-m}$ for all $T \geq T_0$, and set $\mathcal{Z} = \mathcal{Z}(j, k, m) := \mathcal{Z}' \setminus \mathcal{B}_{T_0}^d$. Now $\mathcal{Z}' \setminus \mathcal{Z}$ is finite, and for each $\boldsymbol{q} \in \mathcal{Z}' \setminus \mathcal{Z}$ there exists some $\rho_0^{(\boldsymbol{q})} \in (0, 1)$ such that $|F(j, \boldsymbol{q}, \rho)| < 2^{-k}$ for all $\rho \in (0, \rho_0^{(\boldsymbol{q})})$, by (i). Set

$$\rho_0 = \rho_0(j, k, m) := \min(\{\rho_0'\} \cup \{\rho_0^{(\boldsymbol{q})} \;:\; \boldsymbol{q} \in \mathcal{Z}' \setminus \mathcal{Z}\}).$$

Now for any $j, k, m \in \mathbb{Z}^+$, we have constructed a number $\rho_0(j, k, m) \in (0, 1)$ and a subset $\mathcal{Z}(j, k, m) \subset \mathcal{Q}$, and it is clear from our construction that

$$\#(\mathcal{Z}(j, k, m) \cap \mathcal{B}_T^d) < 2^{-m}T^d, \qquad \forall T > 0 \tag{8.34}$$

and that

$$|F(j, \boldsymbol{q}, \rho)| < 2^{-k}, \qquad \forall \rho \in (0, \rho_0(j, k, m)),\ \boldsymbol{q} \in \mathcal{Q} \setminus \mathcal{Z}(j, k, m). \tag{8.35}$$

Let us now also set, for any $m \in \mathbb{Z}^+$,

$$\widetilde{\mathcal{Z}}(m) := \bigcup_{j\in\mathbb{Z}^+} \bigcup_{k\in\mathbb{Z}^+} \mathcal{Z}(j, k, j + k + m).$$

Then

$$\#(\widetilde{\mathcal{Z}}(m) \cap \mathcal{B}_T^d) \leq \sum_{j,k\in\mathbb{Z}^+} 2^{-j-k-m}T^d = 2^{-m}T^d, \qquad \forall T > 0. \tag{8.36}$$

[13]To conform with the definition of asymptotic density in (1.1), it would be more natural to divide with $\mathrm{vol}(\mathcal{B}_T^d)$ instead of T^d in (8.33); however using (8.33) makes some computations in the following slightly cleaner.

Now let a decreasing function $\mathcal{T}:(0,1)\to\mathbb{R}^+$ be given, as in the statement of the lemma. For any $m\in\mathbb{Z}^+$ we set

$$\widetilde{\rho}_0(m) := \min\{\rho_0(j,k,j+k+m)\;:\;j,k\in\{1,\ldots,m\}\}. \tag{8.37}$$

Next choose numbers $1\le R_1<R_2<\cdots$ so that $R_m\ge\mathcal{T}(\widetilde{\rho}_0(m+1))$ and $R_{m+1}\ge 2R_m$ for all m. Finally set

$$\mathcal{E} := \bigcup_{m=1}^{\infty}\big(\widetilde{\mathcal{Z}}(m)\cap\mathcal{B}^d_{R_m}\big).$$

Let us prove that this set $\mathcal{E}$ has density zero. Given any $T\ge R_1$, choosing n so that $R_n\le T<R_{n+1}$ we have

$$\#(\mathcal{E}\cap\mathcal{B}^d_T)\le\sum_{m=1}^{n}\#(\widetilde{\mathcal{Z}}(m)\cap\mathcal{B}^d_{R_m})+\sum_{m=n+1}^{\infty}\#(\widetilde{\mathcal{Z}}(m)\cap\mathcal{B}^d_T)\le\sum_{m=1}^{n}2^{-m}R_m^d+\sum_{m=n+1}^{\infty}2^{-m}T^d,$$

by (8.36). Here for all $m\le n$ we have $R_m\le 2^{m-n}R_n$ and hence $R_m^d\le 2^{2(m-n)}T^d$, since $d\ge 2$. Plugging in these bounds we get $\#(\mathcal{E}\cap\mathcal{B}^d_T)<3\cdot 2^{-n}T^d$. Hence since $n\to\infty$ as $T\to\infty$, we conclude that $\mathcal{E}$ has density zero.

It remains to prove the uniform convergence stated in the lemma. Thus let $j\in\mathbb{Z}^+$ and $\varepsilon>0$ be given. By (i), we can take $\rho_1\in(0,1)$ so small that $|F(j,\boldsymbol{q},\rho)|<\varepsilon$ for all $\boldsymbol{q}\in\mathcal{Q}\cap\mathcal{B}^d_{R_j}$ and all $\rho\in(0,\rho_1)$. Choose $k\in\mathbb{Z}^+$ so that $2^{-k}<\varepsilon$, and set

$$\rho_0 := \min\big(\{\rho_1\}\cup\{\rho_0(j,k,j+k+m)\;:\;m\in\{1,\ldots,k\}\}\big). \tag{8.38}$$

Now for any $\rho\in(0,\rho_0)$ and $\boldsymbol{q}\in\mathcal{Q}\cap\mathcal{B}^d_{\mathcal{T}(\rho)}\setminus\mathcal{E}$ we can argue as follows: If $\|\boldsymbol{q}\|<R_j$ then $|F(j,\boldsymbol{q},\rho)|<\varepsilon$ since $\rho<\rho_0\le\rho_1$. Next assume instead that $\|\boldsymbol{q}\|\ge R_j$, and let $m>j$ be the minimal positive integer such that $\|\boldsymbol{q}\|<R_m$. Then $\mathcal{T}(\rho)>\|\boldsymbol{q}\|\ge R_{m-1}\ge\mathcal{T}(\widetilde{\rho}_0(m))$, and so $\rho<\widetilde{\rho}_0(m)$. If $m\ge k$ then $\widetilde{\rho}_0(m)\le\rho_0(j,k,j+k+m)$ by (8.37); on the other hand if $m<k$ then $\rho_0\le\rho_0(j,k,j+k+m)$ by (8.38). Hence we always have $\rho<\rho_0(j,k,j+k+m)$. Furthermore, $\boldsymbol{q}\notin\mathcal{E}$ and $\|\boldsymbol{q}\|<R_m$ implies $\boldsymbol{q}\notin\widetilde{\mathcal{Z}}(m)$, and in particular $\boldsymbol{q}\notin\mathcal{Z}(j,k,j+k+m)$. Hence by (8.35), $|F(j,\boldsymbol{q},\rho)|<2^{-k}<\varepsilon$. Summing up, we have proved:

$$|F(j,\boldsymbol{q},\rho)|<\varepsilon,\qquad\forall\rho\in(0,\rho_0),\ \boldsymbol{q}\in\mathcal{Q}\cap\mathcal{B}^d_{\mathcal{T}(\rho)}\setminus\mathcal{E}. \tag{8.39}$$

This completes the proof of the uniform convergence stated in the lemma. □

We are now ready to prove Theorem 6.6.

Proof of Theorem 6.6. Let ψ and $\mathcal{T}$ be given as in the statement of Theorem 6.6.

Let J_1 be a fixed countable dense subset of $\mathrm{C}_c(\mathbb{X}^\psi)$ with respect to the uniform norm. We equip $P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$ the metric d defined by $d(\lambda_1,\lambda_2):=\int_{\mathrm{S}_1^{d-1}}|\lambda_1'-\lambda_2'|\,d\boldsymbol{v}$, where λ_j' is the density of λ_j with respect to σ, that is, $\lambda_j'\in\mathrm{L}^1(\mathrm{S}_1^{d-1})$ and $\lambda_j=\lambda_j'\,d\sigma$ ($j=1,2$). Now let J_2 be a fixed countable dense subset of $P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$ with respect to the metric d; such a set exists since $\mathrm{C}(\mathrm{S}_1^{d-1})$ is dense in $\mathrm{L}^1(\mathrm{S}_1^{d-1})$ [25, Thm. 3.14].

We will apply Lemma 8.6 with $\mathcal{Q}=\mathcal{L}_\psi$, $J:=J_1\times J_2$, and with the function $F:J\times\mathcal{L}_\psi\times(0,1)\to\mathbb{R}$ defined by

$$F\big(\langle f,\lambda\rangle,\boldsymbol{q},\rho\big):=\int_{\mathrm{S}_1^{d-1}}f\big(\Gamma g_0^{(\boldsymbol{q})}\varphi(R(\boldsymbol{v})D_\rho)\big)\,d\lambda(\boldsymbol{v})-\int_{\mathbb{X}^\psi}f\,d\overline{\omega^{(\boldsymbol{q})}}. \tag{8.40}$$

Recall that $\overline{\omega^{(\boldsymbol{q})}}(\mathbb{X}^\psi)=1$ for every $\boldsymbol{q}\in\mathcal{L}_\psi$, by Lemmas 5.4 and 5.2; hence the integral over $\mathbb{X}^\psi$ in (8.40) may just as well be taken over all $\mathbb{X}$. Recall also that for any $\boldsymbol{q}\in\mathbb{R}$ we have $\overline{\omega^{(\boldsymbol{q})}}=\mu^{(\boldsymbol{q})}$, by Proposition 3.7 and (3.2), (5.5), (5.12). Hence by Theorem 3.2 we have $F\big(\langle f,\lambda\rangle,\boldsymbol{q},\rho\big)\to 0$ for any fixed $\langle f,\lambda\rangle\in J$ and $\boldsymbol{q}\in\mathcal{L}_\psi$, i.e. the assumption (i) in Lemma 8.6 holds.

We next verify that also the assumption (ii) in Lemma 8.6 holds. Thus let $\langle f,\lambda\rangle\in J$ and $\varepsilon,\varepsilon'>0$ be given. Choose $k\in\mathbb{Z}^+$ as in Theorem 8.4 (for our given ψ,f,ε). By Proposition 8.5, we can now fix some $\eta>0$ such that the set $\mathcal{Z}:=\{\boldsymbol{q}\in\mathcal{L}_\psi\,:\,\pi(U^{(\boldsymbol{q})})\in\Delta_{\psi,k}^{(\eta)}\}$ has upper density $<\varepsilon'$. We keep $\widetilde{M}\in G'\setminus\mathfrak{D}_{\mathcal{S}}$ fixed as in (2.22), for our given, fixed union of grids $\mathcal{P}$. Because of our choice of k, we may now fix $\rho_0\in(0,1)$ in such a way that (8.24) holds for all $\rho\in(0,\rho_0)$ and all $V\in\mathbb{Y}^\psi\setminus\Delta_{\psi,k}^{(\eta)}$, for our chosen $\widetilde{M},\psi,f,\lambda,\varepsilon,\eta$. Now for any $\boldsymbol{q}\in\mathcal{L}_\psi\setminus\mathcal{Z}$, we may apply (8.24) with $V:=\pi(U^{(\boldsymbol{q})})\in\widetilde{\mathbb{T}}$; indeed, for this V we have $V\in\mathbb{Y}^\psi\setminus\Delta_{\psi,k}^{(\eta)}$ since $\boldsymbol{q}\notin\mathcal{Z}$; also $\Gamma g_0^{(\boldsymbol{q})}=x(V)\widetilde{M}$ (by (2.23)) and $\omega^{(V)}=\omega^{(\boldsymbol{q})}$. Hence we conclude that for all $\boldsymbol{q}\in\mathcal{L}_\psi\setminus\mathcal{Z}$ and all $\rho\in(0,\rho_0)$ we have $\left|F(\langle f,\lambda\rangle,\boldsymbol{q},\rho)\right|<\varepsilon$. Hence assumption (ii) in Lemma 8.6 is indeed fulfilled.

It now follows from Lemma 8.6 that there exists a subset $\mathcal{E}\subset\mathcal{L}_\psi$ of density zero such that for any $f\in J_1$ and $\lambda\in J_2$ we have $F(\langle f,\lambda\rangle,\boldsymbol{q},\rho)\to0$ as $\rho\to0$, uniformly over all $\boldsymbol{q}\in\mathcal{L}_\psi\cap\mathcal{B}^d_{\mathcal{T}(\rho)}\setminus\mathcal{E}$, viz., the uniform convergence in (6.10) in Theorem 6.6 holds. To complete the proof of Theorem 6.6, we will give a (standard) approximation argument to show that the uniform convergence in (6.10) in fact holds for arbitrary $f\in\mathrm{C}_b(\mathbb{X}^\psi)$ and $\lambda\in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$.

Note that equation (8.40) defines $F(\langle f,\lambda\rangle,\boldsymbol{q},\rho)$ for arbitrary $f\in\mathrm{C}_b(\mathbb{X}^\psi)$ and $\lambda\in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$; and for any $f_1,f_2\in\mathrm{C}_b(\mathbb{X}^\psi)$, $\lambda_1,\lambda_2\in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$, $\boldsymbol{q}\in\mathcal{L}_\psi$ and $\rho\in(0,1)$ we have, with $\|\cdot\|_u$ denoting the uniform norm on $\mathrm{C}_b(\mathbb{X}^\psi)$:

$$\begin{aligned}&\Big|F(\langle f_1,\lambda_1\rangle,\boldsymbol{q},\rho)-F(\langle f_2,\lambda_2\rangle,\boldsymbol{q},\rho)\Big|\\&\qquad\le2\|f_1-f_2\|_u+\int_{\mathrm{S}_1^{d-1}}\big|f_1(\Gamma g_0^{(\boldsymbol{q})}\varphi(R(\boldsymbol{v})D_\rho))\big|\cdot|\lambda_1'(\boldsymbol{v})-\lambda_2'(\boldsymbol{v})|\,d\sigma(\boldsymbol{v})\\&\qquad\le2\|f_1-f_2\|_u+\|f_1\|_u\cdot d(\lambda_1,\lambda_2).\end{aligned}$$

Using this bound, and the fact that J_1 and J_2 are dense in $\mathrm{C}_c(\mathbb{X}^\psi)$ and $P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$, respectively, it is immediate to extend the uniform convergence in (6.10) (with the subset $\mathcal{E}\subset\mathcal{L}_\psi$ fixed once and for all) from $f\in J_1$ and $\lambda\in J_2$ to arbitrary $f\in\mathrm{C}_c(\mathbb{X}^\psi)$ and $\lambda\in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$.

It remains to extend to arbitrary functions $f\in\mathrm{C}_b(\mathbb{X}^\psi)$. Thus let $f\in\mathrm{C}_b(\mathbb{X}^\psi)$, $\lambda\in P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$ and $\varepsilon>0$ be given. Set $B:=\|f\|_u$; we may assume $B>0$ since otherwise f is identically zero. Let ν be the $\mathrm{SL}_d(\mathbb{R})$ invariant probability measure on $\mathrm{SL}_d(\mathbb{Z})\backslash\mathrm{SL}_d(\mathbb{R})$; fix a compact subset K' of $\mathrm{SL}_d(\mathbb{Z})\backslash\mathrm{SL}_d(\mathbb{R})$ with $\nu(K')>\big(1-\varepsilon/(4B)\big)^{1/N}$, and then set $K:=\mathbb{X}^\psi\cap\prod_{j=1}^N\widetilde{\iota}_j^{-1}(K')$, where $\widetilde{\iota}_j$ is the projection from $\mathbb{X}_j$ to $\mathrm{SL}_d(\mathbb{Z})\backslash\mathrm{SL}_d(\mathbb{R})$ [14]. Then K is a compact subset of $\mathbb{X}^\psi$, and for every $\boldsymbol{q}\in\mathcal{L}_\psi$ we have

$$\overline{\omega^{(\boldsymbol{q})}}(K)=\overline{\omega^{(\boldsymbol{q})}}\bigg(\prod_{j=1}^N\widetilde{\iota}_j^{-1}(K')\bigg)=\prod_{j=1}^N\overline{\omega_j^{(\boldsymbol{q})}}\big(\widetilde{\iota}_j^{-1}(K')\big)=\nu(K')^N>1-\frac{\varepsilon}{4B},\tag{8.41}$$

where we first used the fact that $\overline{\omega^{(\boldsymbol{q})}}(\mathbb{X}^\psi)=1$ (by Lemmas 5.4 and 5.2), and then used Lemma 3.5. Next fix a function $h\in\mathrm{C}_c(\mathbb{X}_\psi)$ with $0\le h\le1$ and $h|_K\equiv1$. By the uniform convergence which we have already proved, there exists $\rho_0\in(0,1)$ such that for all $\rho\in(0,\rho_0)$ and $\boldsymbol{q}\in\mathcal{L}_\psi\cap\mathcal{B}^d_{\mathcal{T}(\rho)}\setminus\mathcal{E}$ we have $\left|F(\langle h,\lambda\rangle,\boldsymbol{q},\rho)\right|<\varepsilon/(4B)$; but also $\int_{\mathbb{X}^\psi}h\,d\overline{\omega^{(\boldsymbol{q})}}\ge\overline{\omega^{(\boldsymbol{q})}}(K)>1-\varepsilon/(4B)$, and hence $\int_{\mathrm{S}_1^{d-1}}h\big(\Gamma g_0^{(\boldsymbol{q})}\varphi(R(\boldsymbol{v})D_\rho)\big)\,d\lambda(\boldsymbol{v})>1-\varepsilon/(2B)$. It follows that

$$\lambda\big(\big\{\boldsymbol{v}\in\mathrm{S}_1^{d-1}\,:\,\Gamma g_0^{(\boldsymbol{q})}\varphi(R(\boldsymbol{v})D_\rho)\in\operatorname{supp}(h)\big\}\big)>1-\frac{\varepsilon}{2B},$$

for all $\rho\in(0,\rho_0)$ and $\boldsymbol{q}\in\mathcal{L}_\psi\cap\mathcal{B}^d_{\mathcal{T}(\rho)}\setminus\mathcal{E}$.

[14] This is the map which we called "$\widetilde{\iota}$" in (2.26); we now call it $\widetilde{\iota}_j$ for clarity.

Next fix a function $h_1 \in \mathrm{C}_c(\mathbb{X}^\psi)$ with $0 \leq h_1 \leq 1$ and $h_1|_{\mathrm{supp}(h)} \equiv 1$, and set $f_1 := h_1 f \in \mathrm{C}_c(\mathbb{X}^\psi)$. Then note that for all $x \in \mathbb{X}^\psi$ we have

$$\bigl|f(x) - f_1(x)\bigr| \leq \bigl|f(x)\bigr| \cdot \bigl|1 - h_1(x)\bigr| \leq B\bigl|1 - h_1(x)\bigr| \leq B \cdot I(x \notin \mathrm{supp}(h)).$$

Hence for all $\rho \in (0, \rho_0)$ and $\boldsymbol{q} \in \mathcal{L}_\psi \cap \mathcal{B}^d_{\mathcal{T}(\rho)} \setminus \mathcal{E}$ we have

$$\left| \int_{\mathrm{S}_1^{d-1}} f\bigl(\Gamma g_0^{(\boldsymbol{q})} \varphi(R(\boldsymbol{v}) D_\rho)\bigr)\, d\lambda(\boldsymbol{v}) - \int_{\mathrm{S}_1^{d-1}} f_1\bigl(\Gamma g_0^{(\boldsymbol{q})} \varphi(R(\boldsymbol{v}) D_\rho)\bigr)\, d\lambda(\boldsymbol{v}) \right| \\ \leq B \cdot \lambda\bigl(\bigl\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} \,:\, \Gamma g_0^{(\boldsymbol{q})} \varphi(R(\boldsymbol{v}) D_\rho) \notin \mathrm{supp}(h)\bigr\}\bigr) < \frac{\varepsilon}{2},$$

and also

$$\left| \int_{\mathbb{X}^\psi} f \, d\overline{\omega^{(\boldsymbol{q})}} - \int_{\mathbb{X}^\psi} f_1 \, d\overline{\omega^{(\boldsymbol{q})}} \right| \leq B \cdot \overline{\omega^{(\boldsymbol{q})}}\bigl(\mathbb{X}^\psi \setminus \mathrm{supp}(h)\bigr) \leq B \cdot \overline{\omega^{(\boldsymbol{q})}}\bigl(\mathbb{X}^\psi \setminus K\bigr) < \frac{\varepsilon}{4}.$$

Hence for these ρ and $\boldsymbol{q}$ we have $\bigl|F\bigl(\langle f, \lambda\rangle, \boldsymbol{q}, \rho\bigr) - F\bigl(\langle f_1, \lambda\rangle, \boldsymbol{q}, \rho\bigr)\bigr| < 3\varepsilon/4$. Furthermore, by again applying the uniform convergence result which we have already proved it follows that after possibly shrinking ρ_0, we have $\bigl|F\bigl(\langle f_1, \lambda\rangle, \boldsymbol{q}, \rho\bigr)\bigr| < \varepsilon/4$ for all $\rho \in (0, \rho_0)$ and all $\boldsymbol{q} \in \mathcal{L}_\psi \cap \mathcal{B}^d_{\mathcal{T}(\rho)} \setminus \mathcal{E}$. Combining the last two inequalities, we conclude that $\bigl|F\bigl(\langle f, \lambda\rangle, \boldsymbol{q}, \rho\bigr)\bigr| < \varepsilon$ for all $\rho \in (0, \rho_0)$ and all $\boldsymbol{q} \in \mathcal{L}_\psi \cap \mathcal{B}^d_{\mathcal{T}(\rho)} \setminus \mathcal{E}$. This completes the proof of the uniform convergence in (6.10), viz., the proof of Theorem 6.6. □

Note that in view of the results in Section 6, we have now also completed the proof of the main result of the paper, Theorem 1.1.

Index of notation

For the convenience of the reader, we include an index of some of the most important notation. We stress that in Section 7, some of the notation listed below (for example, "Γ", "Γ_j" and "$\widetilde{M}$") is used in a slightly different and more general way; this is explained in the beginning of that section.

$\widetilde{\boldsymbol{c}}_j$	The vector $\bigl(c_{j,1}^{-1} \,\cdots\, c_{j,r_j}^{-1}\bigr)^{\mathrm{T}}$ in $\mathbb{R}^{r_j}$	21
c_ψ	fixed positive real numbers such that (2.4)–(2.6) hold	9
$c_{\mathcal{P}}$	$\sum_{\psi \in \Psi} \overline{n}_\psi$, the asymptotic density of the point set $\mathcal{P}$	10
$\boldsymbol{e}_k$	the kth standard unit vector in $\mathbb{R}^d$ (or in $\mathbb{R}^r$ or $\mathbb{R}^{r_j}$)	4, 19, 27
G	$\mathrm{S}_{r_1}(\mathbb{R}) \times \cdots \times \mathrm{S}_{r_N}(\mathbb{R})$	11
G_j	$\mathrm{S}_{r_j}(\mathbb{R})$	11
G'	$\mathrm{SL}_d(\mathbb{R})^N$ (a subgroup of G)	41
$g_0^{(\boldsymbol{q})}$	$\mathrm{I}_{U(\boldsymbol{q})}\ \widetilde{M}$	12
$\mathfrak{L}(S)$	for $\emptyset \neq S \subset \mathbb{T}^{r_j}$, $\mathfrak{L}(S) := \overline{\langle \pi^{-1}(S) \rangle}^{\circ}$ (a rational subspace of $\mathbb{R}^{r_j}$)	13
L_j	the rational subspace of $\mathbb{R}^{r_j}$ defined in (4.14)	21
$L_j^{(\boldsymbol{q})}$	$L_j^{(\pi(U_j^{(\boldsymbol{q})}))}$	12, 15
$L_j^{(V)}$	$\mathfrak{L}(V_1, \ldots, V_d)$, for $V = (V_1, \ldots, V_d) \in \mathbb{T}_j^d$	14
$\mathcal{L}_\psi$	$c_\psi(\mathcal{L}_{j_\psi} + \boldsymbol{v}_\psi)$	9
L_j^ψ	the rational subspace of $\mathbb{R}^{r_j}$ given by (4.4)	19
M_j	$M_1, \ldots, M_N$ are fixed elements in $\mathrm{SL}_d(\mathbb{R})$ such that (2.4)–(2.6) hold	9
$\widetilde{M}$	$(M_1, \ldots, M_N)$ (an element in G', thus in G)	11
$\overline{n}_\psi$	c_ψ^{-d}	10
$\mathcal{O}_j^{(\boldsymbol{q})}$	$\mathcal{O}_j^{(\pi(U_j^{(\boldsymbol{q})}))}$	15
$\mathcal{O}_j^{(V)}$	the subset of $\bigl(\mathbb{S}_j^{(V)}\bigr)^d$ given by (3.7)	14

$\mathcal{O}_j^\psi$	the subset of $\mathbb{T}_j^d$ given in (4.17)	22
$P(S)$	the set of Borel probability measures on S (for any topological space S)	3
$P_{\mathrm{ac}}(\mathrm{S}_1^{d-1})$	the set of $\lambda \in P(\mathrm{S}_1^{d-1})$ which are absolutely continuous with respect to σ	5
$P(\mathbb{T}_j^d)'$	the subset of $\mathrm{SL}_d(\mathbb{Z})$-invariant measures in $P(\mathbb{T}_j^d)$	14
p_j	the projection map $G \to \mathrm{S}_{r_j}(\mathbb{R})$	11
$\widetilde{p}_j$	the projection map $\mathbb{X} \to \mathbb{X}_j$	11
p_ψ	for $\psi = (i,j) \in \Psi$, p_ψ is the map $\mathsf{a}_i \circ p_j$ from G to $\mathrm{ASL}_d(\mathbb{R})$	11
R	a fixed map $\mathrm{S}_1^{d-1} \to \mathrm{SO}(d)$ such that $\boldsymbol{v}R(\boldsymbol{v}) = \boldsymbol{e}_1$, $\forall \boldsymbol{v} \in \mathrm{S}_1^{d-1}$	4
r_i	the projection map $\mathrm{M}_{r\times d}(\mathbb{R}) \to \mathbb{R}^d$ which takes any matrix to its ith row	11
$\widetilde{\mathsf{r}}_i$	for $1 \leq i \leq r_j$, $\widetilde{\mathsf{r}}_i$ is the projection map $\mathbb{T}_j^d \to (\mathbb{R}/\mathbb{Z})^d$ induced by r_i	12
$\mathbb{S}_j^\psi$	$\pi(L_j^\psi)$	22
$\mathbb{S}_j^{(\boldsymbol{q})}$	$\mathbb{S}_j^{(\pi(U_j^{(\boldsymbol{q})}))}$	15
$\mathbb{S}_j^{(V)}$	$\overline{\langle V_1, \ldots, V_d \rangle}$, a closed subgroup of $\mathbb{T}_j$	14
$\mathbb{T}_j$	$(\mathbb{R}/\mathbb{Z})^{r_j}$	12
$\mathbb{T}_j^d$	$\mathbb{T}_j \times \cdots \times \mathbb{T}_j = \mathrm{M}_{r_j\times d}(\mathbb{R}/\mathbb{Z})$	12
$\widetilde{\mathbb{T}}$	$\mathbb{T}_1^d \times \mathbb{T}_2^d \times \cdots \times \mathbb{T}_N^d$	42
$U_j^{(\boldsymbol{q})}$	the $r_j \times d$ matrix with row vectors $\boldsymbol{w}_{j,i} - c_{j,i}^{-1}\boldsymbol{q}M_j^{-1}$ $(i = 1, \ldots, r_j)$	11
$U^{(\boldsymbol{q})}$	$(U_1^{(\boldsymbol{q})}, \cdots, U_N^{(\boldsymbol{q})})$	11
$\boldsymbol{w}_\psi$	fixed vectors in $\mathbb{R}^d$ such that (2.4)–(2.6) hold	9
W_j	The $r_j \times d$ matrix with row vectors $\boldsymbol{w}_{j,i}$ $(i = 1, \ldots, r_j)$	21
$\mathbb{X}$	$\Gamma\backslash G$	11
$\mathbb{X}_j$	$\Gamma_j\backslash G_j$	11
$\mathbb{X}^\psi$	$\{\Gamma g \in \mathbb{X} \,:\, g \in G,\ \boldsymbol{0} \in \mathbb{Z}^d p_\psi(g)\}$	28
Γ	$\mathrm{S}_{r_1}(\mathbb{Z}) \times \cdots \times \mathrm{S}_{r_N}(\mathbb{Z})$	11
Γ_j	$\mathrm{S}_{r_j}(\mathbb{Z})$	11
ι	the projection $\mathrm{S}_r(\mathbb{R}) \to \mathrm{SL}_d(\mathbb{R})$	11
$\widetilde{\iota}$	the projection $\mathbb{X}_j \to \mathrm{SL}_d(\mathbb{Z})\backslash \mathrm{SL}_d(\mathbb{R})$	12
π	either of the projection maps $\mathbb{R}^{r_j} \to \mathbb{T}_j$ or $(\mathbb{R}^{r_j})^d \to \mathbb{T}_j^d$	12
σ	$\mathrm{vol}_{\mathrm{S}_1^{d-1}}$, Lebesgue measure on S_1^{d-1}	2
σ^ψ	(ψ, ω^ψ) (an element in Σ)	29
φ	the diagonal embedding $\mathrm{SL}_d(\mathbb{R}) \to G$	13
Ψ	$\{(j,i) \,:\, j \in \{1, \ldots, N\},\ i \in \{1, \ldots, r_j\}\}$	9
$\psi(\boldsymbol{q})$	for $\boldsymbol{q} \in \mathcal{P}$, $\psi(\boldsymbol{q})$ is a fixed element in Ψ such that $\boldsymbol{q} \in \mathcal{L}_{\psi(\boldsymbol{q})}$	10
Ω	$\prod_{j=1}^N P(\mathbb{T}_j^d)'$	29
$\omega_j^{(\boldsymbol{q})}$	$\omega_j^{(\pi(U_j(\boldsymbol{q})))}$	15
$\omega_j^{(V)}$	normalized restriction to $\mathcal{O}_j(V)$ of the Haar measure on $\mathbb{S}_j(V)^d$	14
ω_j^ψ	normalized restriction to $\mathcal{O}_j^\psi$ of the Haar measure on $(\widetilde{\mathbb{S}}_j^\psi)^d$	22
$\overline{\omega}$	for $\omega \in P(\mathbb{T}_j^d)'$, $\overline{\omega}$ is the probability measure on $\mathbb{X}_j$ defined below (3.8);	15
	for $\omega \in \Omega$, $\overline{\omega}$ is the probability measure on $\mathbb{X}$ defined in (5.12)	30
ω^ψ	$\langle \omega_1^\psi, \ldots, \omega_N^\psi \rangle$ (an element in Ω)	29

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